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Research Article

Fixed Point Theorems in Cone Banach Spaces

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In this manuscript, a class of self-mappings on cone Banach spaces which have at least one fixed point is considered. More precisely, for a closed and convex subset C of a cone Banach space with the norm $\|x\|_P = d(x, 0)$, if there exist a, b, s and $T : C \to C$ satisfies the conditions $0 \leq s + |a| - 2b < 2(a + b)$ and $4ad(Tx, Ty) + b(d(x, Tx) + d(y, Ty)) \leq sd(x, y)$ for all $x, y \in C$, then T has at least one Fixed point.

1. Introduction and Preliminaries

In 1980, Rzepecki [1] introduced a generalized metric d_E on a set X in a way that $d_E : X \times X \to S$, where E is Banach space and S is a normal cone in E with partial order $\preceq$. In that paper, the author generalized the fixed point theorems of Maia type [2].

Let X be a nonempty set endowed in two metrics d_1, d_2 and T a mapping of X into itself. Suppose that $d_1(x, y) \leq d_2(x, y)$ for all $x, y \in X$, and X is complete space with respect to d_1, and T is continuous with respect to d_1, and T is contraction with respect to d_2, that is, $d_2(Tx, Ty) \leq kd_2(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$. Then f has a unique fixed point in X.

Seven years later, Lin [3] considered the notion of K-metric spaces by replacing real numbers with cone K in the metric function, that is, $d : X \times X \to K$. In that manuscript, some results of Khan and Imdad [4] on fixed point theorems were considered for K-metric spaces. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [5] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality

$$d(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, has a unique fixed point.

Recently, many results on fixed point theorems have been extended to cone metric spaces (see, e.g., [5–9]). Notice also that in ordered abstract spaces, existence of some fixed point theorems is presented and applied the resolution of matrix equations (see, e.g., [10–12]).

In this manuscript, some of known results (see, e.g., [13, 14]) are extended to cone Banach spaces which were defined and used in [15, 16] where the existence of fixed points for self-mappings on cone Banach spaces is investigated.

Throughout this paper $E := (E, \|\cdot\|)$ stands for real Banach space. Let $P := P_E$ always be a closed nonempty subset of E. P is called *cone* if $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b where $P \cap (-P) = \{0\}$ and $P \neq \{0\}$.

For a given cone P, one can define a partial ordering (denoted by $\leq$ or $\leq_P$) with respect to P by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ and $x \neq y$, while $x \ll y$ will show $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. From now on, it is assumed that $\operatorname{int} P \neq \emptyset$.

The cone P is called

(N) *normal* if there is a number $K \geq 1$ such that for all $x, y \in E$:

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|, \tag{1.2}$$

(R) *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n \geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n \to \infty} \|x_n - x\| = 0$.

In (N), the least positive integer K, satisfying (1.2), is called the normal constant of P.

Lemma 1.1 (see [6, 17]). (i) *Every regular cone is normal.*

(ii) *For each $k > 1$, there is a normal cone with normal constant $K > k$.*

(iii) *The cone P is regular if every decreasing sequence which is bounded from below is convergent.*

Proofs of (i) and (ii) are given in [6] and the last one follows from definition.

Definition 1.2 (see [5]). Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

$(M1)$ $0 \leq d(x, y)$ for all $x, y \in X$,

$(M2)$ $d(x, y) = 0$ if and only if $x = y$,

$(M3)$ $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y \in X$,

$(M4)$ $d(x, y) = d(y, x)$ for all $x, y \in X$,

then d is called cone metric on X, and the pair (X, d) is called a cone metric space (CMS).

Example 1.3. Let $E = \mathbb{R}^3$, $P = \{(x, y, z) \in E : x, y, z \geq 0\}$, and $X = \mathbb{R}$. Define $d : X \times X \to E$ by $d(x, \tilde{x}) = (\alpha|x - \tilde{x}|, \beta|x - \tilde{x}|, \gamma|x - \tilde{x}|)$, where α, β, γ are positive constants. Then (X, d) is a CMS. Note that the cone P is normal with the normal constant $K = 1$.

It is quite natural to consider Cone Normed Spaces (CNS).

Definition 1.4 (see [15, 16]). Let X be a vector space over $\mathbb{R}$. Suppose the mapping $\|\cdot\|_P : X \to E$ satisfies

($N1$) $\|x\|_P > 0$ for all $x \in X$,

($N2$) $\|x\|_P = 0$ if and only if $x = 0$,

($N2$) $\|x + y\|_P \leq \|x\|_P + \|y\|_P$ for all $x, y \in X$,

($N2$) $\|kx\|_P = |k|\|x\|_P$ for all $k \in \mathbb{R}$,

then $\|\cdot\|_P$ is called cone norm on X, and the pair $(X, \|\cdot\|_P)$ is called a cone normed space (CNS).

Note that each CNS is CMS. Indeed, $d(x, y) = \|x - y\|_P$.

Definition 1.5. Let $(X, \|\cdot\|_P)$ be a CNS, $x \in X$ and $\{x_n\}_{n\geq 1}$ a sequence in X. Then

(i) $\{x_n\}_{n\geq 1}$ *converges to* x whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;

(ii) $\{x_n\}_{n\geq 1}$ is a *Cauchy sequence* whenever for every $c \in E$ with $0 \ll c$ there is a natural number N, such that $\|x_n - x\|_P \ll c$ for all $n, m \geq N$;

(iii) $(X, \|\cdot\|_P)$ is a complete cone normed space if every Cauchy sequence is convergent.

Complete cone normed spaces will be called cone Banach spaces.

Lemma 1.6. *Let $(X, \|\cdot\|_P)$ be a CNS, P a normal cone with normal constant K, and $\{x_n\}$ a sequence in X. Then,*

(i) *the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_P \to 0$, as $n \to \infty$;*

(ii) *the sequence $\{x_n\}$ is Cauchy if and only if $\|x_n - x_m\|_P \to 0$ as $n, m \to \infty$;*

(iii) *the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y then $\|x_n - y_n\|_P \to \|x - y\|_P$.*

The proof is direct by applying [5, Lemmas 1, 4, and 5] to the cone metric space (X, d), where $d(x, y) = \|x - y\|_P$, for all $x, y \in X$.

Lemma 1.7 (see [7, 8]). *Let $(X, \|\cdot\|_P)$ be a CNS over a cone P in E. Then* (1) $\text{Int}(P) + \text{Int}(P) \subseteq \text{Int}(P)$ *and* $\lambda\text{Int}(P) \subseteq \text{Int}(P)$, $\lambda > 0$. (2) *If $c \gg 0$ then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.* (3) *For any given $c \gg 0$ and $c_0 \gg 0$, there exists $n_0 \in \mathbb{N}$ such that $c_0/n_0 \ll c$.* (4) *If a_n, b_n are sequences in E such that $a_n \to a$, $b_n \to b$, and $a_n \leq b_n$, for all n then $a \leq b$.*

The proofs of the first two parts followed from the definition of $\text{Int}(P)$. The third part is obtained by the second part. Namely, if $c \gg 0$ is given then find $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$. Then find n_0 such that $1/n_0 < \delta/\|c_0\|$ and hence $c_0/n_0 \ll c$. Since P is closed, the proof of fourth part is achieved.

Definition 1.8 (see [17]). P is called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum.

Lemma 1.9 (see [18]). *Every strongly minihedral normal cone is regular.*

Example 1.10. Let $E = C[0,1]$ with the supremum norm and $P = \{f \in E : f \geq 0\}$. Then P is a cone with normal constant $M = 1$ which is not regular. This is clear, since the sequence x^n is monotonically decreasing, but not uniformly convergent to 0. This cone, by Lemma 1.9, is not strongly minihedral. However, it is easy to see that the cone mentioned in Example 1.3 is strongly minihedral.

Definition 1.11. Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_P = d(x,0)$ and $T : C \to C$ a mapping which satisfies the condition

$$\frac{1}{2}\|x - Tx\|_P \leq \|x - y\|_P \Longrightarrow \|Tx - Ty\|_P \leq \|x - y\|_P, \tag{1.3}$$

for all $x, y \in C$. Then, T is said to satisfy the condition (C).

For $T : X \to X$, the set of fixed points of T is denoted by $F(T) := \{z \in X : Tz = z\}$.

Definition 1.12 (see [14]). Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_P = d(x,0)$ and $T : C \to C$ a mapping. Consider the conditions

$$\|Tx - Tz\|_P \leq \|x - z\|_P \quad \forall x, z \in C, \tag{1.4}$$

$$\|Tx - z\|_P \leq \|x - z\|_P \quad \forall x \in C,\ z \in F(T). \tag{1.5}$$

Then T is called nonexpansive (resp., quasi-nonexpansive) if it satisfies the condition (1.4) (resp., (1.5)).

2. Main Results

From now on, $X = (X, \|\cdot\|_P)$ will be a cone Banach space, P a normal cone with normal constant K and T a self-mapping operator defined on a subset C of X.

Theorem 2.1. *Let $a \in \mathbb{R}$ with $a > 1$ and let (X, d) be a complete cone metric space $T : X \to X$ an onto mapping which satisfies the condition*

$$d(Tx, Ty) \geq a d(x, y). \tag{2.1}$$

Then, T has a unique fixed point.

Proof. Let $x \neq y$ and $Tx = Ty$, then by (2.1), one can observe $0 \geq a d(x,y)$ which is a contradiction. Thus, T is one-to-one and it has an inverse, say S. Hence,

$$d(x, y) \geq a d(Sx, Sy) \Longleftrightarrow d(Sx, Sy) \leq \frac{1}{a} d(x, y). \tag{2.2}$$

By [5, Theorem 1], S has a unique fixed point which is equivalent to saying that T has a unique fixed point. □

The following statement is consequence of Definition 1.11.

Proposition 2.2. *Every nonexpansive mapping satisfies the condition* (C).

Proposition 2.3. *Let* T *satisfy the condition* (C) *and* $F(T) \neq \emptyset$, *then* T *is a quasi-nonexpansive.*

Proof. Let $z \in F(T)$ and $x \in C$. Since $(1/2)\|z - Tz\|_P = 0 \leq \|z - x\|_P$ and satisfies the condition (C),

$$\|z - Tx\|_P = \|Tz - Tx\|_P \leq \|z - x\|_P. \tag{2.3}$$

□

Theorem 2.4. *Let* C *be a closed and convex subset of a cone Banach space* X *with the norm* $\|x\|_P = d(x, 0)$ *and* $T : C \to C$ *a mapping which satisfies the condition*

$$d(x, Tx) + d(y, Ty) \leq q d(x, y), \tag{2.4}$$

for all $x, y \in C$, *where* $2 \leq q < 4$. *Then,* T *has at least one fixed point.*

Proof. Let $x_0 \in C$ be arbitrary. Define a sequence $\{x_n\}$ in the following way:

$$x_{n+1} := \frac{x_n + T(x_n)}{2} \quad n = 0, 1, 2, \ldots. \tag{2.5}$$

Notice that

$$x_n - Tx_n = 2\left(x_n - \left(\frac{x_n + Tx_n}{2}\right)\right) = 2(x_n - x_{n+1}), \tag{2.6}$$

which yields that

$$d(x_n, Tx_n) = \|x_n - Tx_n\|_P = 2\|x_n - x_{n+1}\|_P = 2d(x_n, x_{n+1}), \tag{2.7}$$

for $n = 0, 1, 2, \ldots$. Combining this observation with the condition (2.4), one can obtain

$$2d(x_{n-1}, x_n) + 2d(x_n, x_{n+1}) \leq q d(x_{n-1}, x_n). \tag{2.8}$$

Thus, $d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n)$, where $k = (q - 2)/2 < 1$. Hence, $\{x_n\}$ is a Cauchy sequence in C and thus converges to some $z \in C$. Regarding the inequality

$$d(z, Tx_n) \leq d(z, x_n) + d(x_n, Tx_n) = d(z, x_n) + 2d(x_n, x_{n+1}) \tag{2.9}$$

and by the help of Lemma 1.6(iii), one can obtain

$$Tx_n \longrightarrow z. \tag{2.10}$$

Taking into account (2.6) and (2.4), substituting $x = z$ and $y = x_n$ implies that

$$d(z, Tz) + 2d(x_n, x_{n+1}) \leq qd(z, x_n). \tag{2.11}$$

Thus, when $n \to \infty$, one can get $d(z, Tz) \leq 0$, that is, $Tz = z$. □

Notice that identity map, $I(x) = x$, satisfies the condition (2.4). Thus, maps that satisfy the condition (2.4) may have fixed points.

From the triangle inequality,

$$d(x, Tx) + d(y, Ty) \leq d(x, y) + d(y, Tx) + d(y, x) + d(x, Ty). \tag{2.12}$$

By (2.4),

$$d(x, Tx) + d(y, Ty) \leq 2d(x, y) + qd(x, y) = (2 + q)d(x, y), \quad 2 \leq q < 4. \tag{2.13}$$

Thus, letting $p = 2 + q$ implies that

$$d(x, Tx) + d(y, Ty) \leq 2d(x, y) + qd(x, y) = (2 + q)d(x, y), \quad 0 \leq p < 2. \tag{2.14}$$

Hence we have the following conclusion.

Theorem 2.5. *Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_P = d(x, 0)$ and $T : C \to C$ a mapping which satisfies the condition*

$$d(x, Tx) + d(y, Ty) \leq pd(x, y) \tag{2.15}$$

for all $x, y \in C$, where $0 \leq p < 2$. Then T has a fixed point.

Theorem 2.6. *Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_P = d(x, 0)$ and $T : C \to C$ a mapping which satisfies the condition*

$$d(Tx, Ty) + d(x, Tx) + d(y, Ty) \leq rd(x, y) \tag{2.16}$$

for all $x, y \in C$, where $2 \leq r < 5$. Then T has at least one fixed point.

Proof. Construct a sequence $\{x_n\}$ as in the proof of Theorem 2.4, that is, (2.5), (2.6) and also

$$\begin{gathered} x_n - Tx_{n-1} = \frac{x_{n-1} + Tx_{n-1}}{2} - Tx_{n-1} = \frac{x_{n-1} - Tx_{n-1}}{2}, \\ d(x_n, Tx_{n-1}) = \|x_n - Tx_{n-1}\|_P = \frac{1}{2}\|x_{n-1} - Tx_{n-1}\|_P = \frac{1}{2}d(x_{n-1}, Tx_{n-1}) \end{gathered} \tag{2.17}$$

hold. Thus the triangle inequality implies

$$d(x_n, Tx_n) - d(x_n, Tx_{n-1}) \leq d(Tx_{n-1}, Tx_n). \tag{2.18}$$

Then, by (2.17) and (2.7) we obtain

$$2d(x_n, x_{n+1}) - d(x_n, x_{n-1}) \leq d(Tx_{n-1}, Tx_n). \tag{2.19}$$

Replacing $x = x_{n-1}$ and $y = x_n$ in (2.16) and regarding (2.7) and (2.19), one can obtain

$$2d(x_n, x_{n+1}) + d(x_n, x_{n-1}) - 2d(x_n, x_{n-1}) + 2d(x_n, x_{n+1}) \leq rd(x_{n-1}, x_n) \tag{2.20}$$

and thus, $d(x_n, x_{n+1}) \leq ((r-1)/4)d(x_n, x_{n-1})$. Since $1 \leq r < 5$, the sequence $\{x_n\}$ is a Cauchy sequence that converges to some $z \in C$. Since $\{Tx_n\}$ also converges to z as in the proof of Theorem 2.4, the inequality (2.16) (under the assumption $x = z$ and $y = x_n$) by the help of Lemma 1.6(iii) yields that $d(Tz, z)+d(z, Tz) \leq 0$ which is equivalent to saying that $Tz = z$. □

Theorem 2.7. *Let C be a closed and convex subset of a cone Banach space with the norm $\|x\|_P = d(x, 0)$. If there exist a, b, s and $T : C \to C$ satisfies the conditions*

$$0 \leq s + |a| - 2b < 2(a + b), \tag{2.21}$$

$$ad(Tx, Ty) + b\big(d(x, Tx) + d(y, Ty)\big) \leq sd(x, y) \tag{2.22}$$

for all $x, y \in C$. Then, T has at least one fixed point.

Proof. Construct a sequence $\{x_n\}$ as in the proof of Theorem 2.4. We claim that the inequality (2.22) for $x = x_{n-1}$ and $y = x_n$ implies that

$$2ad(x_n, x_{n+1}) - |a|d(x_{n-1}, x_n) + 2b(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq sd(x_{n-1}, x_n) \tag{2.23}$$

for all a, b, s that satisfy (2.21). For the proof of the claim, first recall from (2.7) that

$$d(x_{n-1}, Tx_{n-1}) = 2d(x_{n-1}, x_n), \qquad d(x_n, Tx_n) = 2d(x_n, x_{n+1}). \tag{2.24}$$

The case $a \geq 0$ is trivially true. Indeed, taking into account (2.22) with $x = x_{n-1}$ and $y = x_n$ together with (2.24) and (2.19), one can get

$$2ad(x_n, x_{n+1}) - ad(x_{n-1}, x_n) + 2b(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq sd(x_{n-1}, x_n) \tag{2.25}$$

which is equivalent to (2.23) since $|a| = a$. For the case $a < 0$, consider the inequality $d(Tx_{n-1}, Tx_n) \leq d(x_n, Tx_n) + d(x_n, Tx_{n-1})$ which is equivalent to

$$a(d(x_n, Tx_n) + d(x_n, Tx_{n-1})) \geq ad(Tx_{n-1}, Tx_n). \tag{2.26}$$

By substituting $x = x_{n-1}$ and $y = x_n$ in (2.22) together with (2.24), (2.26) and (2.17), one can get

$$2ad(x_n, x_{n+1}) + ad(x_{n-1}, x_n) + 2b(d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \leq sd(x_{n-1}, x_n) \tag{2.27}$$

which is equivalent to (2.23) since $|a| = -a$. Hence, the claim is proved.

By (2.23), one can obtain

$$d(x_n, x_{n+1}) \leq \frac{|a| - 2b + s}{2(a+b)} d(x_{n-1}, x_n). \tag{2.28}$$

Due to (2.21), we have $0 \leq (|a| - 2b + s)/2(a+b) < 1$. Thus, the sequence $\{x_n\}$ is a Cauchy sequence that converges to some $z \in C$. By substituting x with z and y with x_n in (2.22), one can obtain

$$ad(Tz, z) + bd(z, Tz) \leq 0, \tag{2.29}$$

as $n \to \infty$. This last condition is equivalent to saying that $Tz = z$ as $a + b > 0$. □

References

[1] B. Rzepecki, "On fixed point theorems of Maia type," *Publications de l'Institut Mathématique*, vol. 28(42), pp. 179–186, 1980.

[2] M. G. Maia, "Un'osservazione sulle contrazioni metriche," *Rendiconti del Seminario Matematico della Università di Padova*, vol. 40, pp. 139–143, 1968.

[3] S. D. Lin, "A common fixed point theorem in abstract spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 18, no. 8, pp. 685–690, 1987.

[4] M. S. Khan and M. Imdad, "A common fixed point theorem for a class of mappings," *Indian Journal of Pure and Applied Mathematics*, vol. 14, no. 10, pp. 1220–1227, 1983.

[5] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[6] Sh. Rezapour and R. Hamlbarani, "Some notes on the paper: "Cone metric spaces and fixed point theorems of contractive mappings"," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 2, pp. 719–724, 2008.

[7] D. Turkoglu and M. Abuloha, "Cone metric spaces and fixed point theorems in diametrically contractive mappings," to appear in *Acta Mathematica Sinica*.

[8] D. Turkoglu, M. Abuloha, and T. Abdeljawad, "KKM mappings in cone metric spaces and some fixed point theorems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 348–353, 2010.

[9] I. Şahin and M. Telci, "Fixed points of contractive mappings on complete cone metric spaces," *Hacettepe Journal of Mathematics and Statistics*, vol. 38, no. 1, pp. 59–67, 2009.

[10] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[11] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[12] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, "Fixed point theorems in ordered abstract spaces," *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.

[13] L. Ćirić, *Fixed Point Theory Contraction Mapping principle*, Faculty of Mechanical Engineering Press, Beograd, Serbia, 2003.

[14] T. Suzuki, "Fixed point theorems and convergence theorems for some generalized nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 2, pp. 1088–1095, 2008.

[15] D. Turkoglu, M. Abuloha, and T. Abdeljawad, "Some theorems and examples of cone metric spaces," to appear in *Journal of Computational Analysis and Applications*.

[16] T. Abdeljawad, "Completion of cone metric spaces," to appear in *Hacettepe Journal of Mathematics and Statistics*.

[17] K. Deimling, *Nonlinear Functional Analysis*, Springer, Berlin, Germany, 1985.

[18] T. Abdeljawad and E. Karapinar, "Quasicone Metric Spaces and Generalizations of Caristi Kirk's Theorem," *Fixed Point Theory and Applications*, vol. 2009, p. 9 page, 2009.