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Research Article

Strong Convergence of Monotone Hybrid Method for Maximal Monotone Operators and Hemirelatively Nonexpansive Mappings

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We prove strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a hemirelatively nonexpansive mapping in a Banach space by using monotone hybrid iteration method. By using these results, we obtain new convergence results for resolvents of maximal monotone operators and hemirelatively nonexpansive mappings in a Banach space.

1. Introduction

Let E be a real Banach space and let E^* be the dual space of E. Let A be a maximal monotone operator from E to E^*. It is well known that many problems in nonlinear analysis and optimization can be formulated as follows. Find a point $u \in E$ satisfying

$$0 \in Au. \tag{1.1}$$

We denote by $A^{-1}0$ the set of all points $u \in C$ such that $0 \in Au$. Such a problem contains numerous problems in economics, optimization, and physics and is connected with a variational inequality problem. It is well known that the variational inequalities are equivalent to the fixed point problems. There are many authors who studied the problem of finding a common element of the fixed point of nonlinear mappings and the set of solutions of a variational inequality in the framework of Hilbert spaces see; for instance, [1–11] and the reference therein.

A well-known method to solve problem (1.1) is called the *proximal point algorithm*: $x_0 \in E$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 0, 1, 2, 3, \ldots, \tag{1.2}$$

where $\{r_n\} \subset (0, \infty)$ and J_{r_n} are the resovents of A. Many researchers have studies this algorithm in a Hilbert space; see, for instance, [12–15] and in a Banach space; see, for instance, [16, 17].

In 2005, Matsushita and Takahashi [18] proposed the following hybrid iteration method (it is also called the CQ method) with generalized projection for relatively nonexpansive mapping T in a Banach space E: $x_0 = x \in C$ chosen arbitrarily,

$$\begin{aligned} u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x, \end{aligned} \tag{1.3}$$

where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$. They proved that $\{x_n\}$ generated by (1.3) converges strongly to a fixed point of T under condition that $\limsup_{n\to\infty} \alpha_n < 1$.

In 2008, Su et al. [19] modified the CQ method (1.3) for approximation a fixed point of a closed hemi-relatively nonexpansive mapping in a Banach space. Their method is known as the monotone hybrid method defined as the following. $x_0 = x \in C$ chosen arbitrarily, then

$$\begin{aligned} x_1 &= x \in C, \quad C_{-1} = Q_{-1} = C, \\ u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \ge 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x, \end{aligned} \tag{1.4}$$

where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$. They proved that $\{x_n\}$ generated by (1.4) converges strongly to a fixed point of T under condition that $\limsup_{n\to\infty} \alpha_n < 1$.

Note that the hybrid method iteration method presented by Matsushita and Takahashi [18] can be used for relatively nonexpansive mapping, but it cannot be used for hemi-relatively nonexpansive mapping.

Very recently, Inoue et al. [20] proved the following strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a relatively nonexpansive mapping by using the hybrid method.

Theorem 1.1 (Inoue et al. [20]). *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a monotone operator satisfying*

$D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T : C \to C$ be a relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$ and

$$\begin{aligned}
u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T J_{r_n} x_n), \\
C_n &= \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
Q_n &= \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x
\end{aligned} \tag{1.5}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0} x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection from C onto $F(T) \cap A^{-1}0$.

Employing the ideas of Inoue et al. [20] and Su et al. [19], we modify iterations (1.4) and (1.5) to obtain strong convergence theorems for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a hemi-relatively nonexpansive mapping in a Banach space. Using these results, we obtain new convergence results for resolvents of maximal monotone operators and hemi-relatively nonexpansive mappings in a Banach space. The results of this paper modify and improve the results of Inoue et al. [20], and some others.

2. Preliminaries

Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of all positive integers and real numbers, respectively. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the *weak* convergence of $\{x_n\}$ to x and the *strong* convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Let E be a Banach space. Then the duality mapping J from E into 2^{E^*} is defined by

$$Jx = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \qquad \forall x \in E. \tag{2.1}$$

Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.2}$$

exists for all $x, y \in S(E)$. It is also said to be *uniformly smooth* if the limit exists uniformly in $x, y \in S(E)$. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|(x+y)/2\| < 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$. We know the following (see, [21]):

(i) if E in smooth, then J is single valued;

(ii) if E is reflexive, then J is onto;

(iii) if E is strictly convex, then J is one to one;

(iv) if E is strictly convex, then J is strictly monotone;

(v) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth strictly convex and reflexive Banach space and let C be a closed convex subset of E. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \quad \forall y, x \in E. \tag{2.3}$$

Observe that, in a Hilbert space H, (2.3) reduces to $\phi(x,y) = \|x-y\|^2$, for all $x, y \in H$. It is obvious from the definition of the function ϕ that for all $x, y \in E$,

(1) $(\|x\| - \|y\|)^2 \le \phi(x,y) \le (\|x\| + \|y\|)^2$,

(2) $\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x-z, Jz - Jy\rangle$,

(3) $\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \le \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$.

Following Alber [22], the generalized projection Π_C from E onto C is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y,x)$; that is, $\Pi_C x = \overline{x}$, where $\overline{x}$ is the solution to the minimization problem

$$\phi(\overline{x}, x) = \min_{y \in C} \phi(y,x). \tag{2.4}$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y,x)$ and strict monotonicity of the mapping J. In a Hilbert space, Π_C is the metric projection of H onto C.

Let C be a closed convex subset of a Banach space E, and let T be a mapping from C into itself. We use $F(T)$ to denote the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$. Recall that a self-mapping $T : C \to C$ is *hemi-relatively nonexpansive* if $F(T) \neq \emptyset$ and $\phi(u, Tx) \le \phi(u, x)$ for all $x \in C$ and $u \in F(T)$.

A point $u \in C$ is said to be an *asymptotic* fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to u and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\widehat{F}(T)$. A hemi-relative nonexpansive mapping $T : C \to C$ is said to be *relatively nonexpansive* if $\widehat{F}(T) = F(T) \neq \emptyset$. The asymptotic behavior of a relatively nonexpansive mapping was studied in [23].

Recall that an operator T in a Banach space is call *closed*, if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

We need the following lemmas for the proof of our main results.

Lemma 2.1 (Kamimura and Takahashi [13]). *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.2 (Matsushita and Takahashi [18]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E and let T be a relatively hemi-nonexpansive mapping from C into itself. Then $F(T)$ is closed and convex.*

Lemma 2.3 (Alber [22], Kamimura and Takahashi [13]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space, $x \in E$ and let $z \in C$. Then, $z = \Pi_C x$ if and only if $\langle y - z, Jx - Jz \rangle \leq 0$ for all $y \in C$.*

Lemma 2.4 (Alber [22], Kamimura and Takahashi [13]). *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \quad \forall x \in C, y \in E. \tag{2.5}$$

Let E be a smooth, strictly convex, and reflexive Banach space, and let A be a set-valued mapping from E to E^* with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$, domain $D(A) = \{z \in E : Az \neq \emptyset\}$, and range $R(A) = \cup\{Az : z \in D(A)\}$. We denote a set-valued operator A from E to E^* by $A \subset E \times E^*$.A is said to be *monotone* of $\langle x - y, x^* - y^* \rangle \geq 0$, for all $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is said to be *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator. It is known that a monotone mapping A is maximal if and only if for $(x, x^*) \in E \times E^*$, $\langle x - y, x^* - y^* \rangle \geq 0$ for every $(y, y^*) \in G(A)$ implies that $x^* \in Ax$. We know that if A is a maximal monotone operator, then $A^{-1}0 = \{z \in D(A) : 0 \in Az\}$ is closed and convex; see [19] for more details. The following result is well known.

Lemma 2.5 (Rockafellar [24]). *Let E be a smooth, strictly convex, and reflexive Banach space and let $A \subset E \times E^*$ be a monotone operator. Then A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$.*

Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty closed convex subset of E, and let $A \subset E \times E^*$ be a monotone operator satisfying

$$D(A) \subset C \subset J^{-1}\left(\bigcap_{r>0} R(J + rA)\right). \tag{2.6}$$

Then we can define the resolvent $J_r : C \to D(A)$ by

$$J_r x = \{z \in D(A) : Jx \in Jz + rAz\}, \quad \forall x \in C. \tag{2.7}$$

We know that $J_r x$ consists of one point. For $r > 0$, the Yosida approximation $A_r : C \to E^*$ is defined by $A_r x = (Jx - JJ_r x)/r$ for all $x \in C$.

Lemma 2.6 (Kohsaka and Takahashi [25]). *Let E be a smooth, strictly convex, and reflexive Banach space, let C be a nonempty closed convex subset of E, and let $A \subset E \times E^*$ be a monotone operator satisfying*

$$D(A) \subset C \subset J^{-1}\left(\bigcap_{r>0} R(J + rA)\right). \tag{2.8}$$

Let $r > 0$ and let J_r and A_r be the resolvent and the Yosida approximation of A, respectively. Then, the following hold:

(i) $\phi(u, J_r x) + \phi(J_r x, x) \leq \phi(u, x)$, *for all* $x \in C$, $u \in A^{-1}0$;

(ii) $(J_r x, A_r x) \in A$, *for all* $x \in C$;

(iii) $F(J_r) = A^{-1}0$.

Lemma 2.7 (Kamimura and Takahashi [13]). *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y) \tag{2.9}$$

for all $x, y \in B_r(0)$, where $B_r(0) = \{z \in E : \|z\| \leq r\}$.

3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the fixed point set of a hemi-relatively nonexpansive mapping in a Banach space by using the monotone hybrid iteration method.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1} J$ for all $r > 0$. Let $T : C \to C$ be a closed hemi-relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
x_0 &= x \in C, \quad C_{-1} = Q_{-1} = C,\\
u_n &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T J_{r_n} x_n),\\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\},\\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\},\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x l
\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n\to\infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap A^{-1}0} x_0$, where $\Pi_{F(T)\cap A^{-1}0}$ is the generalized projection from C onto $F(T) \cap A^{-1}0$.

Proof. We first show that C_n and Q_n are closed and convex for each $n \geq 0$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \geq 0$. Next, we prove that C_n is convex.

Since

$$\phi(z, u_n) \leq \phi(z, x_n) \tag{3.2}$$

is equivalent to

$$0 \leq \|x_n\|^2 - \|u_n\|^2 - 2\langle z, Jx_n - Ju_n \rangle, \tag{3.3}$$

which is affine in z, and hence C_n is convex. So, $C_n \cap Q_n$ is a closed and convex subset of E for all $n \geq 0$. Let $u \in F(T) \cap A^{-1}0$. Put $y_n = J_{r_n} x_n$ for all $n \geq 0$. Since T and J_{r_n} are hemi-relatively nonexpansive mappings, we have

$$\begin{aligned}
\phi(u, u_n) &= \phi\Big(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JTy_n)\Big) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n) JTy_n \rangle + \big\|\alpha_n Jx_n + (1-\alpha_n) JTy_n\big\|^2 \\
&\leq \|u\|^2 - 2\alpha_n \langle u, Jx_n \rangle - 2(1-\alpha_n)\langle u, JTy_n \rangle + \alpha_n \|x_n\|^2 + (1-\alpha_n)\big\|Ty_n\big\|^2 \\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, Ty_n) \\
&\leq \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, y_n) \\
&= \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, J_{r_n} x_n) \\
&\leq \alpha_n \phi(u, x_n) + (1-\alpha_n)\phi(u, x_n) \\
&= \phi(u, x_n).
\end{aligned} \tag{3.4}$$

So, $u \in C_n$ for all $n \geq 0$, which implies that $F(T) \cap A^{-1}0 \subset C_n$. Next, we show that $F(T) \cap A^{-1}0 \subset Q_n$ for all $n \geq 0$. We prove that by induction. For $k = 0$, we have $F(T) \cap A^{-1}0 \subset C = Q_{-1}$. Assume that $F(T) \cap A^{-1}0 \subset Q_{k-1}$ for some $k \geq 0$. Because x_k is the projection of x_0 onto $C_{k-1} \cap Q_{k-1}$ by Lemma 2.3, we have

$$\langle x_k - z, Jx_0 - Jx_k \rangle \geq 0, \quad \forall z \in C_{k-1} \cap Q_{k-1}. \tag{3.5}$$

Since $F(T) \cap A^{-1}0 \subset C_{k-1} \cap Q_{k-1}$, we have

$$\langle x_k - z, Jx_0 - Jx_k \rangle \geq 0, \quad \forall z \in F(T) \cap A^{-1}0. \tag{3.6}$$

This together with definition of Q_n implies that $F(T) \cap A^{-1}0 \subset Q_k$ and hence $F(T) \cap A^{-1}0 \subset Q_n$ for all $n \geq 0$. So, we have that $F(T) \cap A^{-1}0 \subset C_n \cap Q_n$ for all $n \geq 0$. This implies that $\{x_n\}$ is well defined. From definition of Q_n we have $x_n = \Pi_{Q_n} x_0$. So, from $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n \cap Q_n \subset Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \quad \forall n \geq 0. \tag{3.7}$$

Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows from Lemma 2.4 and $x_n = \Pi_{Q_n} x_0$ that

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0) \tag{3.8}$$

for all $u \in F(T) \cap A^{-1} \subset Q_n$. Therefore, $\{\phi(x_n, x_0)\}$ is bounded. Moreover, by definition of ϕ, we know that $\{x_n\}$ and $\{J_{r_n} x_n\} = \{y_n\}$ are bounded. So, the limit of $\{\phi(x_n, x_0)\}$ exists. From $x_n = \Pi_{Q_n} x_0$, we have that for any positive integer,

$$\phi(x_{n+k}, x_n) = \phi\left(x_{n+k}, \Pi_{Q_n} x_0\right) \leq \phi(x_{n+k}, x_0) - \phi\left(\Pi_{Q_n} x_0, x_0\right) = \phi(x_{n+k}, x_0) - \phi(x_n, x_0). \quad (3.9)$$

This implies that $\lim_{n\to\infty} \phi(x_{n+k}, x_n) = 0$. Since $\{x_n\}$ is bounded, there exists $r > 0$ such that $\{x_n\} \subset B_r(0)$. Using Lemma 2.7, we have, for m, n with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0), \quad (3.10)$$

where $g : [0, 2r] \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. Then the properties of the function g yield that $\{x_n\}$ is a Cauchy sequence in C. So there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$ and definition of C_n, we also have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \quad (3.11)$$

It follows that $\lim_{n\to\infty} \phi(x_{n+1}, u_n) = \lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \quad (3.12)$$

So, we have $\lim_{n\to\infty} \|x_n - u_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Ju_n\| = \lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \quad (3.13)$$

On the other hand, we have

$$\begin{aligned}
\|Jx_{n+1} - Ju_n\| &= \left\|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n) JTy_n\right\| \\
&= \left\|\alpha_n (Jx_{n+1} - Jx_n) + (1-\alpha_n)\left(Jx_{n+1} - JTy_n\right)\right\| \\
&= \left\|(1-\alpha_n)\left(Jx_{n+1} - JTy_n\right) - \alpha_n (Jx_n - Jx_{n+1})\right\| \\
&\geq (1-\alpha_n)\left\|Jx_{n+1} - JTy_n\right\| - \alpha_n \|Jx_n - Jx_{n+1}\|.
\end{aligned} \quad (3.14)$$

This follows

$$\left\|Jx_{n+1} - JTy_n\right\| \leq \frac{1}{1-\alpha_n}\left(\|Jx_{n+1} - Ju_n\| + \alpha_n \|Jx_n - Jx_{n+1}\|\right). \quad (3.15)$$

From (3.13) and $\liminf_{n\to\infty} (1-\alpha_n) > 0$, we obtain that $\lim_{n\to\infty} \|Jx_{n+1} - JTy_n\| = 0$.

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \left\| x_{n+1} - Ty_n \right\| = 0. \tag{3.16}$$

From

$$\left\| x_n - Ty_n \right\| \le \|x_n - x_{n+1}\| + \left\| x_{n+1} - Ty_n \right\|, \tag{3.17}$$

we have

$$\lim_{n\to\infty} \left\| x_n - Ty_n \right\| = 0. \tag{3.18}$$

From (3.4), we have

$$\phi(u, y_n) \ge \frac{1}{1-\alpha_n} \left(\phi(u, u_n) - \alpha_n \phi(u, x_n)\right). \tag{3.19}$$

Using $y_n = J_{r_n} x_n$ and Lemma 2.6, we have

$$\phi(y_n, x_n) = \phi(J_{r_n} x_n, x_n) \le \phi(u, x_n) - \phi(u, J_{r_n} x_n) = \phi(u, x_n) - \phi(u, y_n). \tag{3.20}$$

It follows that

$$\begin{aligned}
\phi(y_n, x_n) &\le \phi(u, x_n) - \phi(u, y_n) \\
&\le \phi(u, x_n) - \frac{1}{1-\alpha_n}\left(\phi(u, u_n) - \alpha_n \phi(u, x_n)\right) \\
&= \frac{1}{1-\alpha_n}\left(\phi(u, x_n) - \phi(u, u_n)\right) \\
&= \frac{1}{1-\alpha_n}\left(\|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle\right) \\
&\le \frac{1}{1-\alpha_n}\left(\left|\|x_n\|^2 - \|u_n\|^2\right| + 2|\langle u, Jx_n - Ju_n\rangle|\right) \\
&\le \frac{1}{1-\alpha_n}\left(|\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|\right) \\
&\le \frac{1}{1-\alpha_n}\left(\|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Jx_n - Ju_n\|\right).
\end{aligned} \tag{3.21}$$

From (3.13) and $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we have $\lim_{n\to\infty}\phi(y_n, x_n) = 0$.

Since E is uniformly convex and smooth, we have from Lemma 2.1 that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.22}$$

From $\lim_{n\to\infty}\|x_n - Ty_n\| = 0$, we have

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0. \tag{3.23}$$

Since $x_n \to w$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$, we have $y_n \to w$. Since T is a closed operator and $y_n \to w$, w is a fixed point of T. Next, we show $w \in A^{-1}0$. Since J is uniformly norm-to-norm continuous on bounded sets, from (3.22) we have

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.24}$$

From $r_n \geq a$, we have

$$\lim_{n\to\infty}\frac{1}{r_n}\|Jx_n - Jy_n\| = 0. \tag{3.25}$$

Therefore, we have

$$\lim_{n\to\infty}\|A_{r_n}x_n\| = \lim_{n\to\infty}\frac{1}{r_n}\|Jx_n - Jy_n\| = 0. \tag{3.26}$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - y_n, p^* - A_{r_n}x_n\rangle \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we get $\langle p - w, p^*\rangle \geq 0$. From the maximality of A, we have $w \in A^{-1}0$. Finally, we prove that $w = \Pi_{F(T)\cap A^{-1}0}x_0$. From Lemma 2.4, we have

$$\phi\left(w, \Pi_{F(T)\cap A^{-1}0}x_0\right) + \phi\left(\Pi_{F(T)\cap A^{-1}0}x_0, x_0\right) \leq \phi(w, x_0). \tag{3.27}$$

Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0$ and $w \in F(T)\cap A^{-1}0 \subset C_n \cap Q_n$, we get from Lemma 2.4 that

$$\phi\left(\Pi_{F(T)\cap A^{-1}0}x_0, x_{n+1}\right) + \phi(x_{n+1}, x_0) \leq \phi\left(\Pi_{F(T)\cap A^{-1}0}x_0, x_0\right). \tag{3.28}$$

By the definition of ϕ, it follows that $\phi(w, x_0) \leq \phi(\Pi_{F(T)\cap A^{-1}0}x_0, x_0)$ and $\phi(w, x_0) \geq \phi(\Pi_{F(T)\cap A^{-1}0}x_0, x_0)$, whence $\phi(w, x_0) = \phi(\Pi_{F(T)\cap A^{-1}0}x_0, x_0)$. Therefore, it follows from the uniqueness of the $\Pi_{F(T)\cap A^{-1}0}x_0$ that $w = \Pi_{F(T)\cap A^{-1}0}x_0$. □

As direct consequences of Theorem 3.1, we can obtain the following corollaries.

Corollary 3.2. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in E$ and*

$$\begin{aligned}
u_n &= J_{r_n} x_n, \\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x
\end{aligned} \tag{3.29}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$, and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. Then $\{x_n\}$ converges strongly to $\Pi_{A^{-1}0} x_0$, where $\Pi_{A^{-1}0}$ is the generalized projection from C onto $A^{-1}0$.

Proof. Putting $T = I$, $C = E$, and $\alpha_n = 0$ in Theorem 3.1, we obtain Corollary 3.2. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E : f(y) \geq \langle y - x, x^* \rangle + f(x),\ \forall y \in E\} \tag{3.30}$$

for each $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [21] for more details.

Corollary 3.3 (Su et al. [19, Theorem 3.1]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a closed hemi-relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
x_0 &= x \in C, \quad C_{-1} = Q_{-1} = C, \\
u_n &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JT x_n), \\
C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\
Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x
\end{aligned} \tag{3.31}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Set $A = \partial i_C$ in Theorem 3.1, where i_C is the indicator function; that is,

$$i_C = \begin{cases} 0, & x \in C, \\ \infty, & \text{otherwise.} \end{cases} \tag{3.32}$$

Then, we have that A is a maximal monotone operator and $J_r = \Pi_C$ for $r > 0$, in fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.3 that

$$\begin{aligned} z = J_r x &\Longleftrightarrow Jz + r\partial i_C(z) \ni Jx \\ &\Longleftrightarrow Jx - Jz \in r\partial i_C(z) \\ &\Longleftrightarrow i_C(y) \geq \left\langle y - z, \frac{Jx - Jz}{r} \right\rangle + i_C(z), \quad \forall y \in E \\ &\Longleftrightarrow 0 \geq \langle y - z, Jx - Jz \rangle, \quad \forall y \in C \\ &\Longleftrightarrow z = \arg \min_{y \in C} \phi(y, x) \\ &\Longleftrightarrow z = \Pi_C x. \end{aligned} \tag{3.33}$$

So, we obtain the desired result by using Theorem 3.1. □

Since every relatively nonexpansive mapping is a hemi-relatively one, the following theorem is obtained directly from Theorem 3.1.

Theorem 3.4. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed convex subset of E. Let $A \subset E \times E^*$ be a monotone operator satisfying $D(A) \subset C$ and let $J_r = (J + rA)^{-1}J$ for all $r > 0$. Let $T : C \to C$ be a closed relatively nonexpansive mapping such that $F(T) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_0 &= x \in C, \quad C_{-1} = Q_{-1} = C, \\ u_n &= J^{-1}(\alpha_n Jx_n + (1 - \alpha_n) JTJ_{r_n} x_n), \\ C_n &= \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n &= \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0\}, \\ x_{n+1} &= \Pi_{C_n \cap Q_n} x \end{aligned} \tag{3.34}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E, $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $\liminf_{n \to \infty}(1 - \alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T) \cap A^{-1}0} x_0$, where $\Pi_{F(T) \cap A^{-1}0}$ is the generalized projection from C onto $F(T) \cap A^{-1}0$.

Corollary 3.5 (Su et al. [19, Theorem 3.2]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, and let T be a closed relatively nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned}
x_0 &= x \in C, \quad C_{-1} = Q_{-1} = C, \\
u_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\
C_n &= \left\{ z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n) \right\}, \\
Q_n &= \{ z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx - Jx_n \rangle \geq 0 \}, \\
x_{n+1} &= \Pi_{C_n \cap Q_n} x
\end{aligned} \tag{3.35}$$

for all $n \in \mathbb{N} \cup \{0\}$, where J is the duality mapping on E and $\{\alpha_n\} \subset [0,1]$. If $\liminf_{n\to\infty}(1-\alpha_n) > 0$, then $\{x_n\}$ converges strongly to $\Pi_{F(T)} x_0$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

Proof. Set $A = \partial i_C$ in Theorem 3.4, where i_C is the indicator function. So, from Theorem 3.4, we obtain the desired result. □

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References

[1] L.-C. Ceng, C. Lee, and J.-C. Yao, "Strong weak convergence theorems of implicit hybrid steepest-descent methods for variational inequalities," *Taiwanese Journal of Mathematics*, vol. 12, no. 1, pp. 227–244, 2008.

[2] L.-C. Ceng, Q. H. Ansari, and J.-C. Yao, "Viscosity approximation methods for generalized equilibrium problems and fixed point problems," *Journal of Global Optimization*, vol. 43, no. 4, pp. 487–502, 2009.

[3] J.-W. Peng and J.-C. Yao, "A new hybrid-extragradient method for generalized mixed equilibrium problems, fixed point problems and variational inequality problems," *Taiwanese Journal of Mathematics*, vol. 12, no. 6, pp. 1401–1432, 2008.

[4] J.-W. Peng and J.-C. Yao, "Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1816–1828, 2009.

[5] J.-W. Peng and J. C. Yao, "Some new iterative algorithms for generalized mixed equilibrium problems with strict pseudo-contractions and monotone mappings," to appear in *Taiwanese Journal of Mathematics*.

[6] J.-W. Peng and J.-C. Yao, "Some new extragradient-like methods for generalized equilibrium problems, fixed point problems and variational inequality problems," to appear in *Optimization Methods and Software*.

[7] J.-W. Peng and J.-C. Yao, "Ishikawa iterative algorithms for a generalized equilibrium problem and fixed point problems of a pseudo-contraction mapping," *Journal of Global Optimization*. In press.

[8] S. Schaible, J.-C. Yao, and L.-C. Zeng, "A proximal method for pseudomonotone type variational-like inequalities," *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 497–513, 2006.

[9] L.-C. Zeng and J.-C. Yao, “Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems,” *Taiwanese Journal of Mathematics*, vol. 10, no. 5, pp. 1293–1303, 2006.
[10] L. C. Zeng, L. J. Lin, and J. C. Yao, “Auxiliary problem method for mixed variational-like inequalities,” *Taiwanese Journal of Mathematics*, vol. 10, no. 2, pp. 515–529, 2006.
[11] L.-C. Zeng, S.-Y. Wu, and J.-C. Yao, “Generalized KKM theorem with applications to generalized minimax inequalities and generalized equilibrium problems,” *Taiwanese Journal of Mathematics*, vol. 10, no. 6, pp. 1497–1514, 2006.
[12] S. Kamimura and W. Takahashi, “Approximating solutions of maximal monotone operators in Hilbert spaces,” *Journal of Approximation Theory*, vol. 106, no. 2, pp. 226–240, 2000.
[13] S. Kamimura and W. Takahashi, “Strong convergence of a proximal-type algorithm in a Banach space,” *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.
[14] R. T. Rockafellar, “Monotone operators and the proximal point algorithm,” *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.
[15] M. V. Solodov and B. F. Svaiter, “Forcing strong convergence of proximal point iterations in a Hilbert space,” *Mathematical Programming*, vol. 87, no. 1, pp. 189–202, 2000.
[16] S. Kamimura, F. Kohsaka, and W. Takahashi, “Weak and strong convergence theorems for maximal monotone operators in a Banach space,” *Set-Valued Analysis*, vol. 12, no. 4, pp. 417–429, 2004.
[17] F. Kohsaka and W. Takahashi, “Strong convergence of an iterative sequence for maximal monotone operators in a Banach space,” *Abstract and Applied Analysis*, vol. 2004, no. 3, pp. 239–249, 2004.
[18] S. Matsushita and W. Takahashi, “A strong convergence theorem for relatively nonexpansive mappings in a Banach space,” *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.
[19] Y. Su, D. Wang, and M. Shang, “Strong convergence of monotone hybrid algorithm for hemi-relatively nonexpansive mappings,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 284613, 8 pages, 2008.
[20] G. Inoue, W. Takahashi, and K. Zembayashi, “Strong convergence theorems by hybrid methods for maximal monotone operator and relatively nonexpansive mappings in Banach spaces,” to appear in *Journal of Convex Analysis*.
[21] W. Takahashi, *Nonlinear Functional Analysis*, Fixed Point Theory and Its Application, Yokohama Publishers, Yokohama, Japan, 2000.
[22] Y. I. Alber, “Metric and generalized projection operators in Banach spaces: properties and applications,” in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, A. G. Katrosatos, Ed., vol. 178 of *Lecture Notes in Pure and Appl. Math.*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.
[23] D. Butnariu, S. Reich, and A. J. Zaslavski, “Asymptotic behavior of relatively nonexpansive operators in Banach spaces,” *Journal of Applied Analysis*, vol. 7, no. 2, pp. 151–174, 2001.
[24] R. T. Rockafellar, “On the maximality of sums of nonlinear monotone operators,” *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.
[25] F. Kohsaka and W. Takahashi, “Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces,” *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 824–835, 2008.