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Research Article

Bifurcation Results for a Class of Perturbed Fredholm Maps

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We prove a global bifurcation result for an equation of the type $Lx + \lambda(h(x) + k(x)) = 0$, where $L : E \to F$ is a linear Fredholm operator of index zero between Banach spaces, and, given an open subset Ω of E, $h, k : \Omega \times [0, +\infty) \to F$ are C^1 and continuous, respectively. Under suitable conditions, we prove the existence of an unbounded connected set of nontrivial solutions of the above equation, that is, solutions (x, λ) with $\lambda \neq 0$, whose closure contains a trivial solution $(\overline{x}, 0)$. The proof is based on a degree theory for a special class of noncompact perturbations of Fredholm maps of index zero, called α-Fredholm maps, which has been recently developed by the authors in collaboration with M. Furi.

1. Introduction

We study a bifurcation problem for the semilinear operator equation

$$Lx + \lambda\big(h(x) + k(x)\big) = 0 \tag{1.1}$$

in $\Omega \times [0, +\infty)$, where Ω is an open subset of a Banach space E, $L : E \to F$ is a linear Fredholm operator of index zero between real Banach spaces, and the maps $h : \Omega \to F$ and $k : \Omega \to F$ are of class C^1 and continuous, respectively. In addition, we assume that, for any nonnegative real λ, the map $x \mapsto Lx + \lambda h(x)$ is a nonlinear Fredholm map of index zero.

The set of *trivial solutions* of (1.1) is obtained when $\lambda = 0$. It coincides with $(\Omega \cap \operatorname{Ker} L) \times \{0\}$, which, we suppose nonempty. One of the problems related to (1.1) is to establish under which conditions the set of nontrivial solutions is not empty, and to determine topological properties of this set. One of them is the existence of a *bifurcation point*, that is, a point p in

$\Omega \cap \operatorname{Ker} L$ such that $(p, 0)$ lies in the closure of the set of nontrivial solutions. The related bifurcation theory is sometimes called *cobifurcation* [1] or *atypical bifurcation* [2].

Independently, Furi and Pera [1] and Martelli [3] have studied an unperturbed equation of the form

$$Lx + \lambda h(x) = 0, \tag{1.2}$$

with L as in (1.1) and $h : \Omega \to F$ being compact. These authors proved the existence of a connected bifurcating branch of nontrivial solutions of (1.2) that is either unbounded or whose closure contains at least two bifurcation points. More recently, an analogous result has been obtained by Benevieri et al. in [4] by removing the compactness assumption on h, but requiring that such a map is of class C^1. Their proof is based on a degree theory developed in [5] for the class of Fredholm maps of index zero.

A further extension has been obtained by Benevieri and Furi in [6]. They studied (1.1) assuming that the map h is C^1 and the perturbation k is locally compact. To tackle this type of problem, they applied a topological degree theory for the class of compact perturbations of nonlinear Fredholm maps (*quasi-Fredholm maps* in short), which is introduced in [6] and generalizes that given in [5].

In this paper, we extend the domain of investigation of (1.1) by replacing the compactness assumption on the perturbation k with a suitable condition given in terms of measure of noncompactness. Roughly speaking, we suppose that the noncompactness of k is small with respect to a numerical characteristic depending on L and h. Under this assumption, in Theorem 5.3 below we prove the existence of a connected bifurcating branch of nontrivial solutions of (1.1) as in [4]. The technique used here is based on a topological degree theory introduced in [7] (see also [8–10]) for a special class of noncompact perturbations of Fredholm maps, called *α-Fredholm maps*. Such a theory extends that defined in [6] (we recall that any quasi-Fredholm map is also α-Fredholm), and agrees with the Nussbaum degree for the class of locally α-contractive vector fields (see [11]).

Our investigation falls into the research field of continuation results, which goes back to Leray and Schauder and has been widely investigated by many authors. An accurate presentation of this type of problems is due to Mawhin (see, e.g., [12–14] and the references therein).

Concerning the organization of the paper, in Section 2 we recall first the notion (introduced in [5, 15]) of orientability for nonlinear Fredholm maps. Then, following [6], we extend the notion of orientability to quasi-Fredholm maps. This concept is crucial in the definition of the degree for quasi-Fredholm maps. Section 3 is devoted to recall the definition of Kuratowski measure of noncompactness together with some related concepts. In Section 4, we sketch the construction of the degree for α-Fredholm maps given in [7]. Section 5 contains our main result, that is, Theorem 5.3. In Section 6, we give an application to the study of T-periodic solutions of a boundary value problem depending on a parameter. For this problem, we obtain a global bifurcation theorem generalizing analogous results in [4, 6].

2. Orientability and degree for quasi-Fredholm maps

In this section, we recall the definition of quasi-Fredholm maps between Banach spaces, introduced in [6], and we summarize the notions of orientability and degree for this class of maps.

Throughout the paper, E and F will denote two real Banach spaces. The space of bounded linear operators from E to F will be denoted by $L(E, F)$, and $\Phi_0(E, F)$ will be the open subset of Fredholm operators of index zero.

Consider an operator $L \in \Phi_0(E,F)$. A bounded linear operator $A : E \to F$ with finite dimensional image is called a *corrector* of L if $L + A$ is an isomorphism. On the (nonempty) set $\mathcal{C}(L)$ of correctors of L, we define an equivalence relation as follows. Let $A, B \in \mathcal{C}(L)$ be given and consider the following automorphism of E:

$$T = (L+B)^{-1}(L+A) = I - (L+B)^{-1}(B-A). \tag{2.1}$$

The image of $I - T$ has of course finite dimension. Hence, given any nontrivial finite dimensional subspace E_0 of E containing $\operatorname{Im}(I-T)$, the restriction of T to E_0 is an automorphism. Therefore, its determinant is nonzero and independent of the choice of E_0. Denote by $\det T$ this common value. We say that A is *equivalent* to B if

$$\det\left((L+B)^{-1}(L+A)\right) > 0. \tag{2.2}$$

As shown in [5], this is actually an equivalence relation on $\mathcal{C}(L)$ with two equivalence classes.

Definition 2.1. Let $L \in \Phi_0(E,F)$ be given. An *orientation* of L is the choice of one of the two classes of $\mathcal{C}(L)$, and L is *oriented* when an orientation is chosen.

Given an oriented operator L, the elements of its orientation are called *positive correctors* of L.

Since the set of the isomorphisms of E into F is open in $L(E,F)$, a corrector of $L \in \Phi_0(E,F)$ is a corrector of every operator in $\Phi_0(E,F)$ close enough to L. This allows us to give the following definition.

Definition 2.2. Let X be a topological space and $h : X \to \Phi_0(E,F)$ a continuous map. An *orientation* of h is a choice of an orientation $\mathcal{O}(x)$ of $h(x)$ for each $x \in X$, such that for any $x \in X$ there exists $A \in \mathcal{O}(x)$ which is a positive corrector of $h(x')$ for any x' in a neighborhood of x. A map is *orientable* if it admits an orientation and *oriented* when an orientation is chosen.

Remark 2.3. With an abuse of terminology, we can say that if a map h is oriented, the orientation $\mathcal{O}(x)$ of $h(x)$ depends continuously on x.

By Definition 2.2, we can give a notion of orientability for Fredholm maps of index zero between Banach spaces. Recall that, given an open subset Ω of E, a C^1 map $g : \Omega \to F$ is *Fredholm of index n* if its Fréchet derivative $(g'(x))$ is a Fredholm operator of index n for all $x \in \Omega$.

Definition 2.4. An *orientation* of a Fredholm map of index zero $g : \Omega \to F$ is an orientation of the continuous map $g' : x \mapsto g'(x)$, and g is *orientable*, or *oriented*, if so is g' according to Definition 2.2.

The notion of orientability of Fredholm maps of index zero is accurately discussed in [5, 15]. Here, we only recall a property (Theorem 2.6 below) which is a sort of continuous transport of an orientation along a homotopy of Fredholm maps. We need first the following definition.

Definition 2.5. Let $H : \Omega \times [0,1] \to F$ be a C^1 homotopy. Assume that any partial map $H_\lambda = H(\cdot,\lambda)$ is Fredholm of index zero. An *orientation* of H is an orientation of the derivative with respect to the first variable

$$\partial_1 H : \Omega \times [0,1] \longrightarrow \Phi_0(E,F), \qquad \partial_1 H(x,\lambda) = (H_\lambda)'(x); \tag{2.3}$$

H is *orientable*, or *oriented*, if so is $\partial_1 H$ according to Definition 2.2.

Theorem 2.6. *Let $H : \Omega \times [0,1] \to F$ be C^1 and assume that any H_λ is a Fredholm map of index zero. Suppose that, for some $\lambda \in [0,1]$, the partial map H_λ is oriented and call $\mathcal{O}$ its orientation. Then, there exists a unique orientation of H, say β, such that $\beta(x,\lambda) = \mathcal{O}(x)$ for any $x \in \Omega$.*

In the next remark, we show how the orientation of a Fredholm map g is related to the orientations of domain and codomain of suitable restrictions of g. This property plays an important role in the proof of our main result (Theorem 5.3 below).

Remark 2.7. Let $g : \Omega \to F$ be an oriented map and Z a finite dimensional subspace of F, transverse to g. By classical transversality results, $M = g^{-1}(Z)$ is a C^1 manifold of the same dimension as Z. Let Z be oriented. Consider $x \in M$ and a positive corrector A of $g'(x)$ with image contained in Z. Then, orient T_xM in such a way that the isomorphism

$$(g'(x) + A)|_{T_xM} : T_xM \longrightarrow Z \tag{2.4}$$

is orientation-preserving. As proved in [5] (see in particular Remark 2.5 and Lemma 3.1 of that work), the orientation of T_xM does not depend on the choice of the corrector A, but only on the orientations of Z and $g'(x)$. Moreover, such an orientation depends continuously on x; that is, it defines an orientation on M. We will call M the *oriented g-preimage* of Z.

We are now ready to recall the concepts of orientability and degree for quasi-Fredholm maps, defined in [6].

Definition 2.8. Let Ω be an open subset of E, $g : \Omega \to F$ a Fredholm map of index zero, and $k : \Omega \to F$ a locally compact map. The map $f : \Omega \to F$, defined by $f = g - k$, is called a *quasi-Fredholm map* and g is a *smoothing map* of f.

Definition 2.9. A quasi-Fredholm map $f : \Omega \to F$ is *orientable* if it has an orientable smoothing map. If f is orientable, an *orientation* of f is the choice of an orientation of any of its smoothing maps.

The above definition is well posed because, as shown in [6], if f is an orientable quasi-Fredholm map, the following facts hold:

(i) any smoothing map of f is orientable;

(ii) an orientation of a smoothing map f determines uniquely an orientation of any other smoothing map.

In the sequel, if f is oriented and g is an oriented smoothing map that determines the orientation of f, one will refer to g as a *positively oriented smoothing map* of f.

Let us now give a sketch of the construction of the degree.

Definition 2.10. Let $f : \Omega \to F$ be an oriented quasi-Fredholm map and U an open subset of Ω. The triple $(f, U, 0)$ is said to be *qF-admissible* provided that $f^{-1}(0) \cap U$ is compact.

The degree for qF-admissible triples could be defined in two steps. In the first one, the degree is defined for a triple (f, U, y) such that f has a smoothing map g with $(f - g)(U)$ contained in a finite dimensional subspace of F. Then, we remove this assumption, and the degree is given for all qF-admissible triples.

Let $(f, U, 0)$ be a qF-admissible triple, and let g be a positively oriented smoothing map of f such that $(f - g)(U)$ is contained in a finite dimensional subspace of F. As $f^{-1}(0) \cap U$ is compact, let Z be a finite dimensional subspace of F, and let W be an open neighborhood of $f^{-1}(0)$ in U such that g is transverse to Z in W. Assume that Z is oriented and contains $(f - g)(U)$. Let $M = g^{-1}(Z) \cap W$ be the oriented $g|_W$-preimage of Z.

One can easily verify that $(f|_M)^{-1}(0) = f^{-1}(0) \cap U$. Thus, $(f|_M)^{-1}(0)$ is compact, and the Brouwer degree of the triple $(f|_M, M, 0)$ is well defined. Then, the degree of $(f, U, 0)$ is defined as

$$\deg_{qF}(f, U, 0) = \deg_B\big(f|_M, M, 0\big), \tag{2.5}$$

where the right-hand side is the Brouwer degree of the triple $(f|_M, M, 0)$. As proved in [6], this definition is well posed since the right-hand side of (2.5) is independent of the choice of the smoothing map g, the open set W, and the subspace Z.

To define the degree of a general qF-admissible triple $(f, U, 0)$, take a positively oriented smoothing map g of f and a continuous map ξ, with finite dimensional image close enough to $f - g$ in a suitable neighborhood V of $f^{-1}(0) \cap U$. The degree of $(f, U, 0)$ is

$$\deg_{qF}(f, U, 0) = \deg_{qF}(g - \xi, V, 0). \tag{2.6}$$

The reader can find in [6] the details of the construction and the properties verified by the degree.

3. Measures of noncompactness

In this section, we recall the definition of the Kuratowski measure of noncompactness together with some related concepts. For general reference, see, for example, [16] or [17].

From now on, the Banach spaces E and F are assumed to be infinite dimensional.

The *Kuratowski measure of noncompactness* $\alpha(A)$ of a bounded subset A of E is defined as the infimum of real numbers $d > 0$ such that A admits finite covering by sets of diameter less than d. If A is unbounded, we set $\alpha(A) = +\infty$.

Given an open subset Ω of E and a continuous map $f : \Omega \to F$, we recall the definition of the following two extended real numbers (see, e.g., [18]) associated with the map f:

$$\begin{aligned} \alpha(f) &= \sup\left\{ \frac{\alpha\big(f(A)\big)}{\alpha(A)} : A \subseteq \Omega \text{ bounded},\ \alpha(A) > 0 \right\}, \\ \omega(f) &= \inf\left\{ \frac{\alpha\big(f(A)\big)}{\alpha(A)} : A \subseteq \Omega \text{ bounded},\ \alpha(A) > 0 \right\}. \end{aligned} \tag{3.1}$$

We point out that $\alpha(f) = 0$ if and only if f is completely continuous, and $\omega(f) > 0$ only if f is proper on bounded closed sets. For a comprehensive list of properties of $\alpha(f)$ and $\omega(f)$, we refer to [18]. Here, we recall the following one concerning linear operators.

Proposition 3.1. *Let $L : E \to F$ be a bounded linear operator. Then, $\omega(L) > 0$ if and only if* $\operatorname{Im} L$ *is closed and* $\dim \operatorname{Ker} L < +\infty$.

Let $p \in \Omega$ be fixed. We recall the definitions of $\alpha_p(f)$ and $\omega_p(f)$ given in [9] (see also [10]). Roughly speaking, these numbers are the local analogues of $\alpha(f)$ and $\omega(f)$.

Let $B(p,s)$ denote the open ball in E centered at p with radius $s > 0$. Suppose that $B(p,s) \subseteq \Omega$ and consider the number

$$\alpha\bigl(f|_{B(p,s)}\bigr) = \sup\left\{\frac{\alpha\bigl(f(A)\bigr)}{\alpha(A)} : A \subseteq B(p,s), \alpha(A) > 0\right\}, \tag{3.2}$$

which is nondecreasing as a function of s. Hence, we can define

$$\alpha_p(f) = \lim_{s\to 0} \alpha\bigl(f|_{B(p,s)}\bigr). \tag{3.3}$$

Clearly, $\alpha_p(f) \le \alpha(f)$. Analogously, define

$$\omega_p(f) = \lim_{s\to 0} \omega\bigl(f|_{B(p,s)}\bigr). \tag{3.4}$$

Obviously, $\omega_p(f) \ge \omega(f)$.

With only minor changes, it is easy to show that the main properties of α and ω hold for α_p and ω_p as well. In fact, the following proposition holds.

Proposition 3.2 (see [9]). *Let $f : \Omega \to F$ be continuous and $p \in \Omega$. Then,*

(i) $|\alpha_p(f) - \alpha_p(g)| \le \alpha_p(f+g) \le \alpha_p(f) + \alpha_p(g)$;

(ii) $\omega_p(f) - \alpha_p(g) \le \omega_p(f+g) \le \omega_p(f) + \alpha_p(g)$;

(iii) *if f is locally compact, $\alpha_p(f) = 0$;*

(iv) *if $\omega_p(f) > 0$, f is locally proper at p;*

(v) *if f is a local homeomorphism and $\omega_p(f) > 0$, $\alpha_q(f^{-1})\omega_p(f) = 1$, where $q = f(p)$.*

Clearly, for a bounded linear operator $L : E \to F$, the numbers $\alpha_p(L)$ and $\omega_p(L)$ do not depend on the point p and coincide, respectively, with $\alpha(L)$ and $\omega(L)$. Furthermore, for the C^1 case the following result holds.

Proposition 3.3 (see [9]). *Let $f : \Omega \to F$ be of class C^1. Then, for any $p \in \Omega$, one has $\alpha_p(f) = \alpha(f'(p))$ and $\omega_p(f) = \omega(f'(p))$.*

If $f : \Omega \to F$ is a Fredholm map, as a straightforward consequence of Propositions 3.1 and 3.3, we obtain $\omega_p(f) > 0$ for any $p \in \Omega$.

The next property of bounded linear operators is useful for a direct computation of α and ω.

Proposition 3.4. *Let $L : E \to F$ be a bounded linear operator, and let $P : E \to E$ and $Q : F \to F$ be two projectors onto finite codimensional subspaces. Then,*

$$\alpha(L) = \alpha(QLP), \qquad \omega(L) = \omega(QLP). \tag{3.5}$$

Proof. We have, for instance, $L = QL + (I - Q)L$. Observe that the operator $(I - Q)L$ is compact since its image is finite dimensional. Thus, $\alpha((I - Q)L) = 0$. Hence, by property (1) in Proposition 3.2, we have $\alpha(L) = \alpha(QL)$. In an analogous way, one can easily check that $\alpha(L) = \alpha(QLP)$ and $\omega(L) = \omega(QLP)$. □

The next proposition, which will be used in the sequel, is a sort of nonlinear analogue of Proposition 3.4.

Proposition 3.5. *Let $f : \Omega \to F$ be continuous and $p \in \Omega$. Let $Q : F \to F$ be a projector onto a finite codimensional subspace. Then,*

$$\alpha_p(f) = \alpha_p(Qf), \qquad \omega_p(f) = \omega_p(Qf). \tag{3.6}$$

Proof. We have $f = Qf + (I - Q)f$. Note that $\alpha_p((I - Q)f) = 0$ since the map $(I - Q)f$ is compact. Thus, from properties (1) and (2) in Proposition 3.2, it follows that $\alpha_p(f) = \alpha_p(Qf)$ and $\omega_p(f) = \omega_p(Qf)$. □

The following proposition, whose proof can be found in [8, Proposition 4.5], extends to the continuous case an analogous result shown in [9] for C^1 maps.

Proposition 3.6. *Let $g : \Omega \to F$ and $\sigma : \Omega \to \mathbb{R}$ be continuous. Consider the product map $f : \Omega \to F$ defined by $f(x) = \sigma(x)g(x)$. Then, for any $p \in \Omega$, one has $\alpha_p(f) = |\sigma(p)|\alpha_p(g)$ and $\omega_p(f) = |\sigma(p)|\omega_p(g)$.*

In the sequel, we will consider also maps G defined on the product space $E \times \mathbb{R}$. In order to define $\alpha_{(p,\lambda)}(G)$, we consider the norm

$$\|(p, \lambda)\| = \max\{\|p\|, |\lambda|\}. \tag{3.7}$$

The natural projection of $E \times \mathbb{R}$ onto the first factor will be denoted by π_1.

Remark 3.7. With the above norm, π_1 is nonexpansive. Therefore, $\alpha(\pi_1(X)) \leq \alpha(X)$ for any subset X of $E \times \mathbb{R}$. More precisely, since $\mathbb{R}$ is finite dimensional, if $X \subseteq E \times \mathbb{R}$ is bounded, we have $\alpha(\pi_1(X)) = \alpha(X)$.

We conclude the section with the following technical result, which is a straightforward consequence of Proposition 3.6 and which will be useful in the sequel.

Corollary 3.8. *Given a continuous map $\varphi : \Omega \to F$, consider the map*

$$\Phi : \Omega \times [0,1] \longrightarrow F, \quad \Phi(x, \lambda) = \lambda\varphi(x). \tag{3.8}$$

Then, for any fixed pair $(p, \lambda) \in \Omega \times [0,1]$, one has

$$\alpha_{(p,\lambda)}(\Phi) = \lambda\alpha_p(\varphi). \tag{3.9}$$

4. Degree for α-Fredholm maps

In this section, we sketch the construction of the degree for α-Fredholm maps. The interested reader can find the details in [7].

The α-Fredholm maps are special noncompact perturbations of Fredholm maps, defined in terms of the numbers α_p and ω_p. Precisely, an *α-Fredholm map* $f : \Omega \to F$ is of the form $f = g - k$, where g is a Fredholm map of index zero, k is continuous, and $\alpha_p(k) < \omega_p(g)$ for every p.

The degree is given as an integer-valued map defined on a class of triples that we will call *admissible α-Fredholm triples*. This class is recalled in the following two definitions.

Definition 4.1. Let $g : \Omega \to F$ be a Fredholm map of index zero, $k : \Omega \to F$ a continuous map, and U an open subset of Ω. The triple (g, U, k) is said to be *α-Fredholm* if for any $p \in U$ one has

$$\alpha_p(k) < \omega_p(g). \tag{4.1}$$

Definition 4.2. An α-Fredholm triple (g, U, k) is said to be *admissible* if

(i) g is oriented;

(ii) the *solution set* $S = \{x \in U : g(x) = k(x)\}$ is compact.

Let (g, U, k) be an admissible α-Fredholm triple. Given a finite covering $\mathcal{V} = \{V_1, \ldots, V_N\}$ of open balls of S and a compact convex set C, with $S \subseteq C \subseteq U$, the pair $(\mathcal{V}, C)$ is called an α-pair relative to (g, U, k) if, for any $i = 1, \ldots, N$, the following conditions hold:

(1) the ball $\widetilde{V}_i$ of double radius and same center as V_i is contained in U;

(2) $\alpha(k|_{\widetilde{V}_i}) < \omega(g|_{\widetilde{V}_i})$;

(3) $\{x \in V_i : g(x) \in k(\widetilde{V}_i \cap C)\} \subseteq C$.

In [7], it is shown that, given any admissible α-Fredholm triple, it is always possible to find a relative α-pair.

Let $(\mathcal{V}, C)$ be an α-pair relative to (g, U, k). Denote $V = \bigcup_{i=1}^{N} V_i$ and consider a retraction $r : E \to C$, whose existence is ensured by Dugundji's extension theorem (see, e.g., [19]).

Let W be an open subset of V containing S such that, for any $i, x \in W \cap V_i$ implies $r(x) \in \widetilde{V}_i$. Notice that the triple $(g - kr, W, 0)$ is qF-admissible (recall Definition 2.10). The degree of the triple (g, U, k) is

$$\deg(g, U, k) = \deg_{qF}(g - kr, W, 0), \tag{4.2}$$

where the right-hand side is the degree for quasi-Fredholm maps, seen in Section 2. In addition, we show in [7] that the right-hand side of the above equality is independent of the choice of the α-pair $(\mathcal{V}, C)$, of the retraction r, and of the open set W.

As pointed out in [7], this concept of degree extends the degree for quasi-Fredholm maps, and it agrees with the Nussbaum degree for the class of locally α-contractive vector fields (see [11]).

Below we state the most important properties of the degree. Actually, in [7] only the fundamental properties (i.e., normalization, additivity, and homotopy invariance) were stated and proved. The excision and existence properties are easy consequences of the additivity.

Let us introduce the following concept of *α-Fredholm homotopy*.

Definition 4.3. Let W be an open subset of $E \times [0,1]$ and $H : W \to F$ a continuous map of the form

$$H(x,\lambda) = G(x,\lambda) - K(x,\lambda). \tag{4.3}$$

The map H is said to be an *α-Fredholm homotopy* if the following conditions hold:

(i) G is C^1;

(ii) for any $\lambda \in [0,1]$ the partial map G_λ is Fredholm of index zero on the section $W_\lambda = \{x \in E : (x,\lambda) \in W\}$;

(iii) for any pair $(p,\lambda) \in W$ one has $\alpha_{(p,\lambda)}(K) < \omega_{(p,\lambda)}(G)$.

Theorem 4.4. *The following properties hold.*

(1) *Normalization. Let the identity I of E be oriented in such a way that the trivial operator is a positive corrector. Then,*

$$\deg(I,E,0) = 1. \tag{4.4}$$

(2) *Additivity. Given an admissible α-Fredholm triple (g,U,k) and two disjoint open subsets U_1, U_2 of U, assume that $S = \{x \in U : g(x) = k(x)\}$ is contained in $U_1 \cup U_2$. Then,*

$$\deg(g,U,k) = \deg\left(g,U_1,k\right) + \deg\left(g,U_2,k\right). \tag{4.5}$$

(3) *Excision. Given an admissible α-Fredholm triple (g,U,k) and an open subset U_1 of U, assume that S is contained in U_1. Then,*

$$\deg(g,U,k) = \deg\left(g,U_1,k\right). \tag{4.6}$$

(4) *Existence. Given an admissible α-Fredholm triple (g,U,k), if*

$$\deg(g,U,k) \neq 0, \tag{4.7}$$

then the equation $g(x) = k(x)$ has a solution in U.

(5) *Homotopy invariance. Let W be an open subset of $E\times[0,1]$ and $H : W \to F$ an α-Fredholm homotopy of the form $H(x,\lambda) = G(x,\lambda) - K(x,\lambda)$. Assume that G is oriented and that the set $H^{-1}(0)$ is compact. Then,* $\deg(G_\lambda, W_\lambda, K_\lambda)$ *is well defined and does not depend on $\lambda \in [0,1]$.*

5. Nonlinear bifurcation results

In this section, we consider the semilinear operator equation

$$Lx + \lambda\big(h(x) + k(x)\big) = 0 \tag{5.1}$$

in $\Omega \times [0,+\infty)$, where $L : E \to F$ is a linear Fredholm operator of index zero between real Banach spaces, and the maps $h : \Omega \to F$ and $k : \Omega \to F$ are C^1 and continuous, respectively.

Equation (5.1) can be equivalently written as

$$H(x,\lambda) = 0, \tag{5.2}$$

where

$$H : \Omega \times [0,+\infty) \longrightarrow F, \quad H(x,\lambda) = Lx + \lambda\big(h(x) + k(x)\big). \tag{5.3}$$

This map is of the form

$$H(x,\lambda) = G(x,\lambda) - K(x,\lambda), \tag{5.4}$$

where $G(x,\lambda) = Lx + \lambda h(x)$ is of class C^1 and $K(x,\lambda) = -\lambda k(x)$. We will suppose that the following conditions hold:

(H1) for any $\lambda \geq 0$, the partial map G_λ is Fredholm of index zero;

(H2) for any pair $(p,\lambda) \in \Omega \times [0,+\infty)$, we have $\alpha_{(p,\lambda)}(K) < \omega_{(p,\lambda)}(G)$.

Thus, the map H is an α-Fredholm homotopy (see Definition 4.3).

By a solution of (5.1), we mean a pair $(x,\lambda) \in H^{-1}(0)$ and we regard the distinguished subset $(\Omega \cap \operatorname{Ker} L) \times \{0\}$ of $H^{-1}(0)$ as the set of *trivial solutions* of (5.1).

A problem related to (5.1) is that of the existence of a (*atypical*) *bifurcation point* (in the terminology of Prodi-Ambrosetti in [2]), that is, a point p in $\Omega \cap \operatorname{Ker} L$ such that $(p,0)$ lies in the closure of the set of nontrivial solutions (i.e., of the pairs $(x,\lambda) \in H^{-1}(0)$ with $\lambda \neq 0$).

In a recent paper, Benevieri et al. (see [4]) obtained a global bifurcation result for (5.1) in the particular case when $k = 0$. Afterwards, the result in [4] was extended by the first two authors (see [6]) by introducing a locally compact perturbation k. In that case, the map H as in formula (5.4) is such that each $H(\cdot,\lambda)$ is a quasi-Fredholm map.

Theorem 5.3 below is a further extension of the result in [6], by considering a possibly noncompact perturbation k. The compactness assumption of k is replaced by condition (H2) above, which is clearly satisfied when k (and thus K) is locally compact. The proof follows some ideas in [4]. Let us stress that our argument is based on the degree for α-Fredholm maps.

Let F_1 be any fixed (finite dimensional) direct summand of $\operatorname{Im} L$ in F. We consider the decomposition $F = \operatorname{Im} L \oplus F_1$, and we denote by R and π the associated projections onto $\operatorname{Im} L$ and F_1, respectively.

Equation (5.1) is clearly equivalent to the system

$$\begin{gathered} Lx + \lambda\big(Rh(x) + Rk(x)\big) = 0, \\ \lambda\big(\pi h(x) + \pi k(x)\big) = 0. \end{gathered} \tag{5.5}$$

In order to investigate the set of nontrivial solutions of (5.5), it is convenient to consider the system

$$\begin{gathered} Lx + \lambda\big(Rh(x) + Rk(x)\big) = 0, \\ \pi h(x) + \pi k(x) = 0 \end{gathered} \tag{5.6}$$

which is equivalent to (5.5) for $\lambda \neq 0$.

The next result provides a necessary condition for $p \in \Omega \cap \operatorname{Ker} L$ to be a bifurcation point. The easy proof, which is based on a simple continuity argument, is given for completeness.

Theorem 5.1. *Assume that p is a bifurcation point for* (5.1). *Then, $h(p) + k(p) \in \operatorname{Im} L$ or, equivalently, $\pi h(p) + \pi k(p) = 0$.*

Proof. Since p is a bifurcation point, there exists a sequence $\{(\lambda_n, x_n)\}$ of nontrivial solutions of (5.1) converging to $(0, p)$. Hence, (λ_n, x_n) is a solution of the system (5.6) for any n, and the result follows from the continuity of the maps πh and πk. □

Our main result, Theorem 5.3 below, is the analogue of [4, Theorem 3.2], and provides a sufficient condition for the existence of a bifurcation point. The fundamental tools for proving Theorem 5.3 are the homotopy invariance property of the degree for α-Fredholm maps (as in Theorem 4.4), together with the following crucial lemma, whose proof can be found in [20].

Lemma 5.2. *Let Z be a compact subset of a locally compact metric space X. Assume that any compact subset of X containing Z has nonempty boundary. Then, $X \setminus Z$ contains a connected set whose closure is not compact and intersects Z.*

We are now ready to state our main result. The statement involves the Brouwer degree of a map between $\operatorname{Ker} L$ and F_1. Therefore, these spaces should be oriented. However, the result is independent of the chosen orientations.

As in Section 4, given an open subset W of $\Omega \times [0, +\infty)$, by W_λ we denote the section $\{x \in \Omega : (x, \lambda) \in W\}$.

Theorem 5.3. *Let $H : \Omega \times [0, +\infty) \to F$ be defined by $H(x, \lambda) = Lx + \lambda(h(x) + k(x))$, and suppose that conditions (H1) and (H2) above hold. Assume in addition that the map $G : \Omega \times [0, +\infty) \to F$, defined by $G(x, \lambda) = Lx + \lambda h(x)$, is oriented.*

Let $v : \Omega \cap \operatorname{Ker} L \to F_1$ be defined by $v(p) = \pi h(p) + \pi k(p)$. Let W be an open subset of $\Omega \times [0, +\infty)$, and suppose that the Brouwer degree $\deg_B(v, W_0 \cap \operatorname{Ker} L, 0)$ is well defined and nonzero. Then, there exists in W a connected set of nontrivial solutions of (5.1) *whose closure in W is not compact and intersects* $\operatorname{Ker} L \times \{0\}$.

Proof. Notice that, as a consequence of conditions (H1) and (H2), the map H is an α-Fredholm homotopy of the form $H(x, \lambda) = G(x, \lambda) - K(x, \lambda)$.

Let $\widehat{H} : W \to F = \operatorname{Im} L \oplus F_1$ be defined by

$$\widehat{H}(x, \lambda) = Lx + \lambda\big(Rh(x) + Rk(x)\big) + \pi h(x) + \pi k(x). \tag{5.7}$$

This map is clearly an α-Fredholm homotopy which can be written as $\widehat{H} = \widehat{G} - \widehat{K}$, where

$$\widehat{G}(x, \lambda) = Lx + \lambda Rh(x) + \pi h(x) \tag{5.8}$$

is of class C^1 and oriented (with orientation induced by G according to Theorem 2.6), and $\widehat{K}(x, \lambda) = -\lambda Rk(x) - \pi k(x)$. In fact, since $RG = R\widehat{G}$ and $RK = R\widehat{K}$, by Proposition 3.5 we get

$$\alpha_{(p,\lambda)}(\widehat{K}) = \alpha_{(p,\lambda)}(K), \qquad \omega_{(p,\lambda)}(\widehat{G}) = \omega_{(p,\lambda)}(G) \tag{5.9}$$

for any pair $(p, \lambda) \in W$. Thus, $\alpha_{(p,\lambda)}(\widehat{K}) < \omega_{(p,\lambda)}(\widehat{G})$ for any $(p, \lambda) \in W$.

Let now

$$Y = \big\{(x, \lambda) \in W : \widehat{H}(x, \lambda) = 0\big\}. \tag{5.10}$$

Notice that the set Y is locally compact. Indeed, the map $\widehat{H}$ is locally proper at any $(p,\lambda) \in W$ since $\alpha_{(p,\lambda)}(\widehat{K}) < \omega_{(p,\lambda)}(\widehat{G})$. Moreover, $Y_0 = v^{-1}(0) \cap W_0$ is compact because we assumed that $\deg_B(v, W_0 \cap \operatorname{Ker} L, 0)$ is well defined.

We apply Lemma 5.2 with $Y_0 \times \{0\}$ in place of Z, and with Y in place of X. Assume, by contradiction, that there exists a compact set $Y' \subseteq Y$ containing $Y_0 \times \{0\}$ with empty boundary in Y. Thus, Y' is also an open subset of Y. Hence, there exists a bounded open subset U of W such that $Y' = U \cap Y$. Since Y' is compact, the homotopy invariance property of the degree (see Theorem 4.4) implies that $\deg(\widehat{G}_\lambda, U_\lambda, \widehat{K}_\lambda)$ does not depend on $\lambda \geq 0$. Moreover, the slice $Y'_\lambda = U_\lambda \cap Y_\lambda$ is empty for some positive λ. This implies that $\deg(\widehat{G}_\lambda, U_\lambda, \widehat{K}_\lambda) = 0$ for any $\lambda \in [0, +\infty)$ and, in particular, $\deg(\widehat{G}_0, U_0, \widehat{K}_0) = 0$. The inclusions $v^{-1}(0) \cap W_0 \subseteq U_0 \subseteq W_0$ imply, using the excision property of the degree, that $\deg(\widehat{G}_0, W_0, \widehat{K}_0) = 0$.

Now, observe that the map $\widehat{H}_0 = \widehat{G}_0 - \widehat{K}_0$, which is given by

$$\widehat{H}_0(x) = Lx + \pi h(x) + \pi k(x), \tag{5.11}$$

is actually an oriented quasi-Fredholm map (with $\widehat{G}$, and thus $\widehat{G}_0$, being oriented). Consequently, we get

$$0 = \deg\left(\widehat{G}_0, W_0, \widehat{K}_0\right) = \deg_{qF}\left(\widehat{H}_0, W_0, 0\right). \tag{5.12}$$

The subspace F_1 contains the image of $\widehat{K}_0$ and is transverse to $\widehat{G}_0$ being transverse to L. Moreover, $\widehat{H}_0^{-1}(F_1) = \widehat{G}_0^{-1}(F_1) = W_0 \cap \operatorname{Ker} L$. Suppose F_1. Without loss of generality, we assume that $W_0 \cap \operatorname{Ker} L$ is oriented in such a way that it becomes the oriented $\widehat{G}_0$-preimage of F_1. Hence, by definition of degree for quasi-Fredholm maps (see formula (2.5)), we obtain

$$\deg_{qF}\left(\widehat{H}_0, W_0, 0\right) = \deg_B\left(v, W_0 \cap \operatorname{Ker} L, 0\right) \neq 0, \tag{5.13}$$

which contradicts equality (5.12).

Therefore, because of Lemma 5.2, there exists a connected subset of Y whose closure in Y intersects $Y_0 \times \{0\}$ and is not compact. This completes the proof. □

The next consequences of Theorem 5.3 and Corollaries 5.4 and 5.5 below extend analogous results in [4]. The proofs are given for the reader's convenience.

Corollary 5.4. *Let the assumptions of Theorem 5.3 be satisfied. Suppose, moreover, that the map H is proper on bounded and closed subsets of W. Then,* (5.1) *admits a connected set Γ of nontrivial solutions such that its closure in $E \times [0, +\infty)$ intersects* $\operatorname{Ker} L \times \{0\}$ *and is either unbounded or reaches the boundary of W. If, in particular, $\Omega = E$ and $W = E \times [0, +\infty)$, then Γ is unbounded.*

Proof. Let $\overline{\Gamma}$ denote the closure in $E \times [0, +\infty)$ of a connected branch Γ as in Theorem 5.3. Suppose that $\Gamma \cap \partial W = \emptyset$. Thus, the closure of Γ in W coincides with $\overline{\Gamma}$. Hence, $\overline{\Gamma}$ cannot be bounded since the properness of H on bounded closed subsets of W implies that the map $\widehat{H}$ as in the proof of Theorem 5.3 (see (5.7)) has the same property. □

Corollary 5.5. *Let W and v be as in Theorem 5.3. Suppose, moreover, that the map H is proper on bounded and closed subsets of W. Let $p \in W_0 \cap \operatorname{Ker} L$ be such that $v(p) = 0$, and $v'(p)$ is invertible.*

Then, (5.1) *admits a connected set* Γ *of nontrivial solutions such that its closure contains* p *and satisfies at least one of the following three conditions:*

(i) *it is unbounded;*

(ii) *it contains a point* $q \in W_0 \cap \operatorname{Ker} L$, $q \neq p$;

(iii) *it intersects* ∂W.

Proof. The assumptions of $v(p) = 0$ and invertible $v'(p)$ imply the existence of an open neighborhood $\widetilde{W}_0$ of p in W_0 such that $v^{-1}(0) \cap \widetilde{W}_0 = \{p\}$ and $\deg(v, \widetilde{W}_0, 0) = \pm 1$. Now apply Corollary 5.4 with the set $\widetilde{W} = (\widetilde{W}_0 \times \{0\}) \cup \{(x,\lambda) \in W : \lambda \neq 0\}$ in place of W. Observe that $\widetilde{W}$ is open, being obtained from W by removing the closed subset $\{(x,0) \in W : x \notin \widetilde{W}_0\}$, and that the boundary of $\widetilde{W}$ (in $E \times [0,+\infty)$) coincides with the boundary of $\widetilde{W}$ as a subset of $E \times \mathbb{R}$ except for $\widetilde{W}_0 \times \{0\}$. □

6. Applications

In this section, we provide an application of the bifurcation results obtained in Section 5 to the following boundary value problem depending on a parameter $\lambda \geq 0$:

$$\begin{gathered} x'(t) + \lambda\phi\big(t, x(t), x'(t)\big) + \lambda\psi\big(t, x(t), x'(t)\big) = 0, \\ x(0) = x(T), \end{gathered} \tag{6.1}$$

where $\phi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $\psi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. We suppose that ϕ and ψ are T-periodic with respect to the first variable. Under additional assumptions, to be specified in the sequel, we obtain a global bifurcation result for T-periodic solutions of problem (6.1).

Our first step consists in presenting an example of an α-Fredholm homotopy. Let us fix some notation. We denote by $\mathcal{C}^0$ the Banach space $C([0,T],\mathbb{R}^n)$ endowed with the usual supremum norm

$$\|x\|_\infty = \max_{t\in[0,T]} \big|x(t)\big|, \quad x \in \mathcal{C}^0, \tag{6.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, and by $\mathcal{C}^1$ the space $C^1([0,T],\mathbb{R}^n)$ endowed with the norm

$$\|x\|_1 = \max\left\{\left\|x'\right\|_\infty, \big|x(0)\big|\right\}, \quad x \in \mathcal{C}^1. \tag{6.3}$$

We endow the product space $\mathcal{C}^0 \times \mathbb{R}^n$ with the norm

$$\big\|(y,r)\big\| = \max\left\{\|y\|_\infty, |r|\right\}, \quad (y,r) \in \mathcal{C}^0 \times \mathbb{R}^n. \tag{6.4}$$

Given an $n \times n$ matrix M, we denote its norm by $\|M\|$.

For simplicity, we will consider ϕ and ψ defined just on $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

Define

$$\begin{aligned} &\Phi : \mathcal{C}^1 \longrightarrow \mathcal{C}^0, \qquad \Phi(x)(t) = \phi\big(t, x(t), x'(t)\big), \quad t \in [0,T], \\ &\Psi : \mathcal{C}^1 \longrightarrow \mathcal{C}^0, \qquad \Psi(x)(t) = \psi\big(t, x(t), x'(t)\big), \quad t \in [0,T], \end{aligned} \tag{6.5}$$

and set

$$G : \mathcal{C}^1 \times [0,+\infty) \longrightarrow \mathcal{C}^0 \times \mathbb{R}^n, \qquad G(x,\lambda) = \big(x' + \lambda\Phi(x), x(0) - x(T)\big),$$
$$K : \mathcal{C}^1 \times [0,+\infty) \longrightarrow \mathcal{C}^0 \times \mathbb{R}^n, \qquad K(x,\lambda) = -\lambda\big(\Psi(x), 0\big). \tag{6.6}$$

It is convenient to write $G(x,\lambda) = (\widetilde{G}(x,\lambda), x(0) - x(T))$, where

$$\widetilde{G}(x,\lambda) = x' + \lambda\Phi(x). \tag{6.7}$$

That is,

$$\widetilde{G}(x,\lambda)(t) = x'(t) + \lambda\phi\big(t, x(t), x'(t)\big), \quad t \in [0,T]. \tag{6.8}$$

The map G is C^1 (since so is ϕ) and the Fréchet derivative $G'_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ of any partial map G_λ at any $x \in \mathcal{C}^1$ is given by

$$G'_\lambda(x)q = \big(\widetilde{G'_\lambda}(x)q, q(0) - q(T)\big), \tag{6.9}$$

where

$$\big(\widetilde{G'_\lambda}(x)q\big)(t) = q'(t) + \lambda\partial_2\phi\big(t, x(t), x'(t)\big)q(t) + \lambda\partial_3\phi\big(t, x(t), x'(t)\big)q'(t), \quad t \in [0,T] \tag{6.10}$$

for any $q \in \mathcal{C}^1$. Here, $\partial_2\phi$ and $\partial_3\phi$ denote the Jacobian matrices of ϕ with respect to the second and third variables, respectively. In particular, the derivative at any x of $\widetilde{G}_\lambda$ can be written as

$$\big(\widetilde{G'_\lambda}(x)q\big)(t) = \big(I + \lambda M_x(t)\big)q'(t) + \lambda N_x(t)q(t), \quad t \in [0,T], \tag{6.11}$$

where, given $x \in \mathcal{C}^1$, M_x and N_x are $n \times n$ matrices of continuous real functions defined in $[0,T]$ by

$$M_x(t) = \partial_3\phi\big(t, x(t), x'(t)\big), \qquad N_x(t) = \partial_2\phi\big(t, x(t), x'(t)\big). \tag{6.12}$$

If x and λ are such that

$$\det\big(I + \lambda M_x(t)\big) \neq 0 \quad \text{for any } t \in [0,T], \tag{6.13}$$

then $G'_\lambda(x) : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ is a Fredholm operator of index zero. Indeed, it is the sum of the two compact linear operators $q \mapsto (0, -q(T))$ (having finite dimensional image) and $q \mapsto (\lambda N_x(\cdot)q(\cdot), 0)$ (which is compact since so is the inclusion $\mathcal{C}^1 \hookrightarrow \mathcal{C}^0 \times \mathbb{R}^n$) with the isomorphism

$$\mathcal{C}^1 \longrightarrow \mathcal{C}^0 \times \mathbb{R}^n, \quad q \longmapsto \big(\big(I + \lambda M_x(\cdot)\big)q'(\cdot), q(0)\big). \tag{6.14}$$

Let us stress that condition (6.13) holds for any pair (x,λ) if we assume that, for every $(t,a,b) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, the Jacobian matrix $\partial_3\phi(t,a,b)$ has no negative eigenvalues.

Let us now estimate the local measure of noncompactness of the maps G and K. In particular, we look for conditions under which a given pair $(x,\lambda) \in \mathcal{C}^1 \times [0,+\infty)$ verifies the inequality

$$\alpha_{(x,\lambda)}(K) < \omega_{(x,\lambda)}(G). \tag{6.15}$$

Lemma 6.1. *Suppose that* φ *is Lipschitz continuous with respect to the third variable; that is, there exists some* $c > 0$ *such that*

$$\left|\varphi\left(t, a, b_1\right) - \varphi\left(t, a, b_2\right)\right| \leq c\left|b_1 - b_2\right| \tag{6.16}$$

for any $t \in [0, T]$ *and any* $a, b_1, b_2 \in \mathbb{R}^n$. *Then,*

$$\alpha_{(x,\lambda)}(K) \leq \lambda c \tag{6.17}$$

for any pair $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$.

Proof. Let $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$ be fixed. Since $K(x, \lambda) = -\lambda(\Psi(x), 0)$, by Corollary 3.8 we get $\alpha_{(x,\lambda)}(K) = \lambda\alpha_x(\Psi)$. Moreover, the map Ψ is Lipschitz with constant c. Indeed, given $x_1, x_2 \in \mathcal{C}^1$ and $t \in [0, T]$, we have

$$\left|\Psi\left(x_1\right)(t) - \Psi\left(x_2\right)(t)\right| = \left|\varphi\left(t, x_1(t), x_1'(t)\right) - \varphi\left(t, x_2(t), x_2'(t)\right)\right| \leq c\left|x_1'(t) - x_2'(t)\right|, \tag{6.18}$$

and thus

$$\left\|\Psi\left(x_1\right) - \Psi\left(x_2\right)\right\|_{\infty} \leq c\left\|x_1' - x_2'\right\|_{\infty} \leq c\left\|x_1 - x_2\right\|_1. \tag{6.19}$$

It follows that $\alpha_x(\Psi) \leq \alpha(\Psi) \leq c$ and, consequently, $\alpha_{(x,\lambda)}(K) \leq \lambda c$. □

Remark 6.2. The assertion of Lemma 6.1 is still valid when

$$\varphi(t, a, b) = \varphi_1(t, a, b) + \varphi_2(t, a), \tag{6.20}$$

with φ_1 satisfying condition (6.16) and φ_2 being independent of the third variable. In fact, in this case one can easily check that the map $\widetilde{\Psi} : \mathcal{C}^1 \to \mathcal{C}^0$, defined by

$$\widetilde{\Psi}(x)(t) = \varphi_1\left(t, x(t), x'(t)\right) + \varphi_2\left(t, x(t)\right), \quad t \in [0, T], \tag{6.21}$$

is α-Lipschitz with constant c, being the sum of an α-Lipschitz map with constant c and a completely continuous map.

Lemma 6.3. *Assume that for any* $(t, a, b) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$ *the Jacobian matrix* $\partial_3\phi(t, a, b)$ *has no negative eigenvalues. Set*

$$\gamma(\lambda) = \sup_{(t,a,b)} \left\|\left(I + \lambda\partial_3\phi(t, a, b)\right)^{-1}\right\|. \tag{6.22}$$

Then,

$$\omega_{(x,\lambda)}(G) \geq \frac{1}{\gamma(\lambda)} \tag{6.23}$$

for any pair $(x, \lambda) \in \mathcal{C}^1 \times [0, +\infty)$.

Proof. Let $(x,\lambda) \in \mathcal{C}^1 \times [0,+\infty)$ be fixed. First of all, observe that, since G is of class C^1, by Proposition 3.3 we have $\omega_{(x,\lambda)}(G) = \omega(G'(x,\lambda))$ and, by Proposition 3.4, $\omega(G'(x,\lambda)) = \omega(G'_\lambda(x))$. Hence,

$$\omega_{(x,\lambda)}(G) = \omega\left(G'_\lambda(x)\right). \tag{6.24}$$

As we already pointed out, the assumption on the Jacobian matrix $\partial_3\phi(t,a,b)$ implies that condition (6.13) holds for any pair $(x,\lambda) \in \mathcal{C}^1 \times [0,+\infty)$. Consequently, $G'_\lambda(x)$ is a Fredholm operator of index zero.

Now, define the linear operator $\Gamma : \mathcal{C}^1 \to \mathcal{C}^0 \times \mathbb{R}^n$ by $\Gamma q = (\Gamma_1 q, q(0))$, where

$$\Gamma_1 q(t) = \left(I + \lambda M_x(t)\right)q'(t), \quad t \in [0,T]. \tag{6.25}$$

Since the maps $q \mapsto (0,-q(T))$ and $q \mapsto (\lambda N_x(\cdot)q(\cdot),0)$ are compact, by (2) and (3) in Proposition 3.2 we have $\omega(G'_\lambda(x)) = \omega(\Gamma)$. Moreover, condition (6.13) implies that the linear operator Γ is invertible. Thus, by (5) in Proposition 3.2, we get

$$\omega(\Gamma) = \frac{1}{\alpha\left(\Gamma^{-1}\right)}. \tag{6.26}$$

Let us estimate $\alpha(\Gamma^{-1})$. For this purpose, let $P : \mathcal{C}^0 \times \mathbb{R}^n \to \mathcal{C}^0 \times \mathbb{R}^n$ be the natural projection onto $\mathcal{C}^0 \times \{0\}$, defined by $(y,r) \mapsto (y,0)$. By Proposition 3.4, we have $\alpha(\Gamma^{-1}) = \alpha(\Gamma^{-1}P)$. Now, fix $(y,r) \in \mathcal{C}^0 \times \mathbb{R}^n$ and let $q \in \mathcal{C}^1$ be such that $q = \Gamma^{-1}P(y,r)$; that is, q is the solution of the linear problem

$$\begin{gathered} q'(t) = \left(I + \lambda M_x(t)\right)^{-1} y(t), \\ q(0) = 0. \end{gathered} \tag{6.27}$$

We have $|q'(t)| \le \|(I + \lambda M_x(t))^{-1}\|\,|y(t)|$ for any t, and thus

$$\left\|q'\right\|_\infty \le \max_{t\in[0,T]} \left\|\left(I + \lambda M_x(t)\right)^{-1}\right\| \|y\|_\infty \le \sup_{(t,a,b)} \left\|\left(I + \lambda\partial_3\phi(t,a,b)\right)^{-1}\right\| \left\|(y,r)\right\| = \gamma(\lambda)\left\|(y,r)\right\|. \tag{6.28}$$

Consequently, $\|q\|_1 \le \gamma(\lambda)\,\|(y,r)\|$. It follows that

$$\alpha\left(\Gamma^{-1}\right) = \alpha\left(\Gamma^{-1}P\right) \le \gamma(\lambda). \tag{6.29}$$

Hence,

$$\omega_{(x,\lambda)}(G) = \omega(\Gamma) \ge \frac{1}{\gamma(\lambda)}. \tag{6.30}$$

□

The next proposition summarizes the above two lemmas. The statement involves the map

$$H : \mathcal{C}^1 \times [0,+\infty) \longrightarrow \mathcal{C}^0 \times \mathbb{R}^n, \quad H(x,\lambda) = G(x,\lambda) - K(x,\lambda), \tag{6.31}$$

where G and K are as in (6.6).

Proposition 6.4. *Let $\phi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1 and let $\psi : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous. Assume that the following conditions hold:*

(i) *the map ψ is Lipschitz continuous with respect to the third variable with constant $c > 0$;*

(ii) *for any $(t,a,b) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, the Jacobian matrix $\partial_3\phi(t,a,b)$ has no negative eigenvalues;*

(iii) *the constant c is such that*

$$\lambda c < \frac{1}{\gamma(\lambda)} \quad \text{for any } \lambda \in [0,+\infty), \tag{6.32}$$

where $\gamma(\lambda) = \sup_{(t,a,b)} \|(I + \lambda\partial_3\phi(t,a,b))^{-1}\|$.

Then, the map H as in (6.31) *is an α-Fredholm homotopy.*

As an example illustrating condition (iii) in Proposition 6.4, consider the case in which for any (t,a,b) the Jacobian matrix $\partial_3\phi(t,a,b)$ coincides with a diagonal matrix Δ. Suppose that all the eigenvalues of Δ are positive, and let δ be the smallest one. Thus, one can easily check that $\gamma(\lambda) = 1/(1+\lambda\delta)$, and condition (iii) is clearly satisfied if the Lipschitz constant c of the map ψ is smaller than δ.

Let us come back to our study of problem (6.1). For technical reasons, define

$$\begin{aligned} L &: C^1 \longrightarrow C^0 \times \mathbb{R}^n, & Lx &= (x', x(0) - x(T)), \\ h &: C^1 \longrightarrow C^0 \times \mathbb{R}^n, & h(x) &= (\Phi(x), 0), \\ k &: C^1 \longrightarrow C^0 \times \mathbb{R}^n, & k(x) &= (\Psi(x), 0), \end{aligned} \tag{6.33}$$

with Φ and Ψ as in (6.5). Then, problem (6.1) is equivalent to the semilinear operator equation

$$Lx + \lambda(h(x) + k(x)) = 0 \tag{6.34}$$

in $C^1 \times [0,+\infty)$. Observe that (6.34) can be equivalently written as

$$H(x,\lambda) = 0, \tag{6.35}$$

where the map

$$H : C^1 \times [0,+\infty) \longrightarrow C^0 \times \mathbb{R}^n, \qquad H(x,\lambda) = Lx + \lambda(h(x) + k(x)) \tag{6.36}$$

is the same as in (6.31), with $G(x,\lambda) = Lx + \lambda h(x)$ and $K(x,\lambda) = -\lambda k(x)$.

Now, suppose that conditions (i)–(iii) in Proposition 6.4 hold. Hence, by Proposition 6.4, the map H is an α-Fredholm homotopy. Therefore, we can apply the results of Section 5 to (6.34) obtaining a global bifurcation result (see Theorem 6.5 below).

As we already pointed out, Benevieri et al. in [4] obtained a global bifurcation result for (6.34) in the absence of the perturbation k. That is, in [4] they studied a problem analogous to (6.1) with ψ identically zero. Their result was extended by Benevieri and Furi [6] in the case when ψ is nonzero and independent of the third variable. Theorem 6.5 below extends these results, by assuming ψ to be Lipschitz continuous with respect to the third variable, with suitably small Lipschitz constant.

Before stating Theorem 6.5, we need some preliminary remarks. First, to avoid cumbersome notation, any point $p \in \mathbb{R}^n$ is identified with the constant function $t \mapsto p$ so that $\mathbb{R}^n$ can be regarded as the set of trivial solutions of problem (6.1).

Now, it is not difficult to show that the operator L is Fredholm of index zero, with $\operatorname{Ker} L = \mathbb{R}^n$ and

$$\operatorname{Im} L = \left\{ (y, r) \in C^0 \times \mathbb{R}^n : r = -\int_0^T y(t) dt \right\}. \tag{6.37}$$

The reader can easily verify that $C^0 \times \mathbb{R}^n = \operatorname{Im} L \oplus F_1$, where F_1 is an n-dimensional subspace of $C^0 \times \mathbb{R}^n$ which can be identified with $\mathbb{R}^n$. In fact, observe that any pair $(y, r) \in C^0 \times \mathbb{R}^n$ can be uniquely decomposed as

$$(y, r) = \left(y, -\int_0^T y(t) dt \right) + \left(0, r + \int_0^T y(t) dt \right). \tag{6.38}$$

Moreover, the projection π of $C^0 \times \mathbb{R}^n$ onto $F_1 = \mathbb{R}^n$ can be written as

$$\pi(y, r) = r + \int_0^T y(t) dt. \tag{6.39}$$

Thus, the vector field $v : \mathbb{R}^n \to \mathbb{R}^n$, defined by $v(p) = \pi h(p) + \pi k(p)$, can be written as

$$v(p) = \int_0^T \big(\phi(t, p, 0) + \psi(t, p, 0) \big) dt. \tag{6.40}$$

We are now ready to state the main result of this section. The statement involves, instead of v, the *mean value vector field* $w : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$w(p) = \frac{1}{T} \int_0^T \big(\phi(t, p, 0) + \psi(t, p, 0) \big) dt. \tag{6.41}$$

Theorem 6.5. *Let $\phi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be C^1, and let $\psi : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous; suppose that conditions (i)–(iii) in Proposition 6.4 hold.*

Let $w : \mathbb{R}^n \to \mathbb{R}^n$ be the mean value vector field defined in (6.41). *Let W be an open subset of $C^1 \times [0, +\infty)$ and denote $\widetilde{W}_0 = \{ p \in \mathbb{R}^n : (p, 0) \in W \}$. Assume that the Brouwer degree $\deg_B(w, \widetilde{W}_0, 0)$ is defined and is different from zero. Then, W contains a connected set of nontrivial solutions of problem* (6.1), *whose closure in W is not compact and intersects* $\operatorname{Ker} L \times \{0\} \cong \mathbb{R}^n$ *in the compact set $w^{-1}(0) \cap \widetilde{W}_0$.*

Proof. Clearly, $\deg_B(w, \widetilde{W}_0, 0)$ is defined and is different from zero if and only if the same is true for $\deg_B(v, \widetilde{W}_0, 0)$. To apply Theorem 5.3, we need the orientability of the map G defined in (6.6). This is a consequence of the fact, proved in [15], that any Fredholm map defined in a simply connected open set (the whole space in this case) is orientable. Thus, the assertion follows from Theorem 5.3. □

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