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*Research Article*

# Fixed Points of Generalized Contractive Maps

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We prove some results on the existence of fixed points for multivalued generalized $w$-contractive maps not involving the extended Hausdorff metric. Consequently, several known fixed point results are either generalized or improved.



## 1. Introduction

Throughout this paper, unless otherwise specified, $X$ is a metric space with metric $d$. Let $2^X, Cl(X)$, and $CB(X)$ denote the collection of nonempty subsets of $X$, nonempty closed subsets of $X$, and nonempty closed bounded subsets of $X$, respectively. Let $H$ be the Hausdorff metric on $CB(X)$, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad A,B\in CB(X). \tag{1.1}$$

A multivalued map $T : X \to CB(X)$ is called

(i) *contraction* [1] if for a fixed constant $h \in (0,1)$ and for each $x, y \in X$,

$$H(T(x),T(y)) \le hd(x,y); \tag{1.2}$$

(ii) *generalized contraction* [2] if for any $x, y \in X$,

$$H(T(x),T(y)) \le k(d(x,y))d(x,y), \tag{1.3}$$

where $k$ is a function from $[0,\infty)$ to $[0,1)$ with $\limsup_{r\to t^+} k(r) < 1$, for every $t \in [0,\infty)$;

(iii) *contractive* [3] if there exist constants $b, h \in (0,1)$, $h < b$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \leq hd(x,y), \tag{1.4}$$

where $I_b^x = \{y \in T(x) : bd(x,y) \leq d(x,T(x))\}$;

(iv) *generalized contractive* [4] if there exist $b \in (0,1)$ such that for any $x \in X$ there is $y \in I_b^x$ satisfying

$$d(y,T(y)) \leq k(d(x,y))d(x,y), \tag{1.5}$$

where $k$ is a function from $[0,\infty)$ to $[0,b)$ with $\limsup_{r\to t^+} k(r) < b$, for every $t \in [0,\infty)$.

An element $x \in X$ is called a *fixed point* of a multivalued map $T : X \to 2^X$ if $x \in T(x)$. We denote $\mathrm{Fix}(T) = \{x \in X : x \in T(x)\}$.

A sequence $\{x_n\}$ in $X$ is called an *orbit* of $T$ at $x_0 \in X$ if $x_n \in T(x_{n-1})$ for all $n \geq 1$. A map $f : X \to \mathbb{R}$ is called *lower semicontinuous* if for any sequence $\{x_n\} \subset X$ with $x_n \to x \in X$ imply that $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

Using the concept of Hausdorff metric, Nadler Jr. [1] established the following fixed point result for multivalued contraction maps which in turn is a generalization of the well-known Banach contraction principle.

**Theorem 1.1** (see [1]). *Let $X$ be a complete space and let $T : X \to CB(X)$ be a contraction map. Then* $\mathrm{Fix}(T) \neq \emptyset$.

This result has been generalized in many directions. For instance, Mizoguchi and Takahashi [2] have obtained the following general form of the Nadler's theorem.

**Theorem 1.2** (see [2]). *Let $X$ be a complete space and let $T : X \to CB(X)$ be a generalized contraction map. Then* $\mathrm{Fix}(T) \neq \emptyset$.

Another extension of Nadler's result obtained recently by Feng and Liu [3]. Without using the concept of the Hausdorff metric, they proved the following result.

**Theorem 1.3** (see [3]). *Let $X$ be a complete space and let $T : X \to Cl(X)$ be a multivalued contractive map. Suppose that a real-valued function $g$ on $X$, $g(x) = d(x,T(x))$, is lower semicontinuous. Then* $\mathrm{Fix}(T) \neq \emptyset$.

Most recently, Klim and Wardowski [4] generalized Theorem 1.3 as follows:

**Theorem 1.4** (see [4]). *Let $X$ be a complete metric space and let $T : X \to Cl(X)$ be a multivalued generalized contractive map such that a real-valued function $g$ on $X$, $g(x) = d(x,T(x))$ is lower semicontinuous. Then* $\mathrm{Fix}(T) \neq \emptyset$.

Recently, Kada et al. [5] introduced the concept of $w$-distance on a metric space as follows.

A function $\omega : X \times X \to [0, \infty)$ is called *$w$-distance* on $X$ if it satisfies the following for any $x, y, z \in X$:

($w_1$) $\omega(x,z) \leq \omega(x,y) + \omega(y,z)$;

($w_2$) a map $\omega(x,\cdot) : X \to 0, \infty)$ is lower semicontinuous;

($w_3$) for any $\epsilon > 0$, there exists $\delta > 0$ such that $\omega(z,x) \leq \delta$ and $\omega(z,y) \leq \delta$ imply $d(x,y) \leq \epsilon$.

Using the concept of $w$-distance, they improved Caristi's fixed point theorem, Ekland's variational principle, and Takahashi's existence theorem. In [6], Susuki and Takahashi proved a fixed point theorem for contractive type multivalued maps with respect to $w$-distance. See also [7–12].

Let us give some examples of $w$-distance [5].

(a) The metric $d$ is a $w$-distance on $X$.

(b) Let $X$ be normed space with norm $\|\cdot\|$. Then the functions $\omega_1, \omega_2 : X \times X \to [0, \infty)$ defined by $\omega_1(x,y) = \|x\| + \|y\|$ and $\omega_2(x,y) = \|y\|$ for every $x, y \in X$, are $w$-distance.

The following lemmas concerning $w$-distance are crucial for the proofs of our results.

**Lemma 1.5** (see [5]). *Let $\{x_n\}$ and $\{y_n\}$ be sequences in $X$ and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0,\infty)$ converging to $0$. Then, for the $w$-distance $\omega$ on $X$ the following hold for every $x, y, z \in X$:*

(a) *if $\omega(x_n, y) \leq \alpha_n$ and $\omega(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $y = z$; in particular, if $\omega(x,y) = 0$ and $\omega(x,z) = 0$, then $y = z$;*

(b) *if $\omega(x_n, y_n) \leq \alpha_n$ and $\omega(x_n, z) \leq \beta_n$ for any $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$;*

(c) *if $\omega(x_n, x_m) \leq \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*

(d) *if $\omega(y, x_n) \leq \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

**Lemma 1.6** (see [9]). *Let $K$ be a closed subset of $X$ and let $\omega$ be a $w$-distance on $X$. Suppose that there exists $u \in X$ such that $\omega(u,u) = 0$. Then $\omega(u,K) = 0 \Leftrightarrow u \in K$. (where $\omega(u,K) = \inf_{y\in K}\omega(u,y)$.)*

We say a multivalued map $T : X \to 2^X$ is *generalized $w$-contractive* if there exist a $w$-distance $\omega$ on $X$ and a constant $b \in (0,1)$ such that for any $x \in X$ there is $y \in J_b^x$ satisfying

$$\omega(y, T(y)) \leq k(\omega(x,y))\omega(x,y), \tag{1.6}$$

where $J_b^x = \{y \in T(x) : b\omega(x,y) \leq \omega(x, T(x))\}$ and $k$ is a function from $[0,\infty)$ to $[0,b)$ with $\limsup_{r\to t^+} k(r) < b$, for every $t \in [0,\infty)$.

Note that if we take $\omega = d$, then the definition of generalized $w$-contractive map reduces to the definition of generalized contractive map due to Klim and Wardowski [4]. In particular, if we take a constant map $k = h < b$, $h \in (0,1)$ then the map $T$ is weakly contractive (in short, $w$-contractive) [8], and further if we take $\omega = d$, then we obtain $J_b^x = I_b^x$ and $T$ is contractive [3].

In this paper, using the concept of $w$-distance, we first establish key lemma and then obtain fixed point results for multivalued generalized $w$-contractive maps not involving the extended Hausdorff metric. Our results either generalize or improve a number of fixed point results including the corresponding results of Feng and Liu [3], Latif and Albar [8], and Klim and Wardowski [4].

## 2. Results

First, we prove key lemma in the setting of metric spaces.

**Lemma 2.1.** *Let $T : X \to Cl(X)$ be a generalized $w$-contractive map. Then, there exists an orbit $\{x_n\}$ of $T$ in $X$ such that the sequence of nonnegative real numbers $\{\omega(x_n, T(x_n))\}$ is decreasing to zero and the sequence $\{x_n\}$ is Cauchy.*

*Proof.* Since for each $x \in X$, $T(x)$ is closed, the set $J_b^x$ is nonempty for any $b \in (0,1)$. Let $x_o$ be an arbitrary but fixed element of $X$. Since $T$ is generalized $w$-contractive, there is $x_1 \in J_b^{x_o} \subseteq T(x_o)$ such that

$$\omega(x_1, T(x_1)) \leq k(\omega(x_0, x_1))\omega(x_0, x_1), \quad k(\omega(x_0, x_1)) < b, \tag{2.1}$$

$$b\omega(x_0, x_1) \leq \omega(x_0, T(x_0)). \tag{2.2}$$

Using (2.1) and (2.2), we have

$$\begin{aligned} \omega(x_0, T(x_0)) - \omega(x_1, T(x_1)) &\geq b\omega(x_0, x_1) - k(\omega(x_0, x_1))\omega(x_0, x_1) \\ &= [b - k(\omega(x_0, x_1))]\omega(x_0, x_1) > 0. \end{aligned} \tag{2.3}$$

Similarly, there is $x_2 \in J_b^{x_1} \subseteq T(x_1)$ such that

$$\omega(x_2, T(x_2)) \leq k(\omega(x_1, x_2))\omega(x_1, x_2), \quad k(\omega(x_1, x_2)) < b, \tag{2.4}$$

$$b\omega(x_1, x_2) \leq \omega(x_1, T(x_1)). \tag{2.5}$$

Using (2.4) and (2.5), we have

$$\begin{aligned} \omega(x_1, T(x_1)) - \omega(x_2, T(x_2)) &\geq b\omega(x_1, x_2) - k(\omega(x_1, x_2))\omega(x_1, x_2) \\ &= [b - k(\omega(x_1, x_2))]\omega(x_1, x_2) > 0. \end{aligned} \tag{2.6}$$

From (2.5) and (2.1), it follows that

$$\omega(x_1, x_2) \le \frac{1}{b}\omega(x_1, T x_1) \le \frac{1}{b}k(\omega(x_0, x_1))\omega(x_0, x_1) \le \omega(x_0, x_1). \tag{2.7}$$

Continuing this process, we get an orbit $\{x_n\}$ of $T$ in $X$ such that $x_{n+1} \in J_b^{x_n}$,

$$\begin{gathered} b\omega(x_n, x_{n+1}) \le \omega(x_n, T(x_n)), \\ \omega(x_{n+1}, T(x_{n+1})) \le k(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}), \quad k(\omega(x_n, x_{n+1})) < b. \end{gathered} \tag{2.8}$$

Using (2.8), we get

$$\begin{aligned} \omega(x_n, T(x_n)) - \omega(x_{n+1}, T(x_{n+1})) &\ge b\omega(x_n, x_{n+1}) - k(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}) \\ &= [b - k(\omega(x_n, x_{n+1})]\omega(x_n, x_{n+1}) > 0, \end{aligned} \tag{2.9}$$

and thus for all $n$

$$\omega(x_n, T(x_n)) > \omega(x_{n+1}, T(x_{n+1})), \tag{2.10}$$

$$\omega(x_n, x_{n+1}) \le \omega(x_{n-1}, x_n). \tag{2.11}$$

Note that the sequences $\{\omega(x_n, T(x_n))\}$ and $\{\omega(x_n, x_{n+1})\}$ are decreasing, and thus convergent. Now, by the definition of the function $k$ there exists $\alpha \in [0, b)$ such that

$$\limsup_{n\to\infty} k(\omega(x_n, x_{n+1})) = \alpha. \tag{2.12}$$

Thus, for any $b_0 \in (\alpha, b)$, there exists $n_0 \in \mathbb{N}$ such that

$$k(\omega(x_n, x_{n+1})) < b_0, \quad \forall n > n_0, \tag{2.13}$$

and thus for all $n > n_0$, we have

$$k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_{n_0+1}, x_{n_0+2})) < b_0^{n-n_0}. \tag{2.14}$$

Also, it follows from (2.9) that for all $n > n_0$,

$$\omega(x_n, T(x_n)) - \omega(x_{n+1}, T(x_{n+1})) \geq \beta\omega(x_n, x_{n+1}), \tag{2.15}$$

where $\beta = b - b_0$. Note that for all $n > n_0$, we have

$$\begin{aligned}
\omega(x_{n+1}, T(x_{n+1})) &\leq k(\omega(x_n, x_{n+1}))\omega(x_n, x_{n+1}) \\
&\leq \frac{1}{b}k(\omega(x_n, x_{n+1}))\omega(x_n, T(x_n)) \\
&\leq \frac{1}{b}\frac{1}{b}k(\omega(x_n, x_{n+1}))k(\omega(x_{n-1}, x_n))\omega(x_{n-1}, T(x_{n-1})) \\
&\quad\vdots \\
&\leq \frac{1}{b^n}[k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_1, x_2))]\omega(x_1, T(x_1)) \\
&= \frac{k(\omega(x_n, x_{n+1})) \times \cdots \times k(\omega(x_{n_0+1}, x_{n_0+2}))}{b^{n-n_0}} \\
&\quad \times \frac{k(\omega(x_{n_0}, x_{n_0+1})) \times \cdots \times k(\omega(x_1, x_2))\omega(x_1, T(x_1))}{b^{n_0}},
\end{aligned} \tag{2.16}$$

and thus

$$\omega(x_{n+1}, T(x_{n+1})) < \left(\frac{b_0}{b}\right)^{n-n_0} \frac{k(\omega(x_{n_0}, x_{n_0+1})) \times \cdots \times k(\omega(x_1, x_2))\omega(x_1, T(x_1))}{b^{n_0}}. \tag{2.17}$$

Now, since $b_0 < b$, we have $\lim_{n\to\infty}(b_0/b)^{n-n_0} = 0$, and hence the decreasing sequence $\{\omega(x_n, T(x_n))\}$ converges to 0. Now, we show that $\{x_n\}$ is a Cauchy sequence. Note that for all $n > n_0$,

$$\omega(x_n, x_{n+1}) \leq \gamma^n\omega(x_o, x_1), \quad n = 0, 1, 2, \ldots, \tag{2.18}$$

where $\gamma = b_0/b < 1$. Now, for any $n, m \in \mathbb{N}$, $m > n > n_0$,

$$\begin{aligned}
\omega(x_n, x_m) &\leq \sum_{j=n}^{m-1}\omega(x_j, x_{j+1}) \\
&\leq (\gamma^n + \gamma^{n+1} + \cdots + \gamma^{m-1})\omega(x_o, x_1) \\
&\leq \frac{\gamma^n}{1-\gamma}\omega(x_o, x_1),
\end{aligned} \tag{2.19}$$

and thus by Lemma 1.5, $\{x_n\}$ is a Cauchy sequence. □

Using Lemma 2.1, we obtain the following fixed point result which is an improved version of Theorem 1.4 and contains Theorem 1.3 as a special case.

**Theorem 2.2.** *Let $X$ be a complete space and let $T : X \to Cl(X)$ be a generalized $w$-contractive map. Suppose that a real-valued function $g$ on $X$ defined by $g(x) = \omega(x, T(x))$ is lower semicontinous. Then there exists $v_o \in X$ such that $g(v_o) = 0$. Further, if $\omega(v_o, v_o) = 0$, then $v_0 \in \mathrm{Fix}(T)$.*

*Proof.* Since $T : X \to Cl(X)$ is a generalized $w$-contractive map, it follows from Lemma 2.1 that there exists a Cauchy sequence $\{x_n\}$ in $X$ such that the decreasing sequence $\{g(x_n)\} = \{\omega(x_n, T(x_n))\}$ converges to 0. Due to the completeness of $X$, there exists some $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_o$. Since $g$ is lower semicontinuous, we have

$$0 \leq g(v_o) \leq \liminf_{n\to\infty} g(x_n) = 0, \tag{2.20}$$

and thus, $g(v_o) = \omega(v_o, T(v_o)) = 0$. Since $\omega(v_o, v_o) = 0$, and $T(v_o)$ is closed, it follows from Lemma 1.6 that $v_0 \in T(v_0)$. □

As a consequence, we also obtain the following fixed point result.

**Corollary 2.3** (see [8]). *Let $X$ be a complete space and let $T : X \to Cl(X)$ be a $w$-contractive map. If the real-valued function $g$ on $X$ defined by $g(x) = \omega(x, T(x))$ is lower semicontinous, then there exists $v_o \in X$ such that $\omega(v_o, T(v_o)) = 0$. Further, if $\omega(v_o, v_o) = 0$, then $v_0 \in \mathrm{Fix}(T)$.*

Applying Lemma 2.1, we also obtain a fixed point result for multivalued generalized $w$-contractive map satisfying another suitable condition.

**Theorem 2.4.** *Let $X$ be a complete space and let $T : X \to Cl(X)$ be a generalized $w$-contractive map. Assume that*

$$\inf\{\omega(x, v) + \omega(x, T(x)) : x \in X\} > 0, \tag{2.21}$$

*for every $v \in X$ with $v \notin T(v)$. Then $\mathrm{Fix}(T) \neq \emptyset$.*

*Proof.* By Lemma 2.1, there exists an orbit $\{x_n\}$ of $T$, which is a Cauchy sequence in $X$. Due to the completeness of $X$, there exists $v_0 \in X$ such that $\lim_{n\to\infty} x_n = v_o$. Since $\omega(x_n, \cdot)$ is lower semicontinuous and $x_m \to v_0 \in X$, it follows from the proof of Lemma 2.1 that for all $n > n_0$

$$\omega(x_n, v_o) \leq \liminf_{m\to\infty} \omega(x_n, x_m) \leq \frac{\gamma^n}{1-\gamma}\omega(x_o, x_1), \tag{2.22}$$

where $\gamma = b_0/b < 1$. Also, we get

$$\omega(x_n, T(x_n)) \leq \omega(x_n, x_{n+1}) \leq \gamma^n \omega(x_o, x_1). \tag{2.23}$$

Assume that $v_o \notin T(v_o)$. Then, we have

$$
\begin{aligned}
0 &< \inf\{\omega(x, v_o) + \omega(x, T(x)) : x \in X\} \\
&\le \inf\{\omega(x_n, v_o) + \omega(x_n, T(x_n)) : n > n_0\} \\
&\le \inf\left\{\frac{\gamma^n}{1-\gamma}\omega(x_o, x_1) + \gamma^n \omega(x_o, x_1) : n > n_0\right\} \\
&= \left\{\frac{2-\gamma}{1-\gamma}\omega(x_o, x_1)\right\} \inf\{\gamma^n : n > n_0\} = 0,
\end{aligned}
\tag{2.24}
$$

which is impossible and hence $v_o \in \mathrm{Fix}(T)$. □

**Corollary 2.5** (see [8]). *Let $X$ be a complete space and let $T : X \to Cl(X)$ be $w$-contractive map. Assume that*

$$\inf\{\omega(x, u) + \omega(x, T(x)) : x \in X\} > 0, \tag{2.25}$$

*for every $u \in X$ with $u \notin T(u)$. Then* $\mathrm{Fix}(T) \neq \varnothing$.

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