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Research Article

On Strong Convergence by the Hybrid Method for Equilibrium and Fixed Point Problems for an Inifnite Family of Asymptotically Nonexpansive Mappings

Gang Cai and Chang song Hu

Department of Mathematics, Hubei Normal University, Huangshi 435002, China

Correspondence should be addressed to Gang Cai, caigang-aaaa@163.com and
Chang song Hu, huchang1004@yahoo.com.cn

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We introduce two modifications of the Mann iteration, by using the hybrid methods, for equilibrium and fixed point problems for an infinite family of asymptotically nonexpansive mappings in a Hilbert space. Then, we prove that such two sequences converge strongly to a common element of the set of solutions of an equilibrium problem and the set of common fixed points of an infinite family of asymptotically nonexpansive mappings. Our results improve and extend the results announced by many others.

1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $T : C \to C$ is said to be nonexpansive if for all $x, y \in C$ we have $\|Tx-Ty\| \le \|x-y\|$. It is said to be asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \ge 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for all integers $n \ge 1$ and for all $x, y \in C$. The set of fixed points of T is denoted by $F(T)$.

Let $\phi : C \times C \to R$ be a bifunction, where R is the set of real number. The equilibrium problem for the function ϕ is to find a point $x \in C$ such that

$$\phi(x, y) \ge 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(\phi)$. In 2005, Combettes and Hirstoaga [2] introduced an iterative scheme of finding the best approximation to the initial data when $EP(\phi)$ is nonempty, and they also proved a strong convergence theorem.

For a bifunction $\phi : C \times C \to R$ and a nonlinear mapping $A : C \to H$, we consider the following equilibrium problem:

$$\text{Find } z \in C \text{ such that } \phi(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of such that $z \in C$ is denoted by EP, that is,

$$EP = \{z \in C : \phi(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C\}. \tag{1.3}$$

In the case of $A = 0$, $EP = EP(\phi)$. In the case of $\phi \equiv 0$, EP is denoted by $VI(C, A)$. The problem (1.2) is very general in the sense that it includes, as special cases, some optimization problems, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games, and others (see, e.g., [3, 4]).

Recall that a mapping $A : C \to H$ is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0, \quad \forall u,\ v \in C. \tag{1.4}$$

A mapping A of C into H is called α-inverse strongly monotone, see [5–7], if there exists a positive real number α such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2 \tag{1.5}$$

for all $x, y \in C$. It is obvious that any $\alpha-$inverse strongly monotone mapping A is monotone and Lipschitz continuous.

Construction of fixed points of nonexpansive mappings and asymptotically nonexpansive mappings is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see, e.g., [1, 8–10]). Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann [11] and Ishikawa [12] iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities; see, for example, [11–13]. However, Mann and Ishikawa iteration processes have only weak convergence even in Hilbert spaces (see, e.g., [11, 12]).

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. In 2003, Nakajo and Takahashi [14] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{gathered}
x_0 \in\ C \text{ chosen arbitrarily,} \\
y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\
Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0,
\end{gathered} \tag{1.6}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H. They proved that if the sequence $\{\alpha_n\}$ bounded above from one, then $\{x_n\}$ defined by (1.6) converges strongly to $P_{F(T)}x_0$.

Recently, Kim and Xu [15] adapted the iteration (1.6) to an asymptotically nonexpansive mapping in a Hilbert space H:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \left\{v \in C : \left\|y_n - v\right\|^2 \le \|x_n - v\|^2 + \theta_n\right\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x_0, \end{gathered} \tag{1.7}$$

where $\theta_n = (1-\alpha_n)(k_n^2-1)(\operatorname{diam} C)^2 \to 0$, as $n \to \infty$. They proved that if $\alpha_n \le a$ for all n and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.7) converges strongly to $P_{\operatorname{Fix}(T)}(x_0)$.

Very recently, Inchan and Plubtieng [16] introduced the modified Ishikawa iteration process by the shrinking hybrid method [17] for two asymptotically nonexpansive mappings S and T, with C a closed convex bounded subset of a Hilbert space H. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define $\{x_n\}$ as follows:

$$\begin{gathered} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n, \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n, \\ C_{n+1} = \left\{v \in C_n : \left\|y_n - v\right\|^2 \le \|x_n - v\|^2 + \theta_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in N, \end{gathered} \tag{1.8}$$

where $\theta_n = (1-\alpha_n)[(t_n^2-1)+(1-\beta_n)t_n^2(s_n^2-1)](\operatorname{diam} C)^2 \to 0$, as $n \to \infty$ and $0 \le \alpha_n \le a < 1$ and $0 < b \le \beta_n \le c < 1$ for all $n \in N$. They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to a common fixed point of two asymptotically nonexpansive mappings S and T.

Zegeye and Shahzad [18] established the following hybrid iteration process for a finite family of asymptotically nonexpansive mappings in a Hilbert space H:

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \alpha_{n0}x_n + \alpha_{n1}T_1^n x_n + \alpha_{n2}T_2^n x_n + \alpha_{n3}T_3^n x_n + \cdots + \alpha_{nr}T_r^n x_n, \\ C_n = \left\{v \in C : \left\|y_n - v\right\|^2 \le \|x_n - v\|^2 + \theta_n\right\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{gathered} \tag{1.9}$$

where $\theta_n = [(k_{n1}^2 - 1)\alpha_{n1} + (k_{n2}^2 - 1)\alpha_{n2} + \cdots + (k_{nr}^2 - 1)\alpha_{nr}](\operatorname{diam} C)^2 \to 0$, as $n \to \infty$. Under suitable conditions strong convergence theorem is proved which extends and improves the corresponding results of Nakajo and Takahashi [14] and Kim and Xu [15].

On the other hand, for finding a common element of $EP(\phi) \cap F(S)$, Tada and Takahashi [19] introduced the following iterative scheme by the hybrid method in a Hilbert space: $x_0 = x \in H$ and let

$$\begin{gathered}
u_n \in C \text{ such that } \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
w_n = (1 - \alpha_n)x_n + \alpha_n S u_n, \\
C_n = \{z \in H : \|w_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0
\end{gathered} \tag{1.10}$$

for every $n \in N \cup \{0\}$, where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Further, they proved that $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in EP(\phi) \cap F(S)$, where $z = P_{EP(\phi)\cap F(S)} x_0$.

Inspired and motivated by the above facts, it is the purpose of this paper to introduce the Mann iteration process for finding a common element of the set of common fixed points of an infinite family of asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem. Then we prove some strong convergence theorems which extend and improve the corresponding results of Tada and Takahashi [19], Inchan and Plubtieng [16], Zegeye and Shahazad [18], and many others.

2. Preliminaries

We will use the following notations:

(1) "$\rightharpoonup$" for weak convergence and "$\to$" for strong convergence;

(2) $w_\omega(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

Let H be a real Hilbert space. It is well known that

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

for all $x, y \in H$.

It is well known that H satisfies Opial's condition [20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.2}$$

holds for every $y \in H$ with $y \neq x$. Hilbert space H satisfies the Kadec-Klee property [21, 22], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

We need some facts and tools in a real Hilbert space H which are listed as follows.

Lemma 2.1 ([23]). *Let T be an asymptotically nonexpansive mapping defined on a nonempty bounded closed convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup z$ and $Tx_n - x_n \to 0$, then $z \in F(T)$.*

Lemma 2.2 ([24]). *Let C be a nonempty closed convex subset of H and also give a real number $a \in R$. The set $D := \{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.3 ([22]). *Let C be a nonempty closed convex subset of H, and let P_C be the (metric or nearest) projection from H onto C (i.e., $P_C x$ is the only point in C such that $\|x - P_C x\| = \inf\{\|x - z\| : \forall z \in C\}$). Given $x \in H$ and $z \in C$. Then $z = P_C x$ if and only if it holds the relation:*

$$\langle x - z, y - z\rangle \le 0, \quad \forall y \in C. \tag{2.3}$$

For solving the equilibrium problem, let us assume that the bifunction ϕ satisfies the following conditions (see [3]):

(A1) $\phi(x, x) = 0$ for all $x \in C$;

(A2) ϕ is monotone, that is, $\phi(x, y) + \phi(y, x) \le 0$ for any $x, y \in C$;

(A3) ϕ is upper-hemicontinuous, that is, for each $x, y, z \in C$

$$\limsup_{t \to 0^+} \phi(tz + (1 - t)x, y) \le \phi(x, y); \tag{2.4}$$

(A4) $\phi(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$.

The following lemma appears implicity in [3].

Lemma 2.4 ([3]). *Let C be a nonempty closed convex subset of H, and let ϕ be a bifunction of $C \times C$ into R satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \quad \forall y \in C. \tag{2.5}$$

The following lemma was also given in [2].

Lemma 2.5 ([2]). *Assume that $\phi : C \times C \to R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0 \ \forall y \in C\right\} \tag{2.6}$$

for all $x \in H$*. Then, the following holds*

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, that is, for any* $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$.

This implies that $\|T_r x - T_r y\| \leq \|x - y\|, \forall x, y \in H$*, that is,* T_r *is a nonexpansive mapping:*

(3) $F(T_r) = EP(\phi), \forall r > 0$*;*

(4) $EP(\phi)$ *is a closed and convex set.*

Definition 2.6 (see [25]). Let C be a nonempty closed convex subset of H. Let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself, and let $\{\beta_{n,k} : n, k \in N,\ 1 \leq k \leq n\}$ be a sequence of real numbers such that $0 \leq \beta_{i,j} \leq 1$ for every $i, j \in N$ with $i \geq j$. For any $n \geq 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
U_{n,n} &= \beta_{n,n} S_n^n + (1 - \beta_{n,n}) I,\\
U_{n,n-1} &= \beta_{n,n-1} S_{n-1}^n U_{n,n} + (1 - \beta_{n,n-1}) I,\\
&\ \ \vdots\\
U_{n,k} &= \beta_{n,k} S_k^n U_{n,k+1} + (1 - \beta_{n,k}) I,\\
&\ \ \vdots\\
U_{n,2} &= \beta_{n,2} S_2^n U_{n,3} + (1 - \beta_{n,2}) I,\\
W_n = U_{n,1} &= \beta_{n,1} S_1^n U_{n,2} + (1 - \beta_{n,1}) I.
\end{aligned} \tag{2.7}$$

Such a mapping W_n is called the modified W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$.

Lemma 2.7 ([10, Lemma 4.1]). *Let* C *be a nonempty closed convex subset of* H*. Let* $\{S_m\}$ *be a family of asymptotically nonexpansive mappings of* C *into itself with Lipschitz constants* $\{t_{m,n}\}$*, that is,* $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$ *(for all* $m, n \in N$*, for all* $x, y \in C$*) such that* $F := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$*, and let* $\{\beta_{n,k} : n, k \in N,\ 1 \leq k \leq n\}$ *be a sequence of real numbers with* $0 < a \leq \beta_{n,1} \leq 1$ *for all* $n \in N$ *and* $0 < b \leq \beta_{n,i} \leq c < 1$ *for every* $n \in N$ *and* $i = 2, \ldots, n$ *for some* $a, b, c \in (0, 1)$*. Let* W_n *be the modified* W*-mapping generated by* $S_n, S_{n-1}, \ldots, S_1$ *and* $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$*. Let* $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2 - 1)\}$ *for every* $n \in N$ *and* $k = 1, 2, \ldots, n$*. Then, the followings hold:*

(i) $\|W_n x - z\|^2 \leq (1 + r_{n,1})\|x - z\|^2$ *for all* $n \in N$*,* $x \in C$ *and* $z \in \cap_{i=1}^{n} F(S_i)$*;*

(ii) *if* C *is bounded and* $\lim_{n\to\infty} r_{n,1} = 0$*, for every sequence* $\{z_n\}$ *in* C*,*

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0, \quad \lim_{n\to\infty} z_n - W_n z_n = 0 \quad \textit{imply } w_\omega(z_n) \subset F; \tag{2.8}$$

(iii) *if* $\lim_{n\to\infty} r_{n,1} = 0$*,* $F = \cap_{i=1}^{\infty} F(W_n)$ *and* F *is closed convex.*

Lemma 2.8 ([10, Lemma 4.4]). *Let C be a nonempty closed convex subset of H. Let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x-y\|$ (for all $m,n \in N$, for all $x,y \in C$) such that $F := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $T_n = \sum_{k=1}^{n} \beta_{n,k} S_k^n$ for every $n \in N$, where $0 \leq \beta_{n,k} \leq 1$ for every $n = 1,2,3,\ldots$ and $k = 1,2,\ldots,n$ with $\sum_{k=1}^{n} \beta_{n,k} = 1$ for every $n \in N$ and $\lim_{n\to\infty}\beta_{n,k} > 0$ for every $k \in N$, and let $r_n = \sum_{k=1}^{n} \beta_{n,k}(t_{k,n}^2 - 1)$ for every $n \in N$. Then, the following holds:*

(i) *$\|T_n x - z\|^2 \leq (1+r_n)\|x-z\|^2$ for all $n \in N$, $x \in C$ and $z \in \cap_{i=1}^{n} F(S_i)$;*

(ii) *if C is bounded and $\lim_{n\to\infty} r_n = 0$, for every sequence $\{z_n\}$ in C,*

$$\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0, \quad \lim_{n\to\infty}\|z_n - T_n z_n\| = 0 \quad \textit{imply } w_\omega(z_n) \subset F; \tag{2.9}$$

(iii) *if $\lim_{n\to\infty} r_n = 0$, $F = \cap_{i=1}^{\infty} F(T_n)$ and F is closed convex.*

3. Main Results

In this section, we will introduce two iterative schemes by using hybrid approximation method for finding a common element of the set of common fixed points for a family of infinitely asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in Hilbert space. Then we show that the sequences converge strongly to a common element of the two sets.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, let $\phi : C \times C \to R$ be a bifunction satisfying the conditions (A1)–(A4), let A be an α-inverse strongly monotone mapping of C into H, let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x-y\|$ (for all $m,n \in N$, for all $x,y \in C$) such that $F \cap EP \neq \emptyset$, where $F := \cap_{i=1}^{\infty} F(S_i)$, and let $\{\beta_{n,k} : n,k \in N,\ 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in N$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in N$ and $i = 2,\ldots,n$ for some $a,b,c \in (0,1)$. Let W_n be the modified W-mapping generated by $S_n, S_{n-1},\ldots,S_1$ and $\beta_{n,n},\beta_{n,n-1},\ldots,\beta_{n,2},\beta_{n,1}$. Assume that $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2 - 1)\}$ for every $n \in N$ and $k = 1,2,\ldots,n$ such that $\lim_{n\to\infty} r_{n,1} = 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{gathered} x_0 \in C \textit{ chosen arbitrarily}, \\ u_n \in C \textit{ such that } \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n u_n + (1-\alpha_n)W_n u_n, \\ C_{n+1} = \left\{v \in C_n : \left\|y_n - v\right\|^2 \leq \|x_n - v\|^2 + \theta_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in N \cup \{0\}, \end{gathered} \tag{3.1}$$

where $C_0 = C$ and $\theta_n = (1-\alpha_n)r_{n,1}(\operatorname{diam} C)^2$ and $0 \leq \alpha_n \leq d < 1$ and $0 < e \leq r_n \leq f < 2\alpha$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F\cap EP}(x_0)$.

Proof. We show first that the sequences $\{x_n\}$ and $\{u_n\}$ are well defined.

We observe that C_n is closed and convex by Lemma 2.2. Next we show that $F \cap EP \subset C_n$ for all n. we prove first that $(I - r_n A)$ is nonexpansive. Let $x, y \in C$. Since A is α-inverse strongly monotone and $r_n < 2\alpha \ \ \forall n \in N$, we have

$$\begin{aligned}
\|(I - r_n A)x - (I - r_n A)y\|^2 &= \|x - y - r_n(Ax - Ay)\|^2 \\
&= \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2 - 2\alpha r_n\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\
&= \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{3.2}$$

Thus $(I - r_n)A$ is nonexpansive.

Since

$$\phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.3}$$

we obtain

$$\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - (I - r_n A)x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.4}$$

By Lemma 2.5, we have $u_n = T_{r_n}(x_n - r_n A x_n)$, for all $n \in N$.

Let $p \in F \cap EP$, it follows the definition of EP that

$$\phi(p, y) + \langle y - p, Ap\rangle \geq 0, \quad \forall\ y \in C. \tag{3.5}$$

So,

$$\phi(p, y) + \frac{1}{r_n}\langle y - p, p - (p - r_n Ap)\rangle \geq 0, \quad \forall\ y \in C. \tag{3.6}$$

Again by Lemma 2.5, we have $p = T_{r_n}(p - r_n Ap)$, for all $n \in N$.

Since $I - r_n A$ and T_{r_n} are nonexpansive, one has

$$\|u_n - p\| \leq \|T_{r_n}(x_n - r_n A x_n) - T_{r_n}(p - r_n Ap)\| \leq \|x_n - p\|, \quad \forall n \geq 1. \tag{3.7}$$

Then using the convexity of $\|\cdot\|^2$ and Lemma 2.7 we obtain that

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(W_n u_n - p)\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|W_n u_n - p\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)(1+r_{n,1})\|u_n - p\|^2 \\ &= \|u_n - p\|^2 + (1-\alpha_n)r_{n,1}\|u_n - p\|^2 \\ &\le \|u_n - p\|^2 + \theta_n \\ &\le \|x_n - p\|^2 + \theta_n.\end{aligned} \tag{3.8}$$

So $p \in C_n$ for all n and hence $F \cap EP \subset C_n$ for all n. This implies that $\{x_n\}$ is well defined. From Lemma 2.4, we know that $\{u_n\}$ is also well defined.

Next, we prove that $\|x_{n+1}-x_n\| \to 0$, $\|x_n-u_n\| \to 0$, $\|u_{n+1}-u_n\| \to 0$, $\|u_n-W_n u_n\| \to 0$, as $n \to \infty$.

It follows from $x_n = P_{C_n}x_0$ that

$$\langle x_0 - x_n, x_n - v\rangle \ge 0, \quad \text{for each } v \in F \cap EP \subset C_n,\ n \in N. \tag{3.9}$$

So, for $p \in F \cap EP$, we have

$$\begin{aligned}0 \le \langle x_0 - x_n, x_n - p\rangle &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - p\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_n - x_0\|\|x_0 - p\|.\end{aligned} \tag{3.10}$$

This implies that

$$\|x_n - x_0\|^2 \le \|x_n - x_0\|\|x_0 - p\|, \tag{3.11}$$

and hence

$$\|x_n - x_0\| \le \|x_0 - p\|. \tag{3.12}$$

Since C is bounded, then $\{x_n\}$ and $\{u_n\}$ are bounded.

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0 \quad \forall n \in N. \tag{3.13}$$

So,

$$\begin{aligned}0 \le \langle x_0 - x_n, x_n - x_{n+1}\rangle &= -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\le -\|x_n - x_0\|^2 + \|x_n - x_0\|\|x_0 - x_{n+1}\|.\end{aligned} \tag{3.14}$$

This implies that

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \cdot \quad \forall n \in N. \tag{3.15}$$

Hence, $\{\|x_n - x_0\|\}$ is nodecreasing, and so $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

Next, we can show that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Indeed, From (2.1) and (3.13), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.\end{aligned} \tag{3.16}$$

Since $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.17}$$

On the other hand, it follows from $x_{n+1} \in C_{n+1}$ that

$$\|y_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.18}$$

It follows that

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.19}$$

Next, we claim that $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Let $p \in F \cap EP$, it follows from (3.8) that

$$\begin{aligned}\|y_n - p\|^2 &\le \|u_n - p\|^2 + \theta_n \\ &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)p\|^2 + \theta_n \\ &\le \|x_n - p\|^2 + r_n(r_n - 2\alpha)\|Ax_n - Ap\|^2 + \theta_n.\end{aligned} \tag{3.20}$$

This implies that

$$\begin{aligned}e(2\alpha - f)\|Ax_n - Ap\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n \\ &\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.\end{aligned} \tag{3.21}$$

It follows from (3.19) that

$$\lim_{n\to\infty}\|Ax_n - Ap\| = 0. \tag{3.22}$$

From Lemma 2.5, one has

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}(I - r_nA)x_n - T_{r_n}(I - r_nA)p\|^2 \\
&\le \langle (x_n - r_nAx_n) - (p - r_nAp), u_n - p\rangle \\
&= \frac{1}{2}\Big\{ \|x_n - r_nAx_n - (p - r_nAp)\|^2 + \|u_n - p\|^2 \\
&\qquad -\|x_n - r_nAx_n - (p - r_nAp) - (u_n - p)\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Ax_n - Ap)\|^2 \Big\} \\
&= \frac{1}{2}\Big\{ \|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n\|^2 \\
&\qquad +2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2 \Big\}.
\end{aligned} \tag{3.23}$$

This implies that

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle - r_n^2\|Ax_n - Ap\|^2 \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\langle x_n - u_n, Ax_n - Ap\rangle \\
&\le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\|.
\end{aligned} \tag{3.24}$$

By (3.8), we have

$$\|y_n - p\|^2 \le \|u_n - p\|^2 + \theta_n. \tag{3.25}$$

Substituting (3.24) into (3.25), we obtain

$$\|y_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n, \tag{3.26}$$

which implies that

$$\begin{aligned}
\|x_n - u_n\|^2 &\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n \\
&\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + 2r_n\|x_n - u_n\|\|Ax_n - Ap\| + \theta_n.
\end{aligned} \tag{3.27}$$

Noticing that $\lim_{n\to\infty}\|Ax_n - Ap\| = 0$ and (3.19), it follows from (3.27) that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.28}$$

From (3.17) and (3.28), we have

$$\|u_n - u_{n+1}\| \leq \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.29}$$

Similarly, from (3.19) and (3.28), one has

$$\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.30}$$

Noticing that the condition $0 \leq \alpha_n \leq d < 1$, it follows that

$$(1 - \alpha_n)\|W_n u_n - u_n\| = \|y_n - u_n\|, \tag{3.31}$$

which implies that

$$\|W_n u_n - u_n\| = \frac{\|y_n - u_n\|}{1 - \alpha_n} < \frac{\|y_n - u_n\|}{1 - d} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.32}$$

Next, we prove that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z, where $z \in F \cap EP$.

Since $\{x_n\}$ is bounded and C is closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z, where $z \in C$. From (3.28), we have $u_{n_i} \rightharpoonup z$. Noticing (3.29) and (3.32), it follows from Lemma 2.7 that $z \in F$. Next we prove that $z \in EP$. Since $u_n = T_{r_n}(x_n - r_n A x_n)$, for any $y \in C$, we have

$$\phi(u_n, y) + \langle A x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0. \tag{3.33}$$

From (A2), one has

$$\langle A x_n, y - u_n \rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq \phi(y, u_n). \tag{3.34}$$

Replacing n by n_i, we obtain

$$\langle A x_{n_i}, y - u_{n_i} \rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \phi(y, u_{n_i}). \tag{3.35}$$

Put $z_t = ty + (1-t)z$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So we have

$$\begin{aligned} \langle z_t - u_{n_i}, A z_t \rangle &\geq \langle z_t - u_{n_i}, A z_t \rangle - \langle A x_{n_i}, z_t - u_{n_i} \rangle - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \phi(z_t, u_{n_i}) \\ &= \langle z_t - u_{n_i}, A z_t - A u_{n_i} \rangle + \langle z_t - u_{n_i}, A u_{n_i} - A x_{n_i} \rangle \\ &\quad - \left\langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + \phi(z_t, u_{n_i}). \end{aligned} \tag{3.36}$$

Since $\|u_{n_i} - x_{n_i}\| \to 0$, we have $\|Au_{n_i} - Ax_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. So, from (A4) we have

$$\langle z_t - z, Az_t\rangle \geq \phi(z_t, z), \tag{3.37}$$

as $i \to \infty$. From (A1) and (A4), we also have

$$\begin{aligned}
0 = \phi(z_t, z_t) &\leq t\phi(z_t, y) + (1-t)\phi(z_t, z) \\
&\leq t\phi(z_t, y) + (1-t)\langle z_t - z, Az_t\rangle \\
&= t\phi(z_t, y) + (1-t)t\langle y - z, Az_t\rangle,
\end{aligned} \tag{3.38}$$

and hence

$$0 \leq \phi(z_t, y) + (1-t)\langle y - z, Az_t\rangle. \tag{3.39}$$

Letting $t \to 0$, we have, for each $y \in C$,

$$0 \leq \phi(z, y) + \langle y - z, Az\rangle. \tag{3.40}$$

This implies that $z \in EP$. Therefore $z \in F \cap EP$.

Finally we show that $x_n \to z$, $u_n \to z$, where $z = P_{F\cap EP(\phi)}(x_0)$.

Putting $z' = P_{F\cap EP}(x_0)$ and consider the sequence $\{x_0 - x_{n_i}\}$. Then we have $x_0 - x_{n_i} \rightharpoonup x_0 - z$ and by the weak lower semicontinuity of the norm and by the fact that $\|x_0 - x_{n+1}\| \leq \|x_0 - z'\|$ for all $n \geq 0$ which is implied by the fact that $x_{n+1} = P_{C_{n+1}}(x_0)$, we obtain

$$\begin{aligned}
\|x_0 - z'\| &\leq \|x_0 - z\| \\
&\leq \liminf_{i\to\infty} \|x_0 - x_{n_i}\| \\
&\leq \limsup_{i\to\infty} \|x_0 - x_{n_i}\| \\
&\leq \|x_0 - z'\|.
\end{aligned} \tag{3.41}$$

This implies that $\|x_0 - z'\| = \|x_0 - z\|$ (hence $z' = z$ by the uniqueness of the nearest point projection of x_0 onto $F \cap EP$) and that

$$\|x_0 - x_{n_i}\| \longrightarrow \|x_0 - z'\|. \tag{3.42}$$

It follows that $x_0 - x_{n_i} \to x_0 - z'$, and hence $x_{n_i} \to z'$. Since $\{x_{n_i}\}$ is an arbitrary (weakly convergent) subsequence of $\{x_n\}$, we conclude that $x_n \to z'$. From (3.28), we know that $u_n \to z'$ also. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, let $\phi : C \times C \to R$ be a bifunction satisfying the conditions (A1)–(A4), let A be an α-inverse strongly* *monotone mapping of C into H, and let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$ (for all $m, n \in N$, for all $x, y \in C$) such that $F \cap EP \neq \emptyset$, where $F := \cap_{i=1}^{\infty} F(S_i)$. Let $T_n = \sum_{k=1}^{n} \beta_{n,k} S_k^n$ for every $n \in N$, where $0 \leq \beta_{n,k} \leq 1$ for every $n = 1, 2, 3, \ldots$ and $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \beta_{n,k} = 1$ for each $n \in N$ and $\lim_{n\to\infty}\beta_{n,k} > 0$ for every $k \in N$, and assume that $\gamma_n = \sum_{k=1}^{n} \beta_{n,k}(t_{k,n}^2 - 1)$ for every $n \in N$ such that $\lim_{n\to\infty}\gamma_n = 0$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by*

$$
\begin{gathered}
x_0 \in C \text{ chosen arbitrarily,} \\
u_n \in C \text{ such that } \phi(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = \alpha_n u_n + (1 - \alpha_n) T_n u_n, \\
C_n = \left\{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\right\}, \\
Q_n = \{v \in C : \langle x_0 - x_n, x_n - v\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_0, \ n \in N \cup \{0\},
\end{gathered}
\tag{3.43}
$$

where $\theta_n = (1 - \alpha_n)\gamma_n(\operatorname{diam} C)^2$ and $0 \leq \alpha_n \leq d < 1$ and $0 < e \leq r_n \leq f < 2\alpha$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $P_{F\cap EP}(x_0)$.

Proof. We divide the proof of Theorem 3.2 into four steps.

(i) We show first that the sequences $\{x_n\}$ and $\{u_n\}$ are well defined.

From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in N \cup 0$. We prove that C_n is convex. Since

$$
\|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n \tag{3.44}
$$

is equivalent to

$$
2\langle x_n - y_n, v\rangle \leq \|x_n\|^2 - \|y_n\|^2 + \theta_n, \tag{3.45}
$$

it follows that C_n is convex. So, $C_n \cap Q_n$ is a closed convex subset of H for any n.

Next, we show that $F \cap EP \subseteq C_n$. Indeed, let $p \in F \cap EP$, and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.5. Similar to the proof of Theorem 3.1, we have

$$
\|u_n - p\| \leq \|x_n - p\|. \tag{3.46}
$$

By virtue of the convexity of norm $\|\cdot\|^2$, (3.46), and Lemma 2.8, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(u_n - p) + (1-\alpha_n)(T_n u_n - p)\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)\|T_n u_n - p\|^2 \\ &\le \alpha_n\|u_n - p\|^2 + (1-\alpha_n)(1+\gamma_n)\|u_n - p\|^2 \\ &= \|u_n - p\|^2 + (1-\alpha_n)\gamma_n\|u_n - p\|^2 \\ &\le \|u_n - p\|^2 + \theta_n \\ &\le \|x_n - p\|^2 + \theta_n.\end{aligned} \tag{3.47}$$

Therefore, $p \in C_n$ for all n.

Next, we prove that $F \cap EP \subseteq Q_n$, for all $n \ge 0$. For $n = 0$, we have $F \cap EP \subseteq C = Q_0$. Assume that $F \cap EP \subseteq Q_{n-1}$. Since x_n is the projection of x_0 onto $C_{n-1} \cap Q_{n-1}$, by Lemma 2.3, we have

$$\langle x_0 - x_n, x_n - v\rangle \ge 0, \quad \forall v \in C_{n-1} \cap Q_{n-1}. \tag{3.48}$$

In particular, we have

$$\langle x_0 - x_n, x_n - p\rangle \ge 0 \tag{3.49}$$

for each $p \in F \cap EP$ and hence $p \in Q_n$. Hence $F \cap EP \subset Q_n$, for all $n \ge 0$. Therefore, we obtain that

$$F \cap EP \subseteq C_n \cap Q_n, \quad \forall n \ge 0. \tag{3.50}$$

This implies that $\{x_n\}$ is well defined. From Lemma 2.4, we know that $\{u_n\}$ is also well defined.

(ii) We prove that $\|x_{n+1} - x_n\| \to 0$, $\|x_n - u_n\| \to 0$, $\|u_{n+1} - u_n\| \to 0$, $\|u_n - T_n u_n\| \to 0$, as $n \to \infty$.

Since $F \cap EP$ is a nonempty closed convex subset of H, there exists a unique $z' \in F \cap EP$ such that $z' = P_{F\cap EP}x_0$.

From $x_{n+1} = P_{C_n \cap Q_n}x_0$, we have

$$\|x_{n+1} - x_0\| \le \|v - x_0\| \quad \forall v \in C_n \cap Q_n,\ \forall n \in N \cup \{0\}. \tag{3.51}$$

Since $z' \in F \cap EP \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x_0\| \le \|z' - x_0\| \quad \forall n \in N \cup \{0\}. \tag{3.52}$$

Since C is bounded, we have $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ are bounded. From the definition of Q_n, we have $x_n = P_{Q_n}x_0$, which together with the fact that $x_{n+1} \in C_n \cap Q_n \subset Q_n$ implies that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|, \qquad \langle x_0 - x_n, x_{n+1} - x_n \rangle \le 0. \tag{3.53}$$

This shows that the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. So, $\lim_{n\to\infty}\|x_n - x_0\|$ exists.

It follows from (2.1) and (3.53) that

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0 \rangle \\ &\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2. \end{aligned} \tag{3.54}$$

Noticing that $\lim_{n\to\infty}\|x_n - x_0\|$ exists, this implies that

$$\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0. \tag{3.55}$$

Since $x_{n+1} \in C_n$, we have

$$\left\|y_n - x_{n+1}\right\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n. \tag{3.56}$$

So, we have $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$. It follows that

$$\left\|y_n - x_n\right\| \le \left\|y_n - x_{n+1}\right\| + \|x_{n+1} - x_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.57}$$

Similar to the proof of Theorem 3.1, we have

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.58}$$

From (3.55) and (3.58), we have

$$\|u_n - u_{n+1}\| \le \|u_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - u_{n+1}\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.59}$$

Similarly, from (3.57) and (3.58), one has

$$\left\|y_n - u_n\right\| \le \left\|y_n - x_n\right\| + \|x_n - u_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.60}$$

Noticing the condition $0 \le \alpha_n \le d < 1$, it follows that

$$(1 - \alpha_n)\|T_n u_n - u_n\| = \left\|y_n - u_n\right\|, \tag{3.61}$$

which implies that

$$\|T_n u_n - u_n\| = \frac{\|y_n - u_n\|}{1-\alpha_n} < \frac{\|y_n - u_n\|}{1-d} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.62}$$

(iii) We prove that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z, where $z \in F \cap EP$.

Since $\{x_n\}$ is bounded and C is closed, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to z, where $z \in C$. From (3.58), we have $u_{n_i} \rightharpoonup z$. Noticing (3.59) and (3.62), it follows from Lemma 2.8 that $z \in F$. By using the same method as in the proof of Theorem 3.1, we easily obtain that $z \in EP$.

(iv) Finally we show that $x_n \to z$, $u_n \to z$, where $z = P_{F\cap EP}(x_0)$.

Since $x_n = P_{Q_n}x_0$ and $z \in F \cap EP \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|z - x_0\|. \tag{3.63}$$

It follows from $z' = P_{F\cap EP}x_0$ and the weak lower-semicontinuity of the norm that

$$\|z' - x_0\| \leq \|z - x_0\| \leq \liminf_{i\to\infty}\|x_{n_i} - x_0\| \leq \limsup_{i\to\infty}\|x_{n_i} - x_0\| \leq \|z' - x_0\|. \tag{3.64}$$

Thus, we obtain that $\lim_{i\to\infty}\|x_{n_i} - x_0\| = \|z - x_0\| = \|z' - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{i\to\infty} x_{n_i} = z = z'. \tag{3.65}$$

Since $\{x_{n_i}\}$ is an arbitrary subsequence of $\{x_n\}$, we conclude that $\{x_n\}$ converges strongly to $z = P_{F\cap EP}x_0$. By (3.58), we have $u_n \to z = P_{F\cap EP}x_0$ also. This completes the proof of Theorem 3.2. □

Corollary 3.3. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, let $\phi : C\times C \to R$ be a bifunction satisfying the conditions (A1)–(A4), let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$ (for all $m, n \in N$, for all $x, y \in C$) such that $F \cap EP(\phi) \neq \emptyset$, where $F := \cap_{i=1}^{\infty} F(S_i)$, and let $\{\beta_{n,k} : n, k \in N,\ 1 \leq k \leq n\}$ be a sequence of real numbers with $0 < a \leq \beta_{n,1} \leq 1$ for all $n \in N$ and $0 < b \leq \beta_{n,i} \leq c < 1$ for every $n \in N$ and $i = 2, \ldots, n$ for some $a, b, c \in (0,1)$. Let W_n be the modified W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$. Assume that*

$r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2 - 1)\}$ *for every* $n \in N$ *and* $k = 1, 2, \ldots, n$ *such that* $\lim_{n\to\infty} r_{n,1} = 0$. *Let* $\{x_n\}$ *and* $\{u_n\}$ *be sequences generated by the following algorithm:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily}, \\ u_n \in C \text{ such that } \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n u_n + (1 - \alpha_n)W_n u_n, \\ C_{n+1} = \left\{v \in C_n : \left\|y_n - v\right\|^2 \leq \|x_n - v\|^2 + \theta_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in N \cup \{0\}, \end{gathered} \tag{3.66}$$

where $C_0 = C$ *and* $\theta_n = (1 - \alpha_n)r_{n,1}(\operatorname{diam} C)^2$ *and* $0 \leq \alpha_n \leq d < 1$ *and* $\{r_n\} \subset (0, \infty)$ *such that* $\liminf_{n\to\infty} r_n > 0$. *Then* $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $P_{F\cap EP(\phi)}(x_0)$.

Proof. Putting $A = 0$, the conclusion of Corollary 3.3 can be obtained as in the proof of Theorem 3.1. □

Remark 3.4. Corollary 3.3 extends the Theorem of Tada and Takahashi [19] in the following senses:

(1) from one nonexpansive mapping to a family of infinitely asymptotically nonexpansive mappings;

(2) from computation point of view, the algorithm in Corollary 3.3 is also simpler and, more convenient to compute than the one given in [19].

Corollary 3.5. *Let* C *be a nonempty bounded closed convex subset of a real Hilbert space* H, *let* $\{S_m\}$ *be a family of asymptotically nonexpansive mappings of* C *into itself with Lipschitz constants* $\{t_{m,n}\}$, *that is,* $\|S_m^n x - S_m^n y\| \leq t_{m,n}\|x - y\|$ *(for all* $m, n \in N$, *for all* $x, y \in C$*) such that* $F := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$, *and let* $\{\beta_{n,k} : n, k \in N,\ 1 \leq k \leq n\}$ *be a sequence of real numbers with* $0 < a \leq \beta_{n,1} \leq 1$ *for all* $n \in N$ *and* $0 < b \leq \beta_{n,i} \leq c < 1$ *for every* $n \in N$ *and* $i = 2, \ldots, n$ *for some* $a, b, c \in (0, 1)$. *Let* W_n *be the modified* W*-mapping generated by* $S_n, S_{n-1}, \ldots, S_1$ *and* $\beta_{n,n}, \beta_{n,n-1}, \ldots, \beta_{n,2}, \beta_{n,1}$. *Assume that* $r_{n,k} = \{\beta_{n,k}(t_{k,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}t_{k,n}^2(t_{k+1,n}^2 - 1) + \cdots + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n-1}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-2,n}^2(t_{n-1,n}^2 - 1) + \beta_{n,k}\beta_{n,k+1}\cdots\beta_{n,n}t_{k,n}^2 t_{k+1,n}^2 \cdots t_{n-1,n}^2(t_{n,n}^2 - 1)\}$ *for every* $n \in N$ *and* $k = 1, 2, \ldots, n$ *such that* $\lim_{n\to\infty} r_{n,1} = 0$. *Let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$\begin{gathered} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = W_n x_n, \\ C_{n+1} = \left\{v \in C_n : \left\|y_n - v\right\|^2 \leq \|x_n - v\|^2 + \theta_n\right\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in N \cup \{0\}, \end{gathered} \tag{3.67}$$

where $C_0 = C$ *and* $\theta_n = r_{n,1}(\operatorname{diam} C)^2$. *Then* $\{x_n\}$ *converges strongly to* $P_F(x_0)$.

Proof. Putting $\phi(x,y) \equiv 0$, for all $x,y \in C$, $r_n = 1$, $A = 0$ and $\alpha_n = 0$, for all $n \in N$ in Theorem 3.1, we have $u_n = P_C x_n = x_n$, therefore $y_n = W_n u_n = W_n x_n$. The conclusion of Corollary 3.5 can be obtained from Theorem 3.1 immediately. □

Remark 3.6. Corollary 3.5 extends Theorem 3.1 of Inchan and Plubtieng [16] from two asymptotically nonexpansive mappings to an infinite family of asymptotically nonexpansive mappings.

Corollary 3.7. *Let C be a nonempty bounded closed convex subset of a real Hilbert space H, and let $\{S_m\}$ be a family of asymptotically nonexpansive mappings of C into itself with Lipschitz constants $\{t_{m,n}\}$, that is, $\|S_m^n x - S_m^n y\| \le t_{m,n}\|x-y\|$ (for all $m,n \in N$, for all $x,y \in C$) such that $F := \cap_{i=1}^{\infty} F(S_i) \neq \emptyset$. Let $T_n = \sum_{k=1}^{n} \beta_{n,k} S_k^n$ for every $n \in N$, where $0 \le \beta_{n,k} \le 1$ for every $n = 1,2,3,\ldots$ and $k = 1,2,\ldots,n$ with $\sum_{k=1}^{n} \beta_{n,k} = 1$ for each $n \in N$ and $\lim_{n\to\infty} \beta_{n,k} > 0$ for every $k \in N$, and assume that $\gamma_n = \sum_{k=1}^{n} \beta_{n,k}(t_{k,n}^2 - 1)$ for every $n \in N$ such that $\lim_{n\to\infty} \gamma_n = 0$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} &x_0 \in C \text{ chosen arbitrarily}, \\ &y_n = T_n x_n, \\ &C_n = \left\{ v \in C : \|y_n - v\|^2 \le \|x_n - v\|^2 + \theta_n \right\}, \\ &Q_n = \{ v \in C : \langle x_0 - x_n, x_n - v \rangle \ge 0 \}, \\ &x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n \in N \cup \{0\}, \end{aligned} \tag{3.68}$$

where $C_0 = C$ and $\theta_n = \gamma_n (\operatorname{diam} C)^2$. Then $\{x_n\}$ converges strongly to $P_F(x_0)$.

Proof. Putting $\phi(x,y) \equiv 0$, for all $x,y \in C$, $r_n = 1$, $A = 0$ and $\alpha_n = 0$, for all $n \in N$ in Theorem 3.2, we have $u_n = P_C x_n = x_n$, therefore $y_n = T_n u_n = T_n x_n$. The conclusion of Corollary 3.7 can be obtained from Theorem 3.2. □

Remark 3.8. Corollary 3.7 extends Theorem 3.1 of Zegeye and Shahzad [18] from a finite family of asymptotically nonexpansive mappings to an infinite family of asymptotically nonexpansive mappings.

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