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Research Article

Best Proximity Pairs Theorems for Continuous Set-Valued Maps

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A best proximity pair for a set-valued map $F : A \multimap B$ with respect to a set-valued map $G : A \multimap A$ is defined, and a new existence theorem of best proximity pairs for continuous set-valued maps is proved in nonexpansive retract metric spaces. As an application, we derive a coincidence point theorem.

1. Introduction and preliminaries

Let (M,d) be a metric space and let A and B be nonempty subsets of M. Let $d(A,B) = \inf\{d(a,b) : a \in A,\ b \in B\}$, and $\mathrm{Prox}(A,B) = \{(a,b) \in A \times B : d(a,b) = d(A,B)\}$. A is said to be approximately compact if for each $y \in M$ and each sequence (x_n) in A satisfying the condition $d(x_n,y) \to d(y,A)$ there is a subsequence of (x_n) converging to an element of A. Let

$$B_0 := \{b \in B : d(a,b) = d(A,B) \text{ for some } a \in A\},$$
$$A_0 := \{a \in A : d(a,b) = d(A,B) \text{ for some } b \in B\}. \tag{1.1}$$

Let $G : A \multimap A$ and $F : A \multimap B$ be set-valued maps. $(G(x_0), F(x_0))$ is called a *best proximity pair* for F with respect to G if $d(G(x_0), F(x_0)) = d(A,B)$. Best proximity pair theorems analyze the conditions under which the problem of minimizing the real-valued function $x \to d(G(x), F(x))$ has a solution. In the setting of normed linear spaces, the best proximity pair problem has been studied by many authors; see [1–5]. In 2000, Sadiq Basha and Veeramani [4] proved the following theorem.

Theorem 1.1. *Let E be a normed linear space. Let A be a nonempty, approximately compact and convex subset of E and let B be a nonempty, closed and convex subset of E such that* $\operatorname{Prox}(A, B)$ *is nonempty and A_0 is compact. Suppose that*

(a) *$F : A \multimap B$ is a set-valued map such that for every $x \in A_0$, $F(x) \cap B_0 \neq \emptyset$, and for every $y \in B_0$, the fiber $F^{-1}(y)$ is open;*

(b) *for every open set U in A, the set $\cap\{F(u) : u \in U\}$ is convex;*

(c) *$g : A \to A$ is a continuous, proper, quasi-affine, and surjective single-valued map such that $g^{-1}(A_0) \subseteq A_0$.*

Then there exists an element $x_0 \in A_0$ such that

$$d(g(x_0), F(x_0)) = d(A, B). \tag{1.2}$$

In the rest of this section we recall some definitions and theorems which are used in the next section. Let X and Y be topological spaces with $A \subseteq X$ and $B \subseteq Y$. Let $F : X \multimap Y$ be a set-valued map with nonempty values. The image of A under F is the set $F(A) = \bigcup_{x \in A} F(x)$ and the inverse image of B under F is $F^{-}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$. Now F is said to be

(a) closed if its graph, $\operatorname{Gr}(F) = \{(x, y) \in X \times Y : y \in F(x)\}$, is a closed set in product space $X \times Y$;

(b) upper semicontinuous, if for each closed set $B \subseteq Y$, $F^{-}(B)$ is closed in X;

(c) lower semicontinuous, if for each open set $B \subseteq Y$, the set $F^{-}(B)$ is open;

(d) continuous if F is both lower semicontinuous and upper semicontinuous.

We say that $F : X \multimap Y$ is onto if $F(X) = Y$. If $F : X \multimap Y$ is onto then $F^{-} : Y \multimap X$, the lower inverse of F, is defined by $F^{-}(y) = \{x \in X : y \in F(x)\}$. $f : X \to Y$ is called a homeomorphism if f is a bijective, continuous, and open map. We say that the set-valued mapping $F : X \multimap Y$ has a continuous selection if there exists a continuous function $f : X \to Y$ such that $f(x) \in F(x)$ for each $x \in X$. We let

$$\mathcal{S}(X, Y) = \{F : X \multimap Y : F \text{ has a continuous selection}\}. \tag{1.3}$$

For a nonempty finite subset D of X, let $\langle D \rangle$ denote the set of all nonempty finite subsets of D.

Definition 1.2. Let X be a nonempty subset of a topological vector space Y. A set-valued map $F : X \multimap Y$ is said to be a generalized KKM mapping (GKKM) if for each nonempty finite set $\{x_1, \ldots, x_n\} \subseteq X$, there exist a set $\{y_1, \ldots, y_n\}$ of points of Y, not necessarily all different, such that for each subset $\{y_{i_1}, \ldots, y_{i_k}\}$ of $\{y_1, \ldots, y_n\}$, we have

$$\operatorname{conv}\{y_{i_1}, \ldots, y_{i_k}\} \subseteq \bigcup_{j=1}^{k} F(x_{i_j}). \tag{1.4}$$

The following extension of the classical KKM principle in topological vector spaces is due to Chang and Zhang [6].

Theorem 1.3. *Let X be a nonempty subset of a topological vector space Y and let $F : X \multimap Y$ be a GKKM mapping with closed values. Then, the family $\{F(x) : x \in X\}$ has the finite intersection property, that is,*

$$\bigcap_{x\in A} F(x) \neq \varnothing \quad \text{for each } A \in \langle X\rangle. \tag{1.5}$$

Furthermore, if there exists an $x_0 \in X$ such that $F(x_0)$ is a compact set in Y, then

$$\bigcap_{x\in X} F(x) \neq \varnothing. \tag{1.6}$$

Let X be a nonempty subset of a topological vector space Y. Let $F : X \multimap Y$ and $G : Y \multimap Y$ be set-valued mappings such that for each nonempty finite set $\{x_1, \dots, x_n\} \subseteq X$, there exists a set $\{y_1, \dots, y_n\}$ of points of Y, not necessarily all different, such that for each subset $\{y_{i_1}, \dots, y_{i_k}\}$ of $\{y_1, \dots, y_n\}$, we have

$$G(\operatorname{conv}\{y_{i_1}, \dots, y_{i_k}\}) \subseteq \bigcup_{j=1}^{k} F(x_{i_j}). \tag{1.7}$$

Then F is called a generalized KKM mapping with respect to G. If the set-valued mapping $G : Y \multimap Y$ satisfies the requirement that for any generalized KKM mapping $F : X \multimap Y$ with respect to G the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then G is said to be have the KKM property. We denote

$$\text{KKM}(Y) = \{G : Y \multimap Y : G \text{ has the KKM property}\}. \tag{1.8}$$

By Theorem 1.3, the identity map I_Y has the KKM property. It is well known, and easy to see, that the continuous functions have the KKM property. *Thus if a set-valued mapping G has a continuous selection, then G has trivially the KKM property.*

Let (M, d) be a metric space and let $B(x, r) = \{y \in M : d(x, y) \leq r\}$ denote the closed ball with center x and radius r. Let

$$\operatorname{co}(A) = \bigcap\{B \subseteq M : B \text{ is a closed ball in } M \text{ such that } A \subseteq B\}. \tag{1.9}$$

If $A = \operatorname{co}(A)$, we say that A is an admissible subset of M. Note that $\operatorname{co}(A)$ is admissible and the intersection of any family of admissible subsets of M is admissible. The following definition of a hyperconvex metric space is due to Aronszajn and Panitchpakdi [7].

Definition 1.4. A metric space (M, d) is said to be a hyperconvex metric space if for any collection of points x_α of M and any collection r_α of nonnegative real numbers with $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, we have

$$\bigcap_{\alpha} B(x_\alpha, r_\alpha) \neq \varnothing. \tag{1.10}$$

The simplest examples of hyperconvex spaces are finite dimensional real Banach spaces and l_∞ endowed with the maximum norm.

Now we introduce an important class of metric spaces.

Definition 1.5 (see [8]). A nonexpansive retract metric space (i.e., an $\mathcal{NR}$-metric space) (M,E,r) consists of a metric space (M,d), a convex subset (E,ρ) of a metrizable topological vector space (V,ρ) in which every closed ball is convex such that (M,d) can be isometrically embedded into (E,ρ) and $r: E \to M$ is a nonexpansive retraction.

Let $A \subseteq M$. We say that A is r-convex if, for each $D \in \langle A\rangle$, $r(\mathrm{conv}(D)) \subseteq A$ (note we identify M with the isometric embedding image set in E).

Remark 1.6. Every closed ball in (E,ρ) is convex if and only if

$$\rho(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2) \leq \max\left(\rho(x_1,y_1), \rho(x_2,y_2)\right), \tag{1.11}$$

for each $x_1, x_2, y_1, y_2 \in E$, $\alpha + \beta = 1$, $\alpha, \beta \geq 0$.

Examples 1.7. (a) Let $(X,\|\cdot\|)$ be a normed linear space. Let $E = X$, $\rho(x,y) = \|x-y\|$, and $r = I$ the identity mapping. Then $(X,\|\cdot\|)$ is a nonexpansive retract metric space. In this case $A \subseteq X$ is r-convex if and only if A is convex.

(b) Let (M,d) be a hyperconvex metric space. It is well known that there exists an index set I and a natural isometric embedding from M into $l_\infty(I)$. Also there exists a nonexpansive retraction $r: l_\infty(I) \to M$. Thus every hyperconvex metric space is an $\mathcal{NR}$-metric space. In hyperconvex metric spaces, every admissible set is r-convex . To see this, let $A \subseteq M$ be admissible and $D \in \langle A\rangle$. Then $r(\mathrm{conv}(D)) \subseteq \mathrm{co}(D)$ [9]. Since A is admissible, then $\mathrm{co}(D) \subseteq \mathrm{co}(A) = A$. Thus $r(\mathrm{conv}(D)) \subseteq A$, which implies that A is r-convex.

(c) Let (X,d) be a metrizable Hausdorff topological vector space in which every closed ball is convex. Let $E = X$, $\rho(x,y) = d(x,y)$, and $r = I$ be the identity mapping. Then (X,d) is an $\mathcal{NR}$-metric space. In this case, $A \subseteq X$ is r-convex if and only if A is convex.

2. Main theorems

This section is devoted to main results on best proximity pairs.

Theorem 2.1. *Let (M,E,r) be an $\mathcal{NR}$-metric space. Let $A \subseteq M$ be nonempty, compact, r-convex, and let B be a nonempty subset of M. Let $G: A \multimap A$ be a continuous, onto set-valued map with compact values such that $G^- \in \mathcal{S}(A,A)$. Let $F: A \multimap B$ be a continuous set-valued map with r-convex, compact values. Assume that $F(x) \cap B_0 \neq \varnothing$, for each $x \in A$. Then there exists $x_0 \in A$ such that*

$$d\big(G(x_0), F(x_0)\big) = d(A,B). \tag{2.1}$$

Proof. Define a set-valued map $H: A \multimap A$ by

$$H(y) = \big\{x \in A : d\big(G(x), F(x)\big) \leq d\big(G(y), F(x)\big)\big\}. \tag{2.2}$$

Since $y \in H(y)$, then $H(y) \neq \emptyset$ for each $y \in A$. We show that for each $y \in A$, $H(y)$ is closed and therefore is a compact subset of A. Let $x_n \in H(y)$ and $x_n \to x$. Since F and G are compact-valued, then there exist $s \in G(y)$, $t \in F(x)$, $u_n \in G(x_n)$, and $v_n \in F(x_n)$ such that

$$\begin{gathered} d(G(x_n), F(x_n)) = d(u_n, v_n), \\ d(G(y), F(x)) = d(s, t). \end{gathered} \tag{2.3}$$

Now F is lower semicontinuous so for each $n \in \mathbb{N}$, there exists $t_n \in F(x_n)$ such that $t_n \to t$. Since $F(A)$ and $G(A)$ are compact and F and G are closed, without loss of generality, we may assume that $u_n \to u$, $v_n \to v$, $u \in G(x)$ and $v \in F(x)$. Therefore since $x_n \in H(y)$, we have

$$\begin{aligned} d(G(x), F(x)) &\leq d(u, v) \\ &= \lim_n d(u_n, v_n) \\ &= \lim_n d(G(x_n), F(x_n)) \\ &\leq \limsup_n d(G(y), F(x_n)) \\ &\leq \lim_n d(s, t_n) \\ &= d(s, t) = d(G(y), F(x)), \end{aligned} \tag{2.4}$$

which shows that $x \in H(y)$. Now, we prove that

$$H : A \subseteq E \multimap E \tag{2.5}$$

is a generalized KKM mapping with respect to $G^- \circ r$. To show this, suppose that $x_1, \dots, x_n$ are in A and take any y_0 with $y_0 \notin \bigcup_{i=1}^n H(x_i)$. Then we have

$$d(G(y_0), F(y_0)) > d(G(x_k), F(y_0)), \quad \forall k = 1, \dots, n. \tag{2.6}$$

Let

$$S(y_0) := \{x \in A : \exists y \in G(x) \text{ such that } d(G(y_0), F(y_0)) > d(y, F(y_0))\}. \tag{2.7}$$

Clearly $x_k \in S(y_0)$ for $k = 1, \dots, n$. Let $g : A \to A$ be a selection of G (not necessary continuous). We take $z_k \in F(y_0)$ such that

$$d(G(y_0), F(y_0)) > d(g(x_k), z_k), \quad \text{for } 1 \leq k \leq n. \tag{2.8}$$

Let $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$. Now r is nonexpansive and Remark 1.6 yields (note we identify M with the isometric embedding image set in E)

$$d\left(r\left(\sum_{i=1}^{n}\lambda_i g(x_i)\right), r\left(\sum_{i=1}^{n}\lambda_i z_i\right)\right) \leq \rho\left(\sum_{i=1}^{n}\lambda_i g(x_i), \sum_{i=1}^{n}\lambda_i z_i\right) \leq \max_{1\leq i\leq n} \rho(g(x_i), z_i) = \max_{1\leq i\leq n} d(g(x_i), z_i) < d(G(y_0), F(y_0)). \tag{2.9}$$

Since $F(y_0)$ and A are r-convex, then

$$r\left(\sum_{i=1}^{n}\lambda_i z_i\right) \in F(y_0), \qquad r\left(\sum_{i=1}^{n}\lambda_i g(x_i)\right) \in A. \tag{2.10}$$

Thus

$$d\left(r\left(\sum_{i=1}^{n}\lambda_i g(x_i)\right), F(y_0)\right) < d(G(y_0), F(y_0)). \tag{2.11}$$

Hence, we deduce that (note that G is onto and see the definition of $S(y_0)$ with $y = r(\sum_{i=1}^{n}\lambda_i g(x_i))$)

$$G^{-}(r(\text{conv}\{g(x_1), \ldots, g(x_n)\})) \subseteq S(y_0). \tag{2.12}$$

As $y_0 \notin S(y_0)$, we have $y_0 \notin G^{-}(r(\text{conv}\{g(x_1), \ldots, g(x_n)\}))$. Consequently,

$$G^{-} \circ r(\text{conv}\{g(x_1), \ldots, g(x_n)\}) \subseteq \bigcup_{i=1}^{n} H(x_i). \tag{2.13}$$

Since $x_1, \ldots, x_n$ are arbitrary elements of A, then we deduce that for each subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ we have

$$G^{-} \circ r(\text{conv}\{g(x_{i_1}), \ldots, g(x_{i_k})\}) \subseteq \bigcup_{j=1}^{k} H(x_{i_j}). \tag{2.14}$$

Now since $G^{-} \in \mathcal{S}(A, A)$ and r is continuous, then $G^{-} \circ r \in \mathcal{S}(E, A)$ and so $G^{-} \circ r$ has the KKM property. Hence the family $\{H(x) : x \in A\}$ has the finite intersection property. Now since $H(x)$ is compact for any $x \in A$, we have immediately that $\bigcap_{x\in A} H(x) \neq \emptyset$. Therefore, there exists an $x_0 \in A$ such that

$$x_0 \in \bigcap_{x\in A} H(x). \tag{2.15}$$

Then, it is clear that

$$d(G(x_0), F(x_0)) \le d(G(x), F(x_0)) \quad \forall x \in A. \tag{2.16}$$

Since $x_0 \in A$, then

$$d(G(x_0), F(x_0)) = \inf_{x\in A} d(G(x), F(x_0)). \tag{2.17}$$

Since $G : A \multimap A$ is onto, then for each $y \in A$ there exists $x \in A$ such that $y \in G(x)$. Thus

$$d(A, F(x_0)) \le d(G(x), F(x_0)) \le d(y, F(x_0)). \tag{2.18}$$

Hence

$$\inf_{x\in A} d(G(x), F(x_0)) = d(A, F(x_0)). \tag{2.19}$$

Pick $b \in F(x_0) \cap B_0 \neq \emptyset$. Then there exists $a \in A$ such that $d(a,b) = d(A,B)$. Thus

$$d(A, F(x_0)) \le d(A, b) \le d(a, b) = d(A, B). \tag{2.20}$$

By (2.17), (2.19), and (2.20), we get

$$d(G(x_0), F(x_0)) \le d(A, B). \tag{2.21}$$

On the other hand, trivially

$$d(G(x_0), F(x_0)) \ge d(A, B). \tag{2.22}$$

Thus by (2.21) and (2.22), we get

$$d(G(x_0), F(x_0)) = d(A, B). \tag{2.23}$$

□

Remark 2.2. (a) Let $G : A \to A$ be a single-valued homeomorphism. Then G obviously satisfies all conditions of Theorem 2.1.

(b) There are many conditions under which G^- has a continuous selection [10–13].

The following corollary is immediate.

Corollary 2.3. *Let X be a normed linear space. Let $A \subseteq X$ be a nonempty compact convex and let B be a nonempty subset of X. Let $G : A \multimap A$ be a continuous, onto set-valued map with compact values such that $G^- \in \mathcal{S}(A, A)$. Let $F : A \multimap B$ be a continuous set-valued map with convex, compact values. Assume that $F(x) \cap B_0 \neq \emptyset$, for each $x \in A$. Then there exists $x_0 \in A$ such that*

$$d(G(x_0), F(x_0)) = d(A, B). \tag{2.24}$$

Remark 2.4. A similar result to that of Corollary 2.3 holds in every topological vector space in which every closed ball is convex.

Since hyperconvex metric spaces are $\mathcal{NR}$-metric spaces, then we have the following corollary.

Corollary 2.5. *Let (M, d) be a hyperconvex metric space. Let $A \subsetneq M$ be a nonempty compact admissible and let B be a nonempty subset of M. Let $G : A \multimap A$ be a continuous, onto set-valued map with compact values such that $G^{-} \in \mathcal{S}(A, A)$. Let $F : A \multimap B$ be a continuous set-valued map with admissible, compact values. Assume that $F(x) \cap B_0 \neq \emptyset$, for each $x \in A$. Then there exists $x_0 \in A$ such that*

$$d(G(x_0), F(x_0)) = d(A, B). \tag{2.25}$$

Corollary 2.6. *Let (M, d) be a hyperconvex metric space. Let A be a nonempty compact admissible subset of M. Let $G : A \multimap A$ be a continuous, onto set-valued map with compact values such that $G^{-} \in \mathcal{S}(A, A)$. Let $F : A \multimap M$ be a continuous set-valued map with admissible, compact values. Assume that $F(x) \cap A \neq \emptyset$, for each $x \in A$. Then there exists $x_0 \in A$ such that*

$$G(x_0) \cap F(x_0) \neq \emptyset. \tag{2.26}$$

Proof. Let $B = M$ and apply Corollary 2.5 (note $B_0 = A$). □

Remark 2.7. If we take $G = I_A$, Corollary 2.6 reduces to Corollary 3.5 of Kirk and Shin [14].

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