Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 125426, 8 pages
doi:10.1155/2009/125426

Research Article

Some Coupled Fixed Point Theorems in Cone Metric Spaces

F. Sabetghadam,[1] H. P. Masiha,[1] and A. H. Sanatpour[2]

[1] *Department of Mathematics, K. N. Toosi University of Technology, Tehran, 16315-1618, Iran*
[2] *Department of Mathematics, Tarbiat Moallem University, Tehran, 15618-36314, Iran*

Correspondence should be addressed to H. P. Masiha, masiha@kntu.ac.ir

Received 17 July 2009; Revised 23 September 2009; Accepted 28 September 2009

Recommended by Jerzy Jezierski

We prove some coupled fixed point theorems for mappings satisfying different contractive conditions on complete cone metric spaces.

1. Introduction

Recently, Huang and Zhang in [1] generalized the concept of metric spaces by considering vector-valued metrics (cone metrics) with values in an ordered real Banach space. They proved some fixed point theorems in cone metric spaces showing that metric spaces really doesnot provide enough space for the fixed point theory. Indeed, they gave an example of a cone metric space (X, d) and proved existence of a unique fixed point for a selfmap T of X which is contractive in the category of cone metric spaces but is not contractive in the category of metric spaces. After that, cone metric spaces have been studied by many other authors (see [1–9] and the references therein).

Regarding the concept of coupled fixed point, introduced by Bhaskar and Lakshmikantham [10], we consider the corresponding definition for the mappings on complete cone metric spaces and prove some coupled fixed point theorems in the next section. First, we recall some standard notations and definitions in cone metric spaces.

A cone P is a subset of a real Banach space E such that

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;

(iii) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, the partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. The notation $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. Also, we will use $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is called normal if there exists a constant $M > 0$ such that for every $x, y \in E$ if $0 \leq x \leq y$ then $\|x\| \leq M\|y\|$. The least positive number satisfying this inequality is called the normal constant of P (see [1]). The cone P is called regular if every increasing (decreasing) and bounded above (below) sequence is convergent in E. It is known that every regular cone is normal (see [1], or [7, Lemma 1.1]).

Huang and Zhang defined the concept of a cone metric space in [1] as follows.

Definition 1.1 (see [1]). Let X be a nonempty set and let E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:

(d_1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.2 (see [1]). Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ converges to x, denoted by $\lim_{n\to\infty} x_n = x$, if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$;

(ii) $\{x_n\}_{n\geq 1}$ is a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there exists a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

A cone metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent in X. If for any sequence $\{x_n\}$ in X there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X, then the cone metric space (X, d) is called sequentially compact. Clearly, every sequentially compact cone metric space is complete. Huang and Zhang in [1] investigated the existence and uniqueness of the fixed point for a selfmap T on a cone metric space (X, d). They considered different types of contractive conditions on T. They also assumed (X, d) to be complete when P is a normal cone, and (X, d) to be sequentially compact when P is a regular cone. Later, in [7], Rezapour and Hamlbarani improved some of the results in [1] by omitting the normality assumption of the cone P, when (X, d) is complete. See [4, 6, 7, 9] for more related results about (complete) cone metric spaces and fixed point theorems for different types of mappings on these spaces.

In the rest of this paper, we always suppose that E is a real Banach space, $P \subseteq E$ is a cone with $\operatorname{int} P \neq \emptyset$ and $\leq$ is partial ordering with respect to P. We also note that the relations $P + \operatorname{int} P \subseteq \operatorname{int} P$ and $\lambda \operatorname{int} P \subseteq \operatorname{int} P$ $(\lambda > 0)$ always hold true.

2. Main Results

For a given partially ordered set X, Bhaskar and Lakshmikantham in [10] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$. Later in [11] Lakshmikantham and Ćirić investigated some more coupled fixed point theorems in partially ordered sets. The following is the corresponding definition of coupled fixed point in cone metric spaces.

Definition 2.1. Let (X, d) be a cone metric space. An element $(x, y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

In the next theorems of this section, we investigate some coupled fixed point theorems in cone metric spaces.

Theorem 2.2. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \leq kd(x, u) + ld(y, v), \tag{2.1}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0), \ldots, x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$. Then by (2.1) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq kd(x_{n-1}, x_n) + ld(y_{n-1}, y_n), \end{aligned} \tag{2.2}$$

and similarly,

$$\begin{aligned} d(y_n, y_{n+1}) &= d(F(y_{n-1}, x_{n-1}), F(y_n, x_n)) \\ &\leq kd(y_{n-1}, y_n) + ld(x_{n-1}, x_n). \end{aligned} \tag{2.3}$$

Therefore, by letting

$$d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1}), \tag{2.4}$$

we have

$$\begin{aligned} d_n &= d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \\ &\leq kd(x_{n-1}, x_n) + ld(y_{n-1}, y_n) + kd(y_{n-1}, y_n) + ld(x_{n-1}, x_n) \\ &\leq (k + l)(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) \\ &= (k + l)d_{n-1}. \end{aligned} \tag{2.5}$$

Consequently, if we set $\delta = k + l$ then for each $n \in \mathbb{N}$ we have

$$0 \leq d_n \leq \delta d_{n-1} \leq \delta^2 d_{n-2} \leq \cdots \leq \delta^n d_0. \tag{2.6}$$

If $d_0 = 0$ then (x_0, y_0) is a coupled fixed point of F. Now, let $d_0 > 0$. For each $n \geq m$ we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m), \\ d(y_n, y_m) &\leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2}) + \cdots + d(y_{m+1}, y_m). \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) &\leq d_{n-1} + d_{n-2} + \cdots + d_m \\ &\leq \left(\delta^{n-1} + \delta^{n-2} + \cdots + \delta^m\right) d_0 \\ &\leq \frac{\delta^m}{1-\delta} \quad d_0, \end{aligned} \tag{2.8}$$

which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X, and there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. Let $c \in E$ with $0 \ll c$. For every $m \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $d(x_n, x^*) \ll c/2m$ and $d(y_n, y^*) \ll c/2m$ for all $n \geq N$. Thus

$$\begin{aligned} d(F(x^*, y^*), x^*) &\leq d(F(x^*, y^*), x_{N+1}) + d(x_{N+1}, x^*) \\ &= d(F(x^*, y^*), F(x_N, y_N)) + d(x_{N+1}, x^*) \\ &\leq kd(x_N, x^*) + ld(y_N, y^*) + d(x_{N+1}, x^*) \\ &\ll (k+l)\frac{c}{2m} + \frac{c}{2m} \leq \frac{c}{m}. \end{aligned} \tag{2.9}$$

Consequently, $d(F(x^*, y^*), x^*) \ll c/m$ for all $m \geq 1$. Thus, $d(F(x^*, y^*), x^*) = 0$ and hence $F(x^*, y^*) = x^*$. Similarly, we have $F(y^*, x^*) = y^*$ meaning that (x^*, y^*) is a coupled fixed point of F.

Now, if (x', y') is another coupled fixed point of F, then

$$\begin{aligned} d(x', x^*) &= d(F(x', y'), F(x^*, y^*)) \leq kd(x', x^*) + ld(y', y^*), \\ d(y', y^*) &= d(F(y', x'), F(y^*, x^*)) \leq kd(y', y^*) + ld(x', x^*), \end{aligned} \tag{2.10}$$

and therefore,

$$d(x', x^*) + d(y', y^*) \leq (k+l)(d(x', x^*) + d(y', y^*)). \tag{2.11}$$

Since $k + l < 1$, (2.11) implies that $d(x', x^*) + d(y', y^*) = 0$. Hence, we have $(x', y') = (x^*, y^*)$ and the proof of the theorem is complete. □

It is worth noting that when the constants in Theorem 2.2 are equal we have the following corollary.

Corollary 2.3. *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X\times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}(d(x,u)+d(y,v)), \tag{2.12}$$

where $k \in [0,1)$ is a constant. Then F has a unique coupled fixed point.

Example 2.4. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in \mathbb{R}^2 : x,y \geq 0\} \subseteq \mathbb{R}^2$, and $X = [0,1]$. Define $d : X\times X \to E$ with $d(x,y) = (|x-y|,|x-y|)$. Then (X,d) is a complete cone metric space. Consider the mapping $F : X\times X \to X$ with $F(x,y) = (x+y)/6$. Then F satisfies the contractive condition (2.12) for $k = 1/3$, that is,

$$d(F(x,y),F(u,v)) \leq \frac{1}{6}(d(x,u)+d(y,v)). \tag{2.13}$$

Therefore, by Corollary 2.3, F has a unique coupled fixed point, which in this case is $(0,0)$. Note that if the mapping $F : X\times X \to X$ is given by $F(x,y) = (x+y)/2$, then F satisfies the contractive condition (2.12) for $k = 1$, that is,

$$d(F(x,y),F(u,v)) \leq \frac{1}{2}(d(x,u)+d(y,v)). \tag{2.14}$$

In this case, $(0,0)$ and $(1,1)$ are both coupled fixed points of F and hence the coupled fixed point of F is not unique. This shows that the condition $k < 1$ in corollary (2.12) and hence $k+l < 1$ in Theorem 2.2 are optimal conditions for the uniqueness of the coupled fixed point.

Theorem 2.5. *Let (X,d) be a complete cone metric space. Suppose that the mapping $F : X\times X \to X$ satisfies the following contractive condition for all $x,y,u,v \in X$:*

$$d(F(x,y),F(u,v)) \leq kd(F(x,y),x)+ld(F(u,v),u), \tag{2.15}$$

where k,l are nonnegative constants with $k+l<1$. Then F has a unique coupled fixed point.

Proof. Choose $x_0,y_0 \in X$ and set $x_1 = F(x_0,y_0)$, $y_1 = F(y_0,x_0),\ldots,x_{n+1} = F(x_n,y_n)$, $y_{n+1} = F(y_n,x_n)$. Then by applying (2.15) we get

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \delta d(x_n,x_{n-1}), \\ d(y_n,y_{n+1}) &\leq \delta d(y_n,y_{n-1}), \end{aligned} \tag{2.16}$$

where $\delta = k/(1-l) < 1$. This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,d) and therefore by the completeness of X, there exist x^*, $y^* \in X$ such that $\lim_{n\to\infty}x_n = x^*$ and

$\lim_{n\to\infty} y_n = y^*$. Let $m \in \mathbb{N}$ and choose a natural number N such that $d(x_n, x^*) = ((1-l)/4m)c$ for all $n \geq N$. Thus,

$$\begin{aligned} d(F(x^*, y^*), x^*) &\leq d(x_{N+1}, F(x^*, y^*)) + d(x_{N+1}, x^*) \\ &= d(F(x_N, y_N), F(x^*, y^*)) + d(x_{N+1}, x^*) \\ &\leq kd(F(x_N, y_N), x_N) + ld(F(x^*, y^*), x^*) + d(x_{N+1}, x^*), \end{aligned} \tag{2.17}$$

which implies that

$$d(F(x^*, y^*), x^*) \leq \frac{k}{1-l} \; d(x_{N+1}, x_N) + \frac{1}{1-l} \; d(x_{N+1}, x^*) \ll \frac{c}{m}. \tag{2.18}$$

Since $m \in \mathbb{N}$ was arbitrary, $d(F(x^*, y^*), x^*) = 0$ or equivalently $F(x^*, y^*) = x^*$. Similarly, one can get $F(y^*, x^*) = y^*$ showing that (x^*, y^*) is a coupled fixed point of F.

Now, if (x', y') is another coupled fixed point of F, then by applying (2.15) we have

$$\begin{aligned} d(x', x^*) &= d(F(x', y'), F(x^*, y^*)) \\ &\leq kd(F(x', y'), x') + ld(F(x^*, y^*), x^*) = 0, \end{aligned} \tag{2.19}$$

and therefore $x' = x^*$. Similarly, we can get $y' = y^*$ and hence $(x', y') = (x^*, y^*)$. □

Theorem 2.6. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$,*

$$d(F(x, y), F(u, v)) \leq kd(F(x, y), u) + ld(F(u, v), x), \tag{2.20}$$

where k, l are nonnegative constants with $k + l < 1$. Then F has a unique coupled fixed point.

Proof. First, note that the uniqueness of the coupled fixed point is an obvious result of $k+l < 1$ in (2.20). To prove the existence of the fixed point, let $x_0, y_0 \in X$ and choose the sequence $\{x_n\}$ and $\{y_n\}$ like in the proof of Theorem 2.5, that is $x_1 = F(x_0, y_0)$, $y_1 = F(y_0, x_0), \ldots, x_{n+1} = F(x_n, y_n)$, $y_{n+1} = F(y_n, x_n)$. Then by applying (2.20) we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq kd(F(x_{n-1}, y_{n-1}), x_n) + ld(F(x_n, y_n), x_{n-1}) \\ &\leq l(d(F(x_n, y_n), x_n) + d(x_n, x_{n-1})), \end{aligned} \tag{2.21}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{l}{1-l} d(x_n, x_{n-1}). \tag{2.22}$$

Similarly, one can get

$$d(y_n, y_{n+1}) \le \frac{l}{1-l} d(y_n, y_{n-1}). \tag{2.23}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d) and hence by the completeness of X, there exist $x^*, y^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. Let $c \in E$ with $0 \ll c$ and for each $m \in \mathbb{N}$ choose a natural number N such that $d(x_n, x^*) \ll ((1-l)/4m)c$ for all $n \ge N$. Thus,

$$\begin{aligned} d(F(x^*, y^*), x^*) &\le d(x_{N+1}, F(x^*, y^*)) + d(x_{N+1}, x^*) \\ &= d(F(x_N, y_N), F(x^*, y^*)) + d(x_{N+1}, x^*) \\ &\le kd(F(x_N, y_N), x^*) + ld(F(x^*, y^*), x_N) + d(x_{N+1}, x^*), \end{aligned} \tag{2.24}$$

which implies

$$d(F(x^*, y^*), x^*) \le \frac{1+k}{1-l} d(x_{N+1}, x^*) + \frac{l}{1-l} d(x_N, x^*) \ll \frac{c}{m}. \tag{2.25}$$

Since $m \in \mathbb{N}$ was arbitrary, $d(F(x^*, y^*), x^*) = 0$ or equivalently $F(x^*, y^*) = x^*$. Similarly, one can get $F(y^*, x^*) = y^*$ and hence (x^*, y^*) is a coupled fixed point of F. □

When the constants in Theorems 2.5 and 2.6 are equal, we get the following corollaries.

Corollary 2.7. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \le \frac{k}{2} (d(F(x, y), x) + d(F(u, v), u)), \tag{2.26}$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Corollary 2.8. *Let (X, d) be a complete cone metric space. Suppose that the mapping $F : X \times X \to X$ satisfies the following contractive condition for all $x, y, u, v \in X$:*

$$d(F(x, y), F(u, v)) \le \frac{k}{2} (d(F(x, y), u) + d(F(u, v), x)), \tag{2.27}$$

where $k \in [0, 1)$ is a constant. Then F has a unique coupled fixed point.

Remark 2.9. Note that in Theorem 2.5, if the mapping $F : X \times X \to X$ satisfies the contractive condition (2.15) for all $x, y, u, v \in X$, then F also satisfies the following contractive condition:

$$\begin{aligned} d(F(x, y), F(u, v)) &= d(F(u, v), F(x, y)) \\ &\le kd(F(u, v), u) + ld(F(x, y), x). \end{aligned} \tag{2.28}$$

Consequently, by adding (2.15) and (2.28), F also satisfies the following:

$$d(F(x,y),F(u,v)) \leq \frac{k+l}{2}d(F(x,y),x) + \frac{k+l}{2}d(F(u,v),u), \tag{2.29}$$

which is a contractive condition of the type (2.26) in Corollary 2.7 (with equal constants). Therefore, one can also reduce the proof of general case (2.15) in Theorem 2.5 to the special case of equal constants. A similar argument is valid for the contractive conditions (2.20) in Theorem 2.6 and (2.27) in Corollary 2.8.

Acknowledgment

The authors would like to thank the referees for their valuable and useful comments.

References

[1] L. G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[2] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.

[3] R. H. Haghi and Sh. Rezapour, "Fixed points of multifunctions on regular cone metric spaces," *Expositiones Mathematicae*. In press.

[4] D. Ilić and V. Rakočević, "Common fixed points for maps on cone metric space," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 876–882, 2008.

[5] D. Klim and D. Wardowski, "Dynamic processes and fixed points of set-valued nonlinear contractions in cone metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5170–5175, 2009.

[6] S. Radenović, "Common fixed points under contractive conditions in cone metric spaces," *Computers and Mathematics with Applications*, vol. 58, no. 6, pp. 1273–1278, 2009.

[7] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.

[8] Sh. Rezapour and R. H. Haghi, "Fixed point of multifunctions on cone metric spaces," *Numerical Functional Analysis and Optimization*, vol. 30, no. 7-8, pp. 825–832, 2009.

[9] Sh. Rezapour and M. Derafshpour, "Some common fixed point results in cone metric spaces," to appear in *Journal of Nonlinear and Convex Analysis*.

[10] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.

[11] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.