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*Research Article*

# Intuitionistic Fuzzy Stability of a Quadratic Functional Equation

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We consider the intuitionistic fuzzy stability of the quadratic functional equation $f(kx+y)+f(kx-y)=2k^2f(x)+2f(y)$ by using the fixed point alternative, where $k$ is a positive integer.

## 1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [3] for additive mappings. In 1978, Rassias [4] generalized Hyers theorem by obtaining a unique linear mapping near an approximate additive mapping.

Assume that $E_1$ and $E_2$ are real-normed spaces with $E_2$ complete, $f : E_1 \to E_2$ is a mapping such that for each fixed $x \in E_1$, the mapping $t \to f(tx)$ is continuous on $\mathbb{R}$, and there exist $\varepsilon > 0$ and $p \in [0,1)$ such that

$$\left\| f(x+y) - f(x) - f(y) \right\| \le \varepsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.1}$$

for all $x, y \in E_1$. Then there is a unique linear mapping $T : E_1 \to E_2$ such that

$$\left\| f(x) - T(x) \right\| \le \frac{2\epsilon}{|2-2^p|}\|x\|^p \tag{1.2}$$

for all $x \in E_1$.

The paper of Rassias has provided a lot of influence in the development of what we called the generalized Hyers-Ulam-Rassias stability of functional equations. In 1990, Rassias [5] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [6] and Rassias and Semrl [7] that one cannot prove an analogous theorem when $p = 1$. In 1994, Gavruta [8] provided a generalization of Rassias theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. Since then several stability problems for various functional equations have been investigated by many mathematicians [9, 10].

In the following, we first recall some fundamental results in the fixed point theory.

Let $X$ be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies (1) $d(x, y) = 0$ if and only if $x = y$; (2) $d(x, y) = d(y, x)$ for all $x, y \in X$; (3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall the following theorem of Diaz and Margolis [11].

**Theorem 1.1** (see [11]). *Let $(X, d)$ be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $0 < \alpha < 1$. Then for each $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty \tag{1.3}$$

*for all nonnegative integers $n$ or there exists a nonnegative integer $n_0$ such that*

(1) $d(J^n x, J^{n+1} x) < \infty$ *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* $y^*$ *of* $J$;

(3) $y^*$ *is the unique fixed point of* $J$ *in the set* $Y = \{y \in X : d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1 - \alpha)) d(y, Jy)$ *for all* $y \in Y$.

In 2003, Cadariu and Radu used the fixed-point method to the investigation of the Jensen functional equation (see [12, 13]) for the first time. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors.

Using the idea of intuitionistic fuzzy metric spaces introduced by Park [14] and Saadati and Park [15, 16], a new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous $t$-representable was introduced by Shakeri [17]. We refer to [17] for the notions appeared below.

Consider the set $L^*$ and the order relation $\leq_{L^*}$ defined by

$$\begin{gathered} L^* = \left\{(x_1, x_2) : (x_1, x_2) \in [0, 1]^2,\ x_1 + x_2 \leq 1\right\}, \\ (x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1, x_2 \leq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*. \end{gathered} \tag{1.4}$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [18, 19].

A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is said to be a continuous $t$-norm if it satisfies the following conditions: (a) $*$ is associative and commutative; (b) $*$ is continuous; (c) $a * 1 = a$ for all $a \in [0, 1]$; (d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0, 1]$.

An intuitionistic fuzzy set $A_{\xi,\eta}$ in a universal set $U$ is an object $A_{\xi,\eta} = \{(\xi_A(u), \eta_A(u)) : u \in U\}$, where, for all $u \in U$, $\xi_A(u) \in [0, 1]$ and $\eta_A(u) \in [0, 1]$ are called the membership degree and the nonmembership degree, respectively, of $u \in A_{\xi,\eta}$ and, furthermore, they satisfy $\xi_A(u) + \eta_A(u) \leq 1$.

A triangular norm ($t$-norm) on $L^*$ is a mapping $T : (L^*)^2 \to L^*$ satisfying the following conditions: for all $x, y, x', y', z \in L^*$, (a) $(T(x, 1_{L^*}) = x)$ (boundary condition); (b) $(T(x, y) = T(y, x))$ (commutativity); (c) $(T(x, T(y, z)) = T(T(x, y), z))$ (associativity); (d) $(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Rightarrow T(x, y) \leq_{L^*} T(x', y'))$ (monotonicity).

If $(L^*, \leq_{L^*}, T)$ is an abelian topological monoid with unit $1_{L^*}$, then $T$ is said to be a continuous $t$-norm.

The definitions of an intuitionistic fuzzy normed space is given below (see [17]).

*Definition 1.2.* Let $\mu$ and $v$ be the membership and the nonmembership degree of an intuitionistic fuzzy set from $X \times (0, +\infty)$ to $[0, 1]$ such that $\mu_x(t) + v_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The triple $(X, P_{\mu,v}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if $X$ is a vector space, $T$ is a continuous $t$-representable, and $P_{\mu,v}$ is a mapping $X \times (0, +\infty) \to L^*$ satisfying the following conditions: for all $x, y \in X$ and $t, s > 0$,

(a) $P_{\mu,v}(x, 0) = 0_{L^*}$;

(b) $P_{\mu,v}(x, t) = 1_{L^*}$ if and only if $x = 0$;

(c) $P_{\mu,v}(ax, t) = P_{\mu,v}(x, t/a)$ for all $a \neq 0$;

(d) $P_{\mu,v}(x + y, t + s) \geq T(P_{\mu,v}(x, t), P_{\mu,v}(y, s))$.

In this case, $P_{\mu,v}$ is called an intuitionistic fuzzy norm. Here, $P_{\mu,v}(x, t) = (\mu_x(t), v_x(t))$.

Throughout this paper, we assume that $k$ is a fixed positive integer. The functional equation

$$f(kx + y) + f(kx - y) = 2k^2 f(x) + 2f(y) \tag{1.5}$$

was considered in [20]. Suppose $X$ and $Y$ are vector spaces. It is proved in [20] that a mapping $f : X \to Y$ satisfies (1.5) if and only if it satisfies $f(x + y) + f(x - y) = 2f(x) + 2f(y)$.

In this short note, we show the intuitionistic fuzzy stability of the functional equation (1.5) by using the fixed point alternative.

## 2. Main Results

For a given mapping $f : X \to Y$, we define

$$Df(x, y) = f(kx + y) + f(kx - y) - 2k^2 f(x) - 2f(y) \tag{2.1}$$

for all $x, y \in X$.

**Theorem 2.1.** *Let $X$ be a linear space, $(Z, P'_{\mu,v}, M)$ an IFN-space, and $\phi : X \times X \to Z$ a function such that for some $0 \leq \alpha < 1$,*

$$P'_{\mu,v}(\phi(kx, ky), t) \geq_{L^*} P'_{\mu,v}\left(\alpha k^2 \phi(x, y), t\right) \quad (x, y \in X,\ t > 0), \tag{2.2}$$

$$\lim_{n \to \infty} P'_{\mu,v}\left(\phi(k^n x, k^n y), k^{2n} t\right) = 1_{L^*} \tag{2.3}$$

*for all* $x, y \in X$ *and* $t > 0$*. Let* $(Y, P_{\mu,v}, M)$ *be a complete IFN-space. If* $f : X \to Y$ *is a mapping such that for all* $x, y \in X, t > 0$,

$$P_{\mu,v}(Df(x,y),t) \geq_{L^*} P'_{\mu,v}(\phi(x,y),t), \tag{2.4}$$

*and* $f(0) = 0$*, then there is a unique quadratic mapping* $A : X \to Y$ *such that*

$$P_{\mu,v}(f(x) - A(x),t) \geq_{L^*} P'_{\mu,v}\left(\phi(x,0), \left(2k^2 - 2k^2\alpha\right)t\right). \tag{2.5}$$

*Proof.* Put $y = 0$ in (2.4), we have

$$P_{\mu,v}\left(\frac{f(kx)}{k^2} - f(x), t\right) \geq_{L^*} P'_{\mu,v}\left(\frac{1}{2k^2}\phi(x,0), t\right) \tag{2.6}$$

for all $x \in X$ and $t > 0$. Consider the set $E = \{g : X \to Y\}$ and define a generalized metric $d$ on $E$ by

$$d(g,h) = \inf\left\{c \in R^+ : P_{\mu,v}(g(x) - h(x),t) \geq_{L^*} P'_{\mu,v}(c\phi(x,0),t),\ \forall x \in X, t > 0\right\}. \tag{2.7}$$

It is easy to show that $(E, d)$ is complete. Define $J : E \to E$ by $Jg(x) = (1/k^2)g(kx)$ for all $x \in X$. It is not difficult to see that

$$d(Jg, Jh) \leq \alpha d(g,h) \tag{2.8}$$

for all $g, h \in E$. It follows from (2.6) that

$$d(f, Jf) \leq \frac{1}{2k^2} < \infty. \tag{2.9}$$

It follows from Theorem 1.1 that $J$ has a fixed point in the set $E_1 = \{h \in E : d(f,h) < \infty\}$. Let $A$ be the fixed point of $J$. It follows from $\lim_n d(J^n f, A) = 0$ that

$$A(x) = \lim_{n\to\infty} \frac{1}{k^{2n}} f(k^n x) \tag{2.10}$$

for all $x \in X$. Since $d(f, A) \leq 1/(2k^2 - 2k^2\alpha)$,

$$P_{\mu,v}(f(x) - A(x),t) \geq_{L^*} P'_{\mu,v}\left(\phi(x,0), \left(2k^2 - 2k^2\alpha\right)t\right). \tag{2.11}$$

It follows from (2.4) that we have

$$P_{\mu,\nu}\left(\frac{1}{k^{2n}}Df(k^n x, k^n y), t\right) \geq_{L^*} P'_{\mu,\nu}\left(\phi(k^n x, k^n y), k^{2n} t\right). \tag{2.12}$$

It follows from (2.3) and [20] that $A$ is a quadratic mapping.

The uniqueness of $A$ follows from the fact that $A$ is the unique fixed point of $J$ with the property that

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x,y), \left(2k^2 - 2k^2\alpha\right)t\right). \tag{2.13}$$

This completes the proof. □

**Corollary 2.2.** *Let $0 < p < 2$. Let $X$ be a linear space, $(Z, P'_{\mu,\nu}, M)$ an IFN-space, and $(Y, P_{\mu,\nu}, M)$ a complete IFN-space. Suppose $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$, $t > 0$,*

$$P_{\mu,\nu}(Df(x,y), t) \geq_{L^*} P'_{\mu,\nu}((\|x\|^p + \|y\|^p) z_0, t), \tag{2.14}$$

*and $f(0) = 0$, then there is a unique quadratic mapping $A : X \to Y$ such that*

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\nu}\left(\|x\|^p z_0, \left(2k^2 - 2k^p\right)t\right). \tag{2.15}$$

*Proof.* Let

$$\phi(x,y) = (\|x\|^p + \|y\|^p) z_0 \tag{2.16}$$

for all $x, y \in X$. The result follows from Theorem 2.1 with $\alpha = k^{p-2}$. □

**Theorem 2.3.** *Let $X$ be a linear space, $(Z, P'_{\mu,\nu}, M)$ an IFN-space, and $\phi : X \times X \to Z$ a function such that for some $0 \leq \alpha < 1$,*

$$\begin{gathered} P'_{\mu,\nu}(\phi(x,y), t) \geq_{L^*} P'_{\mu,\nu}\left(\frac{\alpha}{k^2}\phi(kx, ky), t\right) \quad (x, y \in X, t > 0), \\ \lim_{n\to\infty} P'_{\mu,\nu}\left(\phi\left(\frac{x}{k^n}, \frac{y}{k^n}\right), \frac{1}{k^{2n}}t\right) = 1_{L^*} \end{gathered} \tag{2.17}$$

*for all $x, y \in X$ and $t > 0$. Let $(Y, P_{\mu,\nu}, M)$ be a complete IFN-space. If $f : X \to Y$ is a mapping such that for all $x, y \in X$, $t > 0$,*

$$P_{\mu,\nu}(Df(x,y), t) \geq_{L^*} P'_{\mu,\nu}(\phi(x,y), t), \tag{2.18}$$

*and* $f(0) = 0$, *then there is a unique quadratic mapping* $A : X \to Y$ *such that*

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\nu}\left(\phi(x,0), \frac{2k^2 - 2k^2\alpha}{\alpha}t\right). \tag{2.19}$$

*Proof.* The proof is similar to that of Theorem 2.1 and we omit it. □

**Corollary 2.4.** *Let* $p > 2$. *Let* $X$ *be a linear space,* $(Z, P'_{\mu,\nu}, M)$ *an IFN-space, and* $(Y, P_{\mu,\nu}, M)$ *a complete IFN-space. If* $f : X \to Y$ *is a mapping such that for all* $x, y \in X, t > 0$,

$$P_{\mu,\nu}(Df(x,y), t) \geq_{L^*} P'_{\mu,\nu}((\|x\|^p + \|y\|^p)z_0, t), \tag{2.20}$$

*and* $f(0) = 0$, *then there is a unique quadratic mapping* $A : X \to Y$ *such that*

$$P_{\mu,\nu}(f(x) - A(x), t) \geq_{L^*} P'_{\mu,\nu}\left(\|x\|^p z_0, \left(2k^p - 2k^2\right)t\right). \tag{2.21}$$

*Proof.* The proof is similar to that of Corollary 2.2. □

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