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Research Article

A Fixed Point Approach to the Stability of the Functional Equation $f(x+y) = F[f(x), f(y)]$

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By applying the fixed point method, we will prove the Hyers-Ulam-Rassias stability of the functional equation $f(x+y) = F[f(x), f(y)]$ under some additional assumptions on the function F and spaces involved.

1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms: "Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?"

The case of approximately additive functions was solved by Hyers [2] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$, for all x and y, can be approximated by an exact solution, say an additive function. Rassias [3] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows:

$$\left\|f(x+y) - f(x) - f(y)\right\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

and derived Hyers' theorem for the stability of the additive mapping as a special case. Thus in [3], a proof of the generalized Hyers-Ulam stability for the linear mapping between Banach spaces was obtained. A particular case of Rassias' theorem regarding the Hyers-Ulam stability of the additive mapping was proved by Aoki (see [4]).

The stability concept that was introduced by Rassias' theorem provided a large influence to a number of mathematicians to develop the notion of what is known today with the term Hyers-Ulam-Rassias stability of the linear mapping. Since then, the stability of several functional equations has been extensively investigated by several mathematicians. The terminology Hyers-Ulam-Rassias stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [5–10].

Solutions of the functional equation

$$f(x+y) = F[f(x), f(y)] \tag{1.2}$$

were investigated in [11, Section 2.2]. The stability problem for a general equation of the form $f[G(x,y)] = H[f(x), f(y)]$ was investigated by Cholewa [12] (see also [13]). Indeed, Cholewa proved the superstability of that equation under some additional assumptions on the functions and spaces involved.

In this paper, we will apply the fixed point method to prove the Hyers-Ulam-Rassias stability of the functional equation (1.2) for a class of functions of a vector space into a Banach space. To the best of authors' knowledge, no one has yet applied the fixed point method for studying the stability problems of (1.2). So, one of the aims of this paper is to apply the fixed point theory to this case.

Throughout this paper, let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X and Y be a vector space over $\mathbb{K}$ and a Banach space over $\mathbb{K}$, respectively.

2. Preliminaries

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(M_1) $d(x,y) = 0$ if and only if $x = y$;

(M_2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(M_3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity. We now introduce one of fundamental results of fixed point theory. For the proof, refer to [14]. For an extensive theory of fixed point theorems and other nonlinear methods the reader is referred to the book of Hyers et al. [15].

Theorem 2.1. *Let* $(\mathbb{X}, d)$ *be a generalized complete metric space. Assume that* $\Lambda : \mathbb{X} \to \mathbb{X}$ *is a strictly contractive operator with the Lipschitz constant* $0 < L < 1$. *If there exists a nonnegative integer* k *such* *that* $d(\Lambda^{k+1}f, \Lambda^k f) < \infty$ *for some* $f \in \mathbb{X}$*, then the followings are true:*

(a) *the sequence* $\{\Lambda^n f\}$ *converges to a fixed point* f^* *of* Λ*;*

(b) f^* *is the unique fixed point of* Λ *in*

$$\mathbb{X}^* = \left\{g \in \mathbb{X} : d\left(\Lambda^k f, g\right) < \infty\right\}; \tag{2.1}$$

(c) *if* $g \in \mathbb{X}^*$*, then*

$$d(g, f^*) \leq \frac{1}{1-L} d(\Lambda g, g). \tag{2.2}$$

Recently, Cădariu and Radu [16] applied the fixed point method to the investigation of the Cauchy additive functional equation [17, 18]. Using such a clever idea, they could present a short, simple proof for the Hyers-Ulam-Rassias stability of Cauchy and Jensen functional equations.

We remark that Isac and Rassias [19] were the first mathematicians who apply the Hyers-Ulam-Rassias stability approach for the proof of new fixed point theorems.

3. Main Results

In this section, by using an idea of Cădariu and Radu (see [16, 17]), we will prove the Hyers-Ulam-Rassias stability of the functional equation $f(x+y) = F[f(x), f(y)]$ under the assumption that F is a bounded linear transformation.

Theorem 3.1. *Let* X *and* $(Y, \|\cdot\|)$ *be a vector space over* $\mathbb{K}$ *and a Banach space over* $\mathbb{K}$*, respectively, and let* $(Y \times Y, \|\cdot\|_2)$ *be a Banach space over* $\mathbb{K}$*. Assume that* $F : Y \times Y \to Y$ *is a bounded linear transformation, whose norm is denoted by* $\|F\|$*, satisfying*

$$F[F(u,u), F(v,v)] = F[F(u,v), F(u,v)] \tag{3.1}$$

for all $u, v \in Y$ *and that there exists a real number* $\kappa > 0$ *with*

$$\|(u,u) - (v,v)\|_2 \leq \kappa \|u - v\| \tag{3.2}$$

for all $u, v \in Y$*. Moreover, assume that* $\varphi : X \times X \to [0, \infty)$ *is a given function satisfying*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \varphi(x, y) \tag{3.3}$$

for all $x, y \in X$*. If* $\kappa\|F\| < 1$ *and a function* $f : X \to Y$ *satisfies the inequality*

$$\left\| f(x+y) - F\left[f(x), f(y)\right] \right\| \leq \varphi(x, y) \tag{3.4}$$

for any $x, y \in X$, *then there exists a unique solution* $f^* : X \to Y$ *of* (1.2) *such that*

$$\|f(x) - f^*(x)\| \le \frac{1}{1 - \kappa\|F\|}\varphi(x, x) \tag{3.5}$$

for all $x \in X$.

Proof. First, we denote by $\mathbb{X}$ the set of all functions $h : X \to Y$ and by d the generalized metric on $\mathbb{X}$ defined as

$$d(g, h) = \inf\{C \in [0, \infty] : \|g(x) - h(x)\| \le C\varphi(x, x) \ \forall x \in X\}. \tag{3.6}$$

Then, as in the proof of [20, Theorem 3.1], we can show that $(\mathbb{X}, d)$ is a generalized complete metric space. Now, let us define an operator $\Lambda : \mathbb{X} \to \mathbb{X}$ by

$$(\Lambda h)(x) = F\left[h\left(\frac{x}{2}\right), h\left(\frac{x}{2}\right)\right] \tag{3.7}$$

for every $x \in X$.

We assert that Λ is strictly contractive on $\mathbb{X}$. Given $g, h \in \mathbb{X}$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$, that is,

$$\|g(x) - h(x)\| \le C\varphi(x, x) \tag{3.8}$$

for each $x \in X$. By (3.2), (3.3), (3.7), and (3.8), we have

$$\begin{aligned} \|(\Lambda g)(x) - (\Lambda h)(x)\| &= \left\|F\left[g\left(\frac{x}{2}\right), g\left(\frac{x}{2}\right)\right] - F\left[h\left(\frac{x}{2}\right), h\left(\frac{x}{2}\right)\right]\right\| \\ &\le \|F\| \left\|\left(g\left(\frac{x}{2}\right), g\left(\frac{x}{2}\right)\right) - \left(h\left(\frac{x}{2}\right), h\left(\frac{x}{2}\right)\right)\right\|_2 \\ &\le \|F\|\kappa \left\|g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| \\ &\le \kappa\|F\|C\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le \kappa\|F\|C\varphi(x, x) \end{aligned} \tag{3.9}$$

for all $x \in X$, that is, in view of (3.6), $d(\Lambda g, \Lambda h) \le \kappa\|F\|d(g, h)$ for any $g, h \in \mathbb{X}$, where $\kappa\|F\|$ is the Lipschitz constant with $0 < \kappa\|F\| < 1$. Thus, Λ is strictly contractive.

We now verify that $d(\Lambda f, f) < \infty$. If we substitute $x/2$ for x and y in (3.4), then it follows from (3.3) and (3.7) that

$$\|f(x) - (\Lambda f)(x)\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \varphi(x, x) \tag{3.10}$$

for every $x \in X$, that is, $d(\Lambda f, f) \le 1$.

Taking $k = 0$ in Theorem 2.1, (a) implies that there exists a function $f^* : X \to Y$, which is a fixed point of Λ, such that

$$\lim_{n\to\infty} d(\Lambda^n f, f^*) = 0. \tag{3.11}$$

Due to Theorem 2.1(c), we get

$$d(f, f^*) \le \frac{1}{1-\kappa\|F\|} d(\Lambda f, f) \le \frac{1}{1-\kappa\|F\|}, \tag{3.12}$$

which implies the validity of (3.5). According to Theorem 2.1(b), f^* is the unique fixed point of Λ with $d(f, f^*) < \infty$.

We now assert that

$$\|(\Lambda^n f)(x+y) - F[(\Lambda^n f)(x), (\Lambda^n f)(y)]\| \le (\kappa\|F\|)^n \varphi(x,y) \tag{3.13}$$

for all $n \in \mathbb{N}$ and $x, y \in X$. Indeed, it follows from (3.1), (3.2), (3.3), (3.4), and (3.7) that

$$\begin{aligned}
&\|(\Lambda f)(x+y) - F[(\Lambda f)(x), (\Lambda f)(y)]\| \\
&\quad = \left\| F\left[f\left(\frac{x+y}{2}\right), f\left(\frac{x+y}{2}\right)\right] - F\left[F\left[f\left(\frac{x}{2}\right), f\left(\frac{y}{2}\right)\right], F\left[f\left(\frac{x}{2}\right), f\left(\frac{y}{2}\right)\right]\right]\right\| \\
&\quad \le \|F\|\kappa \left\| f\left(\frac{x+y}{2}\right) - F\left[f\left(\frac{x}{2}\right), f\left(\frac{y}{2}\right)\right]\right\| \\
&\quad \le \kappa\|F\|\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \\
&\quad \le \kappa\|F\|\varphi(x,y)
\end{aligned} \tag{3.14}$$

for any $x, y \in X$. We assume that (3.13) is true for some $n \in \mathbb{N}$. Then, it follows from (3.1), (3.2), (3.3), (3.7), and (3.13) that

$$\begin{aligned}
&\left\|\left(\Lambda^{n+1} f\right)(x+y) - F\left[\left(\Lambda^{n+1} f\right)(x), \left(\Lambda^{n+1} f\right)(y)\right]\right\| \\
&\quad = \left\| F\left[(\Lambda^n f)\left(\frac{x+y}{2}\right), (\Lambda^n f)\left(\frac{x+y}{2}\right)\right]\right. \\
&\qquad \left. - F\left[F\left[(\Lambda^n f)\left(\frac{x}{2}\right), (\Lambda^n f)\left(\frac{x}{2}\right)\right], F\left[(\Lambda^n f)\left(\frac{y}{2}\right), (\Lambda^n f)\left(\frac{y}{2}\right)\right]\right]\right\| \\
&\quad \le \|F\|\kappa \left\| (\Lambda^n f)\left(\frac{x+y}{2}\right) - F\left[(\Lambda^n f)\left(\frac{x}{2}\right), (\Lambda^n f)\left(\frac{y}{2}\right)\right]\right\| \\
&\quad \le (\kappa\|F\|)^{n+1}\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \\
&\quad \le (\kappa\|F\|)^{n+1}\varphi(x,y),
\end{aligned} \tag{3.15}$$

which proves the validity of (3.13) for all $n \in \mathbb{N}$.

Finally, we prove that $f^*(x+y) = F[f^*(x), f^*(y)]$ for any $x, y \in X$. Since F is continuous as a bounded linear transformation, it follows from (3.11) and (3.13) that

$$\begin{aligned}
&\left\| f^*(x+y) - F\left[f^*(x), f^*(y)\right] \right\| \\
&\quad = \lim_{n\to\infty} \left\| (\Lambda^n f)(x+y) - F\left[(\Lambda^n f)(x), (\Lambda^n f)(y)\right] \right\| \\
&\quad \leq \lim_{n\to\infty} (\kappa\|F\|)^n \varphi(x,y) \\
&\quad = 0,
\end{aligned} \tag{3.16}$$

which ends our proof. □

Obviously, for nonnegative constants θ and p, $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$ satisfies the conditon (3.3).

Corollary 3.2. *Let X and $(Y, \|\cdot\|)$ be a vector space over $\mathbb{K}$ and a Banach space over $\mathbb{K}$, respectively, and let $(Y \times Y, \|\cdot\|_2)$ be a Banach space over $\mathbb{K}$. Assume that $F : Y \times Y \to Y$ is a bounded linear transformation, whose norm is denoted by $\|F\|$, satisfying the condition* (3.1) *and that there exists a real number $\kappa > 0$ satisfying the condition* (3.2). *If $\kappa\|F\| < 1$ and a function $f : X \to Y$ satisfies the inequality*

$$\left\| f(x+y) - F\left[f(x), f(y)\right] \right\| \leq \theta\left(\|x\|^p + \|y\|^p\right) \tag{3.17}$$

for all $x, y \in X$ and for some nonnegative real constants θ and p, then there exists a unique solution $f^* : X \to Y$ of (1.2) *such that*

$$\left\| f(x) - f^*(x) \right\| \leq \frac{2\theta}{1 - \kappa\|F\|} \|x\|^p \tag{3.18}$$

for all $x \in X$.

4. An Example

Assume that $X = Y = \mathbb{C}$ and consider the Banach spaces $(\mathbb{C}, |\cdot|)$ and $(\mathbb{C} \times \mathbb{C}, |\cdot|_2)$, where we define $|(u,v)|_2 = \sqrt{|u|^2 + |v|^2}$ for all $u, v \in \mathbb{C}$. Let A and B be fixed complex numbers with $|A| + |B| < 1/\sqrt{2}$ and let $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a linear transformation defined by

$$F(u,v) = Au + Bv. \tag{4.1}$$

Then it is easy to show that F satisfies the condition (3.1).

If u and v are complex numbers satisfying $|(u,v)|_2 \le 1$, then

$$|F(u,v)| \le |A||u| + |B||v| \le |A| + |B|. \tag{4.2}$$

Thus, we get

$$\|F\| = \sup\{|F(u,v)| : u,v \in \mathbb{C} \text{ with } |(u,v)|_2 \le 1\} \le |A| + |B|, \tag{4.3}$$

which implies the boundedness of the linear transformation F.

On the other hand, we obtain

$$|(u,u) - (v,v)|_2 = |(u-v, u-v)|_2 = \sqrt{2}|u-v| \tag{4.4}$$

for any $u, v \in \mathbb{C}$; that is, we can choose $\sqrt{2}$ for the value of κ and then we have

$$\kappa\|F\| \le \sqrt{2}(|A| + |B|) < 1. \tag{4.5}$$

If a function $f : \mathbb{C} \to \mathbb{C}$ satisfies the inequality

$$\left|f(x+y) - F\left[f(x), f(y)\right]\right| \le \varepsilon \tag{4.6}$$

for all $x, y \in \mathbb{C}$ and for some $\varepsilon > 0$, then our Corollary 3.2 (with $\theta = \varepsilon/2$ and $p = 0$) implies that there exists a unique function $f^* : \mathbb{C} \to \mathbb{C}$ such that

$$f^*(x+y) = F\left[f^*(x), f^*(y)\right] \tag{4.7}$$

for all $x, y \in \mathbb{C}$ and

$$\left|f(x) - f^*(x)\right| \le \frac{\varepsilon}{1 - \sqrt{2}(|A| + |B|)} \tag{4.8}$$

for any $x \in \mathbb{C}$.

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