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Research Article

Best Proximity Point Theorems for p-Cyclic Meir-Keeler Contractions

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We consider a contraction map T of the Meir-Keeler type on the union of p subsets $A_1, \ldots, A_p$, $(p \geq 2)$, of a metric space (X, d) to itself. We give sufficient conditions for the existence and convergence of a best proximity point for such a map.

1. Introduction

Meir and Keeler in [1] considered an extension of the classical Banach contraction theorem on a complete metric space. Kirk et al. in [2] extended the Banach contraction theorem for a class of mappings satisfying cyclical contractive conditions.

Eldred and Veeramani in [3] introduced the following definition. Let A and B be nonempty subsets of a metric space X. A map $T : A \cup B \to A \cup B$, is a cyclic contraction map if it satisfies

(1) $T(A) \subseteq B$ and $T(B) \subseteq A$, and

(2) for some $k \in (0,1)$, $d(Tx, Ty) \leq kd(x, y) + (1-k)\operatorname{dist}(A, B)$ for all $x \in A$, $y \in B$.

In this case, a point $z \in A \cup B$ such that $d(z, Tz) = \operatorname{dist}(A, B)$, called a best proximity point, has been considered. This notion is more general in the sense that if the sets intersect, then every best proximity point is a fixed point. In [3], sufficient conditions for the existence and convergence of a unique best proximity point for a cyclic contraction on a uniformly convex Banach space have been given. Further, in [4], this result is extended by Di Bari et al., where the contraction condition of the map is of the Meir-Keeler-type. That is, in addition to the cyclic condition, if the map satisfies the condition that for a given $\epsilon > 0$, there exists a $\delta > 0$

such that $d(x,y) < \operatorname{dist}(A,B) + \epsilon + \delta$ implies that $d(Tx,Ty) < \operatorname{dist}(A,B) + \epsilon$, $x \in A$, $y \in B$. Then, such a map is called a cyclic Meir-Keeler map. In [4], sufficient conditions are given to obtain a unique best proximity point for such maps. One may refer to [5, 6] for similar types of notion of best proximity points. A question that naturally arises is whether the main results in [4] can be extended to p subsets, $p \geq 2$? From a geometrical point of view, for the cyclic Meir-Keeler contraction defined on the union of two sets, there is no question concerning the position of the sets. But in the case of more than two sets, the map is defined on the union of p sets, $\{A_i\}_{i=1}^{p}$ (Definition 3.5), so that the image of A_i is contained in A_{i+1}, and the image of A_{i+1} is contained in A_{i+2} but not in A_i ($1 \leq i \leq p$ and $A_{p+1} = A_1$). Hence, it is interesting to extend the notion of the cyclic Meir-Keeler contraction to p sets, $p \geq 2$, and we call this map a p-cyclic Meir-Keeler contraction. In this paper, we give sufficient conditions for the existence and convergence of a best proximity point for such a map (Theorem 3.13). Here, we observe that the distances between the adjacent sets are equal under this map, and this fact plays an important role in obtaining a best proximity point. Also, the obtained best proximity point is a periodic point of T with period p. Moreover, if $x \in A_i$ is a best proximity point in A_i, then $T^j x$ is a best proximity point in A_{i+j} for $j = 0, 1, 2, \ldots, p-1$.

2. Preliminaries

In this section, we give some basic definitions and concepts related to the main results. We begin with a definition due to Lim [7].

Definition 2.1. A function $\phi : [0,\infty) \to [0,\infty)$ is called an L-function if $\phi(0) = 0$, $\phi(s) > 0$ for $s > 0$, and for every $s > 0$, there exists $\delta > 0$ such that $\phi(t) \leq s$ for all $t \in [s, s+\delta]$.

Lemma 2.2 (see [7, 8]). *Let Y be a nonempty set, and let $f, g : Y \to [0,\infty)$. Then, the following are equivalent.*

(1) *For each $\epsilon > 0$, there exists $\delta > 0$ such that $x \in Y$, $f(x) < \epsilon + \delta \Rightarrow g(x) < \epsilon$.*

(2) *There exists an L-function ϕ (nondecreasing, continuous) such that $x \in Y$, $f(x) > 0 \Rightarrow g(x) < \phi(f(x))$, and $f(x) = 0 \Rightarrow g(x) = 0$.*

Lemma 2.3 (see [8]). *Let ϕ be an L-function. Let $\{s_n\}$ be a nonincreasing sequence of nonnegative real numbers. Suppose $s_{n+1} < \phi(s_n)$ for all $n \in \mathbb{N}$ with $s_n > 0$, then, $s_n \to 0$ as $n \to \infty$.*

It is well known that if X_0 is a convex subset of a strictly convex normed linear space X and $x \in X$, then a best approximation of x from X_0, if it exists, is unique.

We use the following lemmas proved in [3].

Lemma 2.4. *Let A be a nonempty closed and convex subset and B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A, and let $\{y_n\}$ be a sequence in B satisfying*

(1) $\|z_n - y_n\| \to \operatorname{dist}(A,B)$,

(2) *for every $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$, such that for all $m > n \geq N_0$, $\|x_m - y_n\| \leq \operatorname{dist}(A,B) + \epsilon$.*

Then, for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$, such that for all $m > n \geq N_1$, $\|x_m - z_n\| \leq \epsilon$.

Lemma 2.5. *Let A be a nonempty closed and convex subsets and let B be a nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\} and \{z_n\}$ be sequences in A and let $\{y_n\}$ be a sequence in B satisfying*

(1) $\|x_n - y_n\| \to \operatorname{dist}(A, B)$,

(2) $\|z_n - y_n\| \to \operatorname{dist}(A, B)$.

Then, $\|x_n - z_n\|$ converges to zero.

3. Main Results

Definition 3.1. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space. Then, $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ is called a p-cyclic mapping if

$$T(A_i) \subseteq A_{i+1} \quad \text{for } 1 \leq i \leq p, \text{ where } A_{p+1} = A_1. \tag{3.1}$$

A point $x \in A_i$ is said to be a best proximity point if $d(x, Tx) = \operatorname{dist}(A_i, A_{i+1})$.

Definition 3.2. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X, and $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic mapping. T is called a p-cyclic nonexpansive mapping if

$$d(Tx, Ty) \leq d(x, y) \quad \forall x \in A_i,\ y \in A_{i+1},\ 1 \leq i \leq p. \tag{3.2}$$

It is an interesting fact to note that the distances between the adjacent sets are equal under the p-cyclic nonexpansive mapping.

Lemma 3.3. *Let $X, A_1, \ldots, A_p, T$ be as in Definition 3.2. Then,* $\operatorname{dist}(A_i, A_{i+1}) = \operatorname{dist}(A_{i+1}, A_{i+2}) = \operatorname{dist}(A_1, A_2)$ *for all i, $1 \leq i \leq p$.*

Proof. For $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq p$, $\operatorname{dist}(A_{i+1}, A_{i+2}) \leq d(Tx, Ty) \leq d(x, y)$ implies $\operatorname{dist}(A_{i+1}, A_{i+2}) \leq \operatorname{dist}(A_i, A_{i+1})$. That is, $\operatorname{dist}(A_1, A_2) \leq \operatorname{dist}(A_p, A_1) \leq \cdots \leq \operatorname{dist}(A_1, A_2)$. □

Remark 3.4. If $\xi \in A_i$ is a best proximity point, then since $d(T^p\xi, T^{p-1}\xi) \leq d(T^{p-1}\xi, T^{p-2}\xi) \leq \cdots \leq d(\xi, T\xi)$ and since the distances between the adjacent sets are equal, $T^j\xi$ is a best proximity point of T in A_{i+j} for $j = 0$ to $p - 1$.

Definition 3.5. Let $A_1, \ldots, A_p$ be nonempty subsets of a metric space X. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic mapping. T is called a p-cyclic Meir-Keeler contraction if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \operatorname{dist}\left(A_i, A_{i+1}\right) + \epsilon + \delta \Longrightarrow d(Tx, Ty) < \operatorname{dist}\left(A_i, A_{i+1}\right) + \epsilon \tag{3.3}$$

for all $x \in A_i$, $y \in A_{i+1}$, for $1 \leq i \leq p$.

Remark 3.6. From Lemma 2.2, we see that T is a p-cyclic Meir-Keeler contraction if and only if there exists an L-function ϕ (nondecreasing and continuous) such that for all $x \in A_i$,

$y \in A_{i+1}$, $1 \leq i \leq p$, $d(x,y) - \operatorname{dist}(A_i, A_{i+1}) > 0 \Rightarrow d(Tx,Ty) - \operatorname{dist}(A_i, A_{i+1}) < \phi(d(x,y) - \operatorname{dist}(A_i, A_{i+1}))$, $d(x,y) - \operatorname{dist}(A_i, A_{i+1}) = 0 \Rightarrow d(Tx,Ty) - \operatorname{dist}(A_i, A_{i+1}) = 0$.

Remark 3.7. From Remark 3.6, if T is a p-cyclic Meir-Keeler contraction, then for $x \in A_i$, $y \in A_{i+1}$, $1 \leq i \leq p$, the following hold:

(1) $d(Tx,Ty) - \operatorname{dist}(A_i, A_{i+1}) \leq \phi(d(x,y) - \operatorname{dist}(A_i, A_{i+1}))$,

(2) $d(Tx,Ty) \leq d(x,y)$.

Hence, every p-cyclic Meir-Keeler contraction is a p-cyclic nonexpansive map.

Lemma 3.8. *Let $X, A_1, \ldots, A_p, T$ be as in Definition 3.5, where each A_i is closed. Then, for every $x, y \in A_i$, for $1 \leq i \leq p$,*

(1) $d(T^{pn}x, T^{pn+1}y) \to \operatorname{dist}(A_i, A_{i+1})$ *as* $n \to \infty$,

(2) $d(T^{p(n\pm1)}x, T^{pn+1}y) \to \operatorname{dist}(A_i, A_{i+1})$ *as* $n \to \infty$.

Proof. To prove (1), Lemma 2.3 is used. Let $s_n = d(T^{pn}x, T^{pn+1}y) - \operatorname{dist}(A_i, A_{i+1})$. If $s_n = 0$ for some n, then $d(T^{p(n+k)}x, T^{p(n+k)+1}y) \leq d(T^{pn}x, T^{pn+1}y)$ for all $k \in \mathbb{N}$. Since $d(T^{pn}x, T^{pn+1}y) = \operatorname{dist}(A_i, A_{i+1})$, we find that $d(T^{p(n+k)}x, T^{p(n+k)+1}y) = \operatorname{dist}(A_i, A_{i+1})$ and this proves (1). Hence, assume $s_n > 0$ for all n. By Remark 3.7, $s_{n+1} \leq s_n$, and by Remark 3.6, there exists an L-function ϕ such that

$$\begin{aligned} d\left(T^{p(n+1)}x, T^{p(n+1)+1}y\right) - \operatorname{dist}\left(A_i, A_{i+1}\right) &< \phi\left(d\left(T^{p(n+1)-1}x, T^{p(n+1)}y\right) - \operatorname{dist}\left(A_i, A_{i+1}\right)\right) \\ &\leq d\left(T^{p(n+1)-2}x, T^{p(n+1)-1}y\right) - \operatorname{dist}\left(A_i, A_{i+1}\right) \\ &\leq \cdots \\ &\leq d\left(T^{pn+1}x, T^{pn+2}y\right) - \operatorname{dist}\left(A_i, A_{i+1}\right) \\ &< \phi\left(d\left(T^{pn}x, T^{pn+1}y\right) - \operatorname{dist}\left(A_i, A_{i+1}\right)\right). \end{aligned} \tag{3.4}$$

Hence, $s_{n+1} < \phi(s_n)$. Therefore, $s_n \to 0$ as $n \to \infty$.

Similarly, (2) can easily be proved. □

Remark 3.9. From Lemma 3.8, if X is a uniformly convex Banach space and if each A_i is convex, then for $x \in A_i$, $\|T^{pn}x - T^{pn+1}x\| \to \operatorname{dist}(A_i, A_{i+1})$ as $n \to \infty$, $\|T^{p(n\pm1)}x - T^{pn+1}x\| \to \operatorname{dist}(A_i, A_{i+1})$, as $n \to \infty$. Then, by Lemma 2.5, $\|T^{pn}x - T^{p(n\pm1)}x\| \to 0$, as $n \to \infty$. Similarly, $\|T^{pn+1}x - T^{p(n\pm1)+1}x\| \to 0$, as $n \to \infty$.

Theorem 3.10. *Let $X, A_1, \ldots, A_p, T$ be as in Definition 3.5. If for some i and for some $x \in A_i$, the sequence $\{T^{pn}x\}$ in A_i contains a convergent subsequence $\{T^{pn_j}x\}$ converging to $\xi \in A_i$, then ξ is a best proximity point in A_i.*

Proof.

$$\begin{aligned}
\operatorname{dist}(A_i, A_{i+1}) &\leq d(\xi, T\xi) \\
&= \lim_{j\to\infty} d\left(T^{pn_j}x, T\xi\right) \\
&\leq \lim_{j\to\infty} d\left(T^{pn_j-1}x, \xi\right) \\
&= \lim_{j\to\infty} d\left(T^{pn_j-1}x, T^{pn_j}x\right) \\
&= \operatorname{dist}(A_{i-1}, A_i) \\
&= \operatorname{dist}(A_i, A_{i+1}).
\end{aligned} \tag{3.5}$$

Therefore, $d(\xi, T\xi) = \operatorname{dist}(A_i, A_{i+1})$. □

Let X be a metric space. Let $A_1, \ldots, A_p$ be nonempty subsets of X, and let T be a p-cyclic map which satisfies the following condition. For given $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\epsilon \leq d(x, y) < \epsilon + \delta \text{ implies } d(Tx, Ty) < \epsilon \tag{3.6}$$

for all $x \in A_i,\ y \in A_{i+1},\ 1 \leq i \leq p$.

It follows from Lemma 2.2 that a p-cyclic map T satisfies the condition (3.6), if and only if there exists an L-function ϕ (nondecreasing and continuous) such that for all $x \in A_i$, $y \in A_{i+1}$ and for all i, $1 \leq i \leq p$, $d(x, y) > 0 \Rightarrow d(Tx, Ty) < \phi(d(x, y))$, $d(x, y) = 0 \Rightarrow d(Tx, Ty) = 0$, and T satisfies the p-cyclic nonexpansive property.

We use the following result due to Meir and Keeler [1] in the proof of Theorem 3.12.

Theorem 3.11. *Let X be a complete metric space, and let $T : X \to X$ be such that for given $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,*

$$\epsilon \leq d(x, y) < \epsilon + \delta \textit{ implies } d(Tx, Ty) < \epsilon. \tag{3.7}$$

Then, T has a unique fixed point $\xi \in X$. Moreover, for any $x \in X$, the sequence $\{T^n x\}$ converges to ξ.

Theorem 3.12. *Let X be a complete metric space. Let $A_1, \ldots, A_p$ be nonempty closed subsets of X. Let $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ be a p-cyclic map satisfying (3.6). Then, $\bigcap_{i=1}^p A_i$ is nonempty and for any $x \in A_i$, $1 \leq i \leq p$, the sequence $\{T^{pn}x\}$ converges to a unique fixed point in $\bigcap_{i=1}^p A_i$.*

Proof. Let $x \in A_i$. Let $s_n = d(T^n x, T^{n+1}x)$. If $s_n = 0$, for some n, then by the p-cyclic nonexpansive property of T, $\lim_n s_n = 0$. Therefore, assume $s_n > 0$ for all n. We note that the sequence $\{s_n\}$ is nonincreasing, and there exists an L-function ϕ such that $s_{n+1} < \phi(s_n)$, $s_n > 0$ and by Lemma 2.3, $\lim_n s_n = 0$. Now,

$$\begin{aligned}
d\left(T^{pn}x, T^{p(n+1)+1}x\right) &\leq \left\{d\left(T^{pn}x, T^{pn+1}x\right) + d\left(T^{pn+1}x, T^{pn+2}x\right) + \cdots + d\left(T^{pn+p}x, T^{(pn+1)+1}x\right)\right\} \\
&= s_{pn} + s_{pn+1} + \cdots + s_{pn+p} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{3.8}$$

Also, consider

$$d\left(T^{pn+1}x, T^{p(n+1)}x\right) \le \left\{d\left(T^{pn+1}x, T^{pn+2}x\right) + d\left(T^{pn+2}x, T^{pn+3}x\right) + \cdots + d\left(T^{pn+p-1}x, T^{pn+p}x\right)\right\}$$
$$= s_{pn+1} + s_{pn+2} + \cdots + s_{pn+p-1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.9}$$

Fix $\epsilon > 0$. By the definition of L-function, there exists $\delta \in (0, \epsilon)$ such that $\phi(\epsilon + \delta) \le \epsilon$. Choose an $n_0 \in \mathbb{N}$ satisfying

$$d\left(T^{p(n+1)+1}x, T^{pn}x\right) < \frac{\delta}{3}, \quad \forall n \ge n_0, \tag{3.10}$$

$$d\left(T^{pn}x, T^{pn+1}x\right) < \frac{\delta}{3}, \quad \forall n \ge n_0, \tag{3.11}$$

$$d\left(T^{p(n+1)}x, T^{pn+1}x\right) < \frac{\delta}{3}, \quad \forall n \ge n_0. \tag{3.12}$$

Let us show that

$$d\left(T^{pm}x, T^{pn+1}x\right) < \epsilon + \delta < 2\epsilon, \quad \forall m > n \ge n_0. \tag{3.13}$$

Let us do this by the method of induction. From (3.12), it is clear that (3.13) holds for $m = n+1$. Fix $n \ge n_0$. Assume that (3.7) is true for $m > n$. Now,

$$\begin{aligned}
d\left(T^{p(m+1)}x, T^{pn+1}x\right) &\le d\left(T^{p(m+1)}x, T^{p(n+1)+1}x\right) + d\left(T^{p(n+1)+1}x, T^{pn}x\right) + d\left(T^{pn}x, T^{pn+1}x\right) \\
&< \phi\left(d\left(T^{pm}x, T^{pn+1}x\right)\right) + \left(\frac{\delta}{3}\right) + \left(\frac{\delta}{3}\right), \text{ by (3.11) and (3.12)} \\
&< \phi(\epsilon + \delta) + \left(\frac{2}{3}\right)\delta \\
&\le \epsilon + \left(\frac{2}{3}\right)\delta \\
&< \epsilon + \delta \\
&< 2\epsilon.
\end{aligned} \tag{3.14}$$

By induction, (3.13) holds for all $m > n \ge n_0$. Now, for all $m > n > n_0$,

$$\begin{aligned}
d\left(T^{pn}x, T^{pm}x\right) &\le d\left(T^{pn}x, T^{pn+1}x\right) + d\left(T^{pn+1}x, T^{pm}x\right) \\
&< \left(\frac{\delta}{3}\right) + \epsilon + \delta \\
&< 3\epsilon.
\end{aligned} \tag{3.15}$$

Therefore, $\{T^{pn}x\}$ is a Cauchy sequence and converges to a point $z \in \bigcup_{i=1}^{p} A_i$. Consider

$$\begin{aligned} d(z, Tz) &= \lim_n d\left(T^{pn}x, Tz\right) \\ &\le \lim_n d\left(T^{pn-1}x, z\right) \\ &= \lim_n d\left(T^{pn-1}x, T^{pn}x\right) \\ &= \lim_n s_{pn-1} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.16}$$

Therefore, $z = Tz$. Since $T^j z = z$ for all j, $1 \le j \le p$, and since $T(A_i) \subseteq A_{i+1}$, $z \in A_i$ for all i, $1 \le i \le p$. Therefore, $z \in \bigcap_{i=1}^{p} A_i$ is a fixed point. Let $A = \bigcap_{i=1}^{p} A_i$. Restricting $T : A \to A$, we see that T is a Meir-Keeler contraction on the complete metric space A. Hence, by Theorem 3.11, z is the unique fixed point in A. □

Now, we prove our main result.

Theorem 3.13. *Let $A_1, \dots, A_p$ be nonempty, closed, and convex subsets of a uniformly convex Banach space. Let $T : \bigcup_{i=1}^{p} A_i \to \bigcup_{i=1}^{p} A_i$ be a p-cyclic Meir-Keeler contraction. Then, for each i, $1 \le i \le p$, there exists a unique $z_i \in A_i$ such that for any $x \in A_i$, the sequence $\{T^{pn}x\}$ converges to $z_i \in A_i$, which is a best proximity point in A_i. Moreover, z_i is a periodic point of period p, and $T^j z_i$ is a best proximity point in A_{i+j} for $j = 1, 2, \dots, p-1$.*

Proof. If $\operatorname{dist}(A_i, A_{i+1}) = 0$ for some i, then $\operatorname{dist}(A_i, A_{i+1}) = 0$ for all i, and hence, $\bigcap_{i=1}^{p} A_i$ is nonempty. In this case, T has a unique fixed point in the intersection. Therefore, assume $\operatorname{dist}(A_i, A_{i+1}) > 0$ for all i. Let $x \in A_i$. There exists an L-function ϕ as given in Remark 3.6. Fix $\epsilon > 0$. Choose $\delta \in (0, \epsilon)$ satisfying $\phi(\epsilon + \delta) \le \epsilon$. By Remark 3.9, $\lim_n \|T^{pn+1}x - T^{p(n+1)+1}x\| = 0$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\left\|T^{pn+1}x - T^{p(n+1)+1}x\right\| < \delta. \tag{3.17}$$

Let us prove that

$$\left\|T^{pn+1}x - T^{pm}x\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right) < \epsilon + \delta < 2\epsilon, \quad \forall m \ge n \ge n_0. \tag{3.18}$$

Fix $n \ge n_0$. It is clear that (3.18) is true for $m = n$. Assume that (3.18) is true for $m \ge n$. Now,

$$\begin{aligned} \left\|T^{pn+1}x - T^{p(m+1)}x\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right) &\le \Big\{\left\|T^{pn+1}x - T^{p(n+1)+1}x\right\| + \left\|T^{p(n+1)+1}x - T^{p(m+1)}x\right\| \\ &\qquad - \operatorname{dist}\left(A_i, A_{i+1}\right)\Big\} \\ &< \delta + \phi\left(\left\|T^{pn+1}x - T^{pm}x\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right)\right) \\ &< \delta + \phi(\epsilon + \delta) \\ &\le \delta + \epsilon < 2\epsilon. \end{aligned} \tag{3.19}$$

Hence, (3.18) holds for $m+1$. Therefore, by induction, (3.18) is true for all $m \geq n \geq n_0$. Note that $\lim_n \|T^{pn}x - T^{pn+1}x\| = \operatorname{dist}(A_i, A_{i+1})$. Now, by Lemma 2.4, for every $\epsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that for every $m > n \geq n_1$, $\|T^{pn}x - T^{pm}x\| < \epsilon$. Hence, $\{T^{pn}x\}$ is a Cauchy sequence and converges to $z \in A_i$. By Theorem 3.10, z is a best proximity point in A_i. That is, $\|z - Tz\| = \operatorname{dist}(A_i, A_{i+1})$. Let $y \in A_i$ such that $y \neq x$ and such that $\{T^{pn}y\} \to z_1$. Then, by Theorem 3.10, z_1 is a best proximity point. That is, $\|z_1 - Tz_1\| = \operatorname{dist}(A_i, A_{i+1})$. Let us show that $z_1 = z$. To do this,

$$\begin{aligned}\left\|z - T^{p+1}z\right\| &= \lim_n \left\|T^{pn}x - T^{p+1}z\right\| \\ &\leq \lim_n \left\|T^{p(n-1)}x - Tz\right\| \\ &= \|z - Tz\| \\ &= \operatorname{dist}\left(A_i, A_{i+1}\right).\end{aligned} \tag{3.20}$$

Since A_{i+1} is a convex set and X is a uniformly convex Banach space, $Tz = T^{p+1}z$. Similarly, we can prove that $Tz_1 = T^{p+1}z_1$. Now,

$$\left\|T^p z - Tz\right\| = \left\|T^p z - T^{p+1}z\right\| \leq \|z - Tz\| = \operatorname{dist}\left(A_i, A_{i+1}\right). \tag{3.21}$$

Since A_i is convex, $T^p z = z$. Now, $\|z - Tz_1\| = \|T^p z - T^{p+1}z_1\|$. If $\|z - Tz_1\| \leq \operatorname{dist}(A_i, A_{i+1})$, then there is nothing to prove. Therefore, let $\|z - Tz_1\| - \operatorname{dist}(A_i, A_{i+1}) > 0$. This implies that

$$\begin{aligned}\left\|Tz - T^2 z_1\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right) &< \phi\left(\left\|z - Tz_1\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right)\right) \\ &\leq \left\|z - Tz_1\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right) \\ &= \left\|T^p z - T^{p+1}z_1\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right) \\ &\leq \left\|Tz - T^2 z_1\right\| - \operatorname{dist}\left(A_i, A_{i+1}\right).\end{aligned} \tag{3.22}$$

Thus, $\|Tz - T^2 z_1\| < \|Tz - T^2 z_1\|$ a contradiction. Hence, $\|z - Tz_1\| = \operatorname{dist}(A_i, A_{i+1})$. Since $\|z_1 - Tz_1\| = \operatorname{dist}(A_i, A_{i+1})$ and A_i is convex, $z_1 = z$. □

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