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Research Article

Fixed Point Theorems for Nonlinear Operators with and without Monotonicity in Partially Ordered Banach Spaces

Hui-Sheng Ding,[1] Jin Liang,[2] and Ti-Jun Xiao[3]

[1] *College of Mathematics and Information Science, Jiangxi Normal University, Nanchang, Jiangxi 330022, China*
[2] *Department of Mathematics, Shanghai Jiao Tong University, Shanghai 200240, China*
[3] *Shanghai Key Laboratory for Contemporary Applied Mathematics, School of Mathematical Sciences, Fudan University, Shanghai 200433, China*

Correspondence should be addressed to Ti-Jun Xiao, xiaotj@ustc.edu.cn

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We establish two fixed point theorems for nonlinear operators on Banach spaces partially ordered by a cone. The first fixed point theorem is concerned with a class of mixed monotone operators. In the second fixed point theorem, the nonlinear operators are neither monotone nor mixed monotone. We also provide an illustrative example for our second result.

1. Introduction

Fixed point theorems for nonlinear operators on partially ordered Banach spaces have many applications in nonlinear equations and many other subjects (cf., e.g., [1–7] and references therein); in particular, various kinds of fixed point theorems for mixed monotone operators are proved and applied (see, e.g., [1, 3, 5, 7] and references therein).

Stimulated by [7, 8], we investigate further, in this paper, the existence of fixed points of nonlinear operators with and without monotonicity in partially ordered Banach spaces.

In Section 2, a fixed point theorem for a class of mixed monotone operators is established. In Section 3, without any monotonicity assumption for a class of nonlinear operators, we obtain a fixed point theorem by using Hilbert's projection metric.

Let us recall some basic notations about cone (for more details, we refer the reader to [2]). Let X be a real Banach space. A closed convex set P in X is called a convex cone if the following conditions are satisfied:

(i) if $x \in P$, then $\lambda x \in P$ for any $\lambda \geq 0$,

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

A cone P induces a partial ordering $\leq$ in X by

$$x \leq y \quad \text{iff } y - x \in P. \tag{1.1}$$

For any given $u, v \in P$,

$$[u, v] := \{x \in X \mid u \leq x \leq v\}. \tag{1.2}$$

A cone P is called normal if there exists a constant $k > 0$ such that

$$0 \leq x \leq y \text{ implies that } \|x\| \leq k\|y\|, \tag{1.3}$$

where $\|\cdot\|$ is the norm on X.

Throughout this paper, we denote by $\mathbb{N}$ the set of nonnegative integers, $\mathbb{R}$ the set of real numbers, X a real Banach space, P a convex cone in X, e an element in $P \setminus \{\theta\}$ (θ is the zero element of X), and P_e the following set:

$$P_e = \{x \in P : \exists \alpha, \beta > 0 \text{ such that } \alpha e \leq x \leq \beta e\}. \tag{1.4}$$

2. Monotonic Operators

Theorem 2.1. *Suppose that the operator $A : P_e \times P_e \times P_e \to P_e$ satisfies the following.*

(S1) *$A(\cdot, y, z)$ is increasing, $A(x, \cdot, z)$ is decreasing, and $A(x, y, \cdot)$ is decreasing.*

(S2) *There exist a constant $t_0 \in [0, 1)$ and a function $\phi : (0, 1) \times P_e \times P_e \to (0, +\infty)$ such that for each $x, y, z \in P_e$ and $t \in (t_0, 1)$, $\phi(t, x, y) > t$ and*

$$A\left(tx, t^{-1}y, z\right) \geq \phi(t, x, y) A(x, y, z). \tag{2.1}$$

(S3) *There exist $x_0, y_0 \in P_e$ such that $x_0 \leq y_0$, $x_0 \leq A(x_0, y_0, x_0)$, $A(y_0, x_0, y_0) \leq y_0$ and*

$$\inf_{x, y \in [x_0, y_0]} \phi(t, x, y) > t, \quad \forall t \in (t_0, 1). \tag{2.2}$$

(S4) *There exists a constant $L > 0$ such that, for all $x, y, z_1, z_2 \in P_e$ with $z_1 \geq z_2$,*

$$A(x, y, z_1) - A(x, y, z_2) \geq -L \cdot (z_1 - z_2). \tag{2.3}$$

Then A has a unique fixed point x^* in $[x_0, y_0]$, that is, $A(x^*, x^*, x^*) = x^*$.

Proof. The proof is divided into 4 steps.

Step 1. Let $t_1 \in (t_0, 1)$ and

$$\psi(t,x,y) = \frac{\phi(t_1,x,y)}{t_1} t, \quad t \in (0,1), \quad x,y \in P_e. \tag{2.4}$$

For each $t \in (0,1)$, there exists a nonnegative integer k such that $t_1^{k+1} \le t < t_1^k$, that is, $t_1 \le t/t_1^k < 1$. Now, by (S2), we deduce, for all $x,y,z \in P_e$,

$$\begin{aligned} A\left(tx, t^{-1}y, z\right) &= A\left(\frac{t}{t_1^k} \cdot t_1^k x, \frac{t_1^k}{t} \cdot t_1^{-k} y, z\right) \\ &\ge \phi\left(\frac{t}{t_1^k}, t_1^k x, t_1^{-k} y\right) A\left(t_1^k x, t_1^{-k} y, z\right) \\ &\ge \frac{t}{t_1^k} A\left(t_1^k x, t_1^{-k} y, z\right) \\ &\ge \frac{t}{t_1} A\left(t_1 x, t_1^{-1} y, z\right) \\ &\ge \frac{t}{t_1} \phi(t_1,x,y) A(x,y,z) \\ &= \psi(t,x,y) A(x,y,z). \end{aligned} \tag{2.5}$$

Moreover, by (S3), we get

$$\inf_{x,y\in[x_0,y_0]} \psi(t,x,y) = \frac{\inf_{x,y\in[x_0,y_0]} \phi(t_1,x,y)}{t_1} \cdot t > t, \quad \forall t \in (0,1). \tag{2.6}$$

Hence, in the following proof, one can assume that $t_0 = 0$ in (S2) and (S3) without loss.

Step 2. Fix $x,y \in P_e$. Then, there exists $\alpha \in (0,1]$ such that $x,y \in [\alpha x_0, \alpha^{-1} y_0]$. Let

$$\Psi_{xy}(z) = \frac{A(x,y,z) + Lz}{1+L}, \quad z \in P_e. \tag{2.7}$$

Then Ψ_{xy} is an operator from P_e to P_e, and by (S4), Ψ_{xy} is increasing in P_e. Combining (S1)–(S3), we have

$$A(x,y,\alpha x_0) \ge A\left(\alpha x_0, \alpha^{-1} y_0, x_0\right) \ge \phi(\alpha, x_0, y_0) A(x_0,y_0,x_0) \ge \alpha x_0, \tag{2.8}$$

provided that $\alpha \in (0,1)$. Moreover, it is easy to see that (2.8) holds when $\alpha = 1$. Similarly, one can show that

$$A\left(x, y, \alpha^{-1} y_0\right) \leq \alpha^{-1} y_0. \tag{2.9}$$

Then, it follows that

$$\Psi_{xy}(\alpha x_0) \geq \alpha x_0, \quad \Psi_{xy}\left(\alpha^{-1} y_0\right) \leq \alpha^{-1} y_0. \tag{2.10}$$

Let

$$\begin{gathered} x_{xy}^n = \Psi_{xy}\left(x_{xy}^{n-1}\right), \quad y_{xy}^n = \Psi_{xy}\left(y_{xy}^{n-1}\right), \quad n = 1, 2, \ldots, \\ x_{xy}^0 = \alpha x_0, y_{xy}^0 = \alpha^{-1} y_0. \end{gathered} \tag{2.11}$$

Then, using arguments similar to those in the proof of [7, Theorem 2.1], one can show that Ψ_{xy} has a unique fixed point x_{xy}^* in $[\alpha x_0, \alpha^{-1} y_0]$, and

$$x_{xy}^n \longrightarrow x_{xy}^*, \quad y_{xy}^n \longrightarrow x_{xy}^* \quad (n \longrightarrow \infty). \tag{2.12}$$

We claim that x_{xy}^* is the unique fixed point of Ψ_{xy} in P_e. In fact, let y_{xy}^* be a fixed point of Ψ_{xy} in P_e, and $\beta \in (0, \alpha)$ such that $y_{xy}^* \in [\beta x_0, \beta^{-1} y_0]$. By the above proof, Ψ_{xy} has a unique fixed point in $[\beta x_0, \beta^{-1} y_0]$, which means that $x_{xy}^* = y_{xy}^*$. In addition, it follows from

$$x_{xy}^* = \Psi_{xy}\left(x_{xy}^*\right) = \frac{A\left(x, y, x_{xy}^*\right) + L x_{xy}^*}{1 + L} \tag{2.13}$$

that $x_{xy}^* = A(x, y, x_{xy}^*)$.

Step 3. By Step 2, we can define an operator $\Phi : P_e \times P_e \to P_e$ by

$$\Phi(x, y) = x_{xy}^* = \Psi_{xy}\left(x_{xy}^*\right) = A\left(x, y, x_{xy}^*\right). \tag{2.14}$$

Let $x, x' \in [x_0, y_0]$ with $x \leq x'$ and $\alpha \in (0,1]$ with $x, x', y \in [\alpha x_0, \alpha^{-1} y_0]$. Denote by $\{x_{xy}^n\}, \{x_{x'y}^n\}$ the corresponding sequences in the proof of Step 2. Then

$$\begin{aligned} x_{xy}^1 = \Psi_{xy}\left(x_{xy}^0\right) = \Psi_{xy}(\alpha x_0) = \frac{A(x, y, \alpha x_0) + L\alpha x_0}{1 + L} \\ \leq \frac{A(x', y, \alpha x_0) + L\alpha x_0}{1 + L} = \Psi_{x'y}(\alpha x_0) = x_{x'y}^1. \end{aligned} \tag{2.15}$$

Next, by induction and Ψ_{xy} being increasing, one can show that $x_{xy}^n \le x_{x'y}^n$ for all $n \in \mathbb{N}$. So

$$x_{xy}^* = \lim_{n\to\infty} x_{xy}^n \le \lim_{n\to\infty} x_{x'y}^n = x_{x'y}^*, \tag{2.16}$$

that is, $\Phi(x,y) \le \Phi(x',y)$. Thus, $\Phi(\cdot,y)$ is increasing. By a similar method, one can prove that $\Phi(x,\cdot)$ is decreasing. On the other hand, by (S3), for $x,y \in P_e$ and $t \in (0,1)$,

$$\begin{aligned}\Phi\left(tx,t^{-1}y\right) &= A\left(tx,t^{-1}y,\Phi\left(tx,t^{-1}y\right)\right) \\ &\ge A\left(tx,t^{-1}y,\Phi(x,y)\right) \\ &\ge \phi(t,x,y)A(x,y,\Phi(x,y)) \\ &= \phi(t,x,y)\Phi(x,y).\end{aligned} \tag{2.17}$$

Let $u_0 = x_0, v_0 = y_0$, and

$$u_n = \Phi(u_{n-1},v_{n-1}), \quad v_n = \Phi(v_{n-1},u_{n-1}), \quad \text{for} \quad n = 1,2,\ldots. \tag{2.18}$$

By choosing $\alpha = 1$ in Step 1, we get $x_{x_0y_0}^* \in [x_0,y_0]$. Then

$$u_1 = \Phi(x_0,y_0) = x_{x_0y_0}^* \ge x_0 = u_0, \quad v_1 = \Phi(y_0,x_0) = x_{y_0x_0}^* \le y_0 = v_0. \tag{2.19}$$

As $\Phi(\cdot,y)$ is increasing and $\Phi(x,\cdot)$ is decreasing, it follows immediately that

$$u_0 \le u_1 \le \cdots \le u_n \le \cdots \le v_n \le \cdots \le v_0. \tag{2.20}$$

Next, by making some needed modifications in the proof of [3, Theorem 2.11], one can show that Φ has a fixed point $x^* \in [x_0,y_0]$. Suppose that $y^* \in [x_0,y_0]$ is a fixed point of Φ. It follows from the definition of u_n and v_n that $u_n \le y^* \le v_n$ for all $n \in \mathbb{N}$. Then, by the normality of Φ, we get $y^* = x^*$. So x^* is the unique fixed point of Φ in $[x_0,y_0]$.

Step 4. By Steps 2 and 3, we get

$$x^* = \Phi(x^*,x^*) = A(x^*,x^*,\Phi(x^*,x^*)) = A(x^*,x^*,x^*). \tag{2.21}$$

Let $\overline{x} \in [x_0,y_0]$ such that $\overline{x} = A(\overline{x},\overline{x},\overline{x})$. Then it follows from Step 2 that $\Phi(\overline{x},\overline{x}) = \overline{x}$, that is, $\overline{x}$ is a fixed point of Φ in $[x_0,y_0]$. Thus, by Step 3, $\overline{x} = x^*$, which means that x^* is the unique fixed point of A in $[x_0,y_0]$.

□

Remark 2.2. Compared with [7, Remark 2.4], the nonlinear operator A in Theorem 2.1 is more general, and so Theorem 2.1 may have a wider range of applications.

3. Nonmonotonic Case

First, let us recall some definitions and basic results about Hilbert's projection metric (for more details, see [6]).

Definition 3.1. Elements x and y belonging to P (not both zero) are said to be **linked** if there exist $\lambda, \mu > 0$ such that

$$\lambda x \leq y \leq \mu x. \tag{3.1}$$

This defines an equivalence relation on P and divides P into disjoint subsets which we call **constituents** of P.

Let x and y be linked. Define

$$\begin{aligned} M(x,y) &= \inf\{\mu > 0 : y \leq \mu x\}, \\ d(x,y) &= \ln[\max\{M(x,y), M(y,x)\}]. \end{aligned} \tag{3.2}$$

Then, the following holds.

Theorem 3.2. *$d(\cdot,\cdot)$ defines a complete metric on each constituent of P.*

Proof. See [6]. □

We will also need the following result.

Theorem 3.3. *[9] Let M be a complete metric space and suppose that $f : M \to M$ satisfies*

$$d(f(x), f(y)) \leq \Psi(d(x,y)), \quad \forall x, y \in M, \tag{3.3}$$

where $\Psi : [0,+\infty) \to [0,+\infty)$ is upper semicontinuous from the right and satisfies $\Psi(t) < t$ for all $t > 0$. Then f has a unique fixed point in M.

Theorem 3.3 is a generalization of the classical Banach's contraction mapping principle. There are many generalizations of the classical Banach's contraction mapping principle (see, e.g., [10, 11] and references therein), and these generalizations play an important role in research work about fixed points of nonlinear operators in partially ordered Banach spaces; see, for example, [1] and the proof of the following theorem.

Now, we are ready to present our fixed point theorem, in which no monotone condition is assumed on the nonlinear operator.

Theorem 3.4. *Let T be an operator from P_e to P_e. Assume that there exist a constant $\varepsilon \in (0,1)$ and a function $\phi : [\varepsilon, 1) \to (0,+\infty)$ such that $\phi(\lambda) > \lambda$ for all $\lambda \in [\varepsilon, 1)$, and*

$$Ty \geq \phi(\lambda) Tx, \tag{3.4}$$

for all $x, y \in P_e$ and $\lambda \in [\varepsilon, 1)$ satisfying $\lambda x \leq y \leq \lambda^{-1} x$. Then T has a unique fixed point in P_e.

Proof. We divided the proof into 2 steps.

Step 1. Let $\lambda \in (0,1)$, $x, y \in P_e$, and $\lambda x \leq y \leq \lambda^{-1} x$. Then, there exists $k \in \mathbb{N}$ such that

$$\varepsilon \leq \frac{\lambda}{\varepsilon^k} < 1. \tag{3.5}$$

In view of

$$\frac{\lambda}{\varepsilon^k} \cdot \left(\varepsilon^k x\right) = \lambda x \leq y \leq \lambda^{-1} x \leq \varepsilon^{2k} \lambda^{-1} x = \left(\frac{\lambda}{\varepsilon^k}\right)^{-1} \cdot \left(\varepsilon^k x\right), \tag{3.6}$$

by the assumptions, we have

$$Ty \geq \phi\left(\frac{\lambda}{\varepsilon^k}\right) \cdot T\left(\varepsilon^k x\right) \geq \frac{\lambda}{\varepsilon^k} \cdot T\left(\varepsilon^k x\right). \tag{3.7}$$

Similar to the above proof, since $\varepsilon \cdot \varepsilon^{k-1} x = \varepsilon^k x \leq \varepsilon^{-1} \cdot \varepsilon^{k-1} x$, one can deduce

$$Ty \geq \frac{\lambda}{\varepsilon^k} \cdot T\left(\varepsilon^k x\right) \geq \frac{\lambda}{\varepsilon^k} \cdot \phi(\varepsilon) \cdot T\left(\varepsilon^{k-1} x\right) \geq \frac{\lambda}{\varepsilon^{k-1}} \cdot T\left(\varepsilon^{k-1} x\right). \tag{3.8}$$

Continuing by this way, one can get

$$Ty \geq \frac{\lambda}{\varepsilon} \cdot T(\varepsilon x) \geq \frac{\phi(\varepsilon)}{\varepsilon} \lambda \cdot Tx. \tag{3.9}$$

Let

$$\psi(\lambda) = \frac{\phi(\varepsilon)}{\varepsilon} \lambda, \quad \lambda \in (0,1). \tag{3.10}$$

Then ψ is continuous, $\psi(\lambda) > \lambda$ for all $\lambda \in (0,1)$, and

$$Ty \geq \psi(\lambda) Tx, \tag{3.11}$$

for all $x, y \in P_e$ and $\lambda \in (0,1)$ satisfying $\lambda x \leq y \leq \lambda^{-1} x$.

Step 2. Next, let $x, y \in P_e$ with $x \neq y$ and

$$\lambda = \frac{1}{\max\{M(x,y), M(y,x)\}}. \tag{3.12}$$

Then $\lambda \in (0,1)$, $\lambda x \leq y \leq \lambda^{-1} x$, and $d(x,y) = \ln(\lambda^{-1})$. Moreover, by Step 1, we have

$$Ty \geq \psi(\lambda) Tx. \tag{3.13}$$

On the other hand, since $\lambda y \le x \le \lambda^{-1} y$, we also have

$$Tx \ge \varphi(\lambda) Ty. \tag{3.14}$$

Thus, we get

$$\varphi(\lambda) Tx \le Ty \le \frac{Tx}{\varphi(\lambda)}. \tag{3.15}$$

Now, by the definition of $d(\cdot,\cdot)$, we have

$$d(Tx, Ty) \le \ln\left(\frac{1}{\varphi(\lambda)}\right). \tag{3.16}$$

Let

$$\Psi(t) = \begin{cases} -\ln\left[\varphi\left(e^{-t}\right)\right], & t \in (0, +\infty), \\ 0, & t = 0. \end{cases} \tag{3.17}$$

Then, Ψ is a continuous function from $[0, +\infty)$ to $[0, +\infty)$, and

$$d(Tx, Ty) \le \Psi(d(x, y)). \tag{3.18}$$

Moreover, since $\varphi(\lambda) > \lambda$ for all $\lambda \in (0, 1)$, we get

$$\Psi(t) = \ln \frac{1}{\varphi(e^{-t})} < \ln \frac{1}{e^{-t}} = t, \quad t > 0. \tag{3.19}$$

On the other hand, P_e is obviously a constituent of P, and thus (P_e, d) is complete by Theorem 3.2. Now, Theorem 3.3 yields that T has a unique fixed point in P_e. □

Corollary 3.5. *Assume that $A : P_e \times P_e \to P_e$ is a mixed monotone operator, that is, $A(\cdot, y)$ is increasing and $A(x, \cdot)$ is decreasing. Moreover, there exist a constant $\varepsilon \in (0, 1)$ and a function $\phi : [\varepsilon, 1) \to (0, +\infty)$ such that $\phi(\lambda) > \lambda$ for all $\lambda \in [\varepsilon, 1)$, and*

$$A\left(\lambda x, \lambda^{-1} y\right) \ge \phi(\lambda) A(x, y), \tag{3.20}$$

for all $x, y \in P_e$ and $\lambda \in [\varepsilon, 1)$. Then A has a unique fixed point in P_e.

Proof. Let $Tz = A(z, z)$, $z \in P_e$. Then, since A is a mixed monotone operator, we have

$$Ty = A(y, y) \ge A\left(\lambda x, \lambda^{-1} x\right) \ge \phi(\lambda) A(x, x) = \phi(\lambda) Tx, \tag{3.21}$$

for all $x, y \in P_e$ and $\lambda \in [\varepsilon, 1)$ satisfying $\lambda x \leq y \leq \lambda^{-1}x$. Then, Theorem 3.4 yields the conclusion. □

Remark 3.6. Corollary 3.5 is an improvement of [1, Corollary 3.2] in the sense that there ϕ is lower semicontinuous on $(0, 1)$, and the corresponding conditions need to hold on the whole interval $(0, 1)$.

4. An Example

In this section, we give an example to illustrate Theorem 3.4. Let us consider the following nonlinear delay integral equation:

$$x(t) = \int_{t-\tau}^{t} f(s, x(s))ds, \tag{4.1}$$

which is a classical model for the spread of some infectious disease (cf. [12]). In fact, (4.1) has been of great interest for many authors (see, e.g., [3, 8] and references therein).

In the rest of this paper, let $\tau = 1$ and

$$f(t, x) = \begin{cases} \left(1 + \sin^2 t + \sin^2 \pi t\right)\sqrt{x}, & t \in (-\infty, +\infty), \quad 0 \leq x \leq 1, \\ \dfrac{1 + \sin^2 t + \sin^2 \pi t}{\sqrt{x}}, & t \in (-\infty, +\infty), \quad x \geq 1. \end{cases} \tag{4.2}$$

Next, let us investigate the existence of positive almost periodic solution to (4.1). For the reader's convenience, we recall some definitions and basic results about almost periodic functions (for more details, see [13]).

Definition 4.1. A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called almost periodic if for each $\varepsilon > 0$ there exists $l(\varepsilon) > 0$ such that every interval I of length $l(\varepsilon)$ contains a number τ with the property that

$$\|f(t + \tau) - f(t)\| < \varepsilon \quad \forall\, t \in \mathbb{R}. \tag{4.3}$$

Denote by $AP(\mathbb{R})$ the set of all such functions.

Lemma 4.2. *Assume that $f, g \in AP(\mathbb{R})$. Then the following hold.*

(a) *The range $\mathcal{R}_f = \{f(t) : t \in \mathbb{R}\}$ is precompact in $\mathbb{R}$, and so f is bounded.*

(b) *$F(f) \in AP(\mathbb{R})$ provided that F is continuous on $\overline{\mathcal{R}_f}$.*

(c) *$f + g$, $f \cdot g \in AP(\mathbb{R})$. Moreover, $f/g \in AP(\mathbb{R})$ provided that* $\inf_{t\in\mathbb{R}}|g(t)| > 0$.

(d) *Equipped with the sup norm*

$$\|f\| = \sup_{t\in\mathbb{R}}|f(t)|, \tag{4.4}$$

$AP(\mathbb{R})$ turns out to be a Banach space.

Now, let $P = \{x \in AP(\mathbb{R}) : x(t) \geq 0, \forall t \in \mathbb{R}\}$, and $e \in P$ is defined by $e(t) \equiv 1$. It is not difficult to verify that P is a normal cone in $AP(\mathbb{R})$, and

$$P_e = \{x \in AP(\mathbb{R}) : \exists\, \varepsilon > 0 \text{ such that } x(t) > \varepsilon, \quad \forall t \in \mathbb{R}\}. \tag{4.5}$$

Define a nonlinear operator T on P_e by

$$(Tx)(t) = \int_{t-1}^{t} f(s, x(s))ds, \quad x \in P_e, \quad t \in \mathbb{R}. \tag{4.6}$$

By Lemma 4.2 and [3, Corollary 3.3], it is not difficult to verify that T is an operator from P_e to P_e. In addition, in view of (4.2), one can verify that

$$(Ty)(t) = \int_{t-1}^{t} f(s, y(s))ds \geq \sqrt{\lambda}\int_{t-1}^{t} f(s, x(s))ds = \sqrt{\lambda}(Tx)(t), \quad t \in \mathbb{R}, \tag{4.7}$$

that is, $Ty \geq \sqrt{\lambda}Tx$ for all $x, y \in P_e$ and $\lambda \in (0,1)$ with $\lambda x \leq y \leq \lambda^{-1}x$. Then, by Theorem 3.4, T has a unique fixed point in P_e, that is, (4.1) has a unique almost periodic solution with positive infimum.

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