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Research Article

Coupled Coincidence Point and Coupled Common Fixed Point Theorems in Partially Ordered Metric Spaces with w-Distance

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We introduce the concept of a w-compatible mapping to obtain a coupled coincidence point and a coupled point of coincidence for nonlinear contractive mappings in partially ordered metric spaces equipped with w-distances. Related coupled common fixed point theorems for such mappings are also proved. Our results generalize, extend, and unify several well-known comparable results in the literature.

1. Introduction and Preliminaries

In 1996, Kada et al. [1] introduced the notion of w-distance. They elaborated, with the help of examples, that the concept of w-distance is general than that of metric on a nonempty set. They also proved a generalization of Caristi fixed point theorem employing the definition of w-distance on a complete metric space. Recently, Ilić and Rakočević [2] obtained fixed point and common fixed point theorems in terms of w-distance on complete metric spaces (see also [3–9]).

Definition 1.1. Let (X, d) be a metric space. A mapping $p : X \times X \to [0, \infty)$ is called a w-distance on X if the following are satisfied:

(w_1) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$,

(w_2) for any $x \in X$, $p(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous,

(w_3) for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $p(x,y) \leq \varepsilon$, for any $x,y,z \in X$.

The metric d is a w-distance on X. For more examples of w-distances, we refer to [10].

Definition 1.2. Let X be a nonempty set with a w-distance on X. Ones denotes the w-closure of a subset B of X by $\mathrm{cl}_\omega(B)$ which is defined as

$$\mathrm{cl}_\omega(B) = \{x \in X : p(x_n, x) \longrightarrow 0 \text{ for some sequence } \{x_n\} \text{ in } B\} \cup B. \tag{1.1}$$

The next Lemma is crucial in the proof of our results.

Lemma 1.3 (see [1]). *Let (X,d) be a metric space, and let p be a w-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let α_n and β_n be sequences in $[0,\infty)$ converging to 0, and let $x,y,z \in X$. Then the following hold.*

(1) *If $p(x_n,y) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then $y = z$. In particular, if $p(x,y) = 0$, $p(x,z) = 0$ then $y = z$.*

(2) *If $p(x_n,y_n) \leq \alpha_n$ and $p(x_n,z) \leq \beta_n$ for any $n \in N$, then y_n converges to z.*

(3) *If $p(x_n,x_m) \leq \alpha_n$ for any $m,n \in N$ with $n \prec m$, then x_n is a Cauchy sequence.*

(4) *If $p(y,x_n) \leq \alpha_n$ for any $n \in N$, then x_n is a Cauchy sequence.*

Bhaskar and Lakshmikantham in [11] introduced the concept of coupled fixed point of a mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered sets. They also discussed an application of their result by investigating the existence and uniqueness of solution for a periodic boundary value problem. Sabetghadam et al. in [12] introduced this concept in cone metric spaces. They investigated some coupled fixed point theorems in cone metric spaces. Recently, Lakshmikantham and Ćirić [13] proved coupled coincidence and coupled common fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces which extend the coupled fixed point theorem given in [11]. The following are some other definitions needed in the sequel.

Definition 1.4 (see [12]). Let X be any nonempty set. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. An ordered pair $(x,y) \in X \times X$ is called

(1) a coupled fixed point of a mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$,

(2) a coupled coincidence point of hybrid pair $\{F,g\}$ if $g(x) = F(x,y)$ and $g(y) = F(y,x)$ and (gx,gy) is called coupled point of coincidence,

(3) a common coupled fixed point of hybrid pair $\{F,g\}$ if $x = g(x) = F(x,y)$ and $y = g(y) = F(y,x)$.

Note that if (x,y) is a coupled fixed point of F, then (y,x) is also a coupled fixed point of the mapping F.

Definition 1.5. Let X be any nonempty set. Mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $g(F(x,y)) = F(gx,gy)$ whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$.

Definition 1.6. Let (X,d) be a metric space with w-distance p. A mapping $F : X \times X \to X$ is said to be w-continuous at a point $(x,y) \in X \times X$ with respect to mapping $g : X \to X$ if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $p(gu,gx) + p(gv,gy) < \delta$ implies that $p(F(x,y),F(u,v)) < \varepsilon$ for all $u,v \in X$.

Definition 1.7. Let X be a partially ordered set. Mapping $g : X \to X$ is called strictly monotone increasing mapping if

$$x \preccurlyeq y \Longleftrightarrow gx \preccurlyeq gy \text{ or equivalently} x \succcurlyeq y \Longleftrightarrow gx \succcurlyeq gy. \tag{1.2}$$

Definition 1.8. Let X be a partially ordered set. A mapping $F : X \times X \to X$ is said to be a mixed monotone if $F(x,y)$ is monotone nondecreasing in x and monotone nonincreasing in y, that is, for any $x,y \in X$,

$$\begin{aligned} &x_1,x_2 \in X, \quad x_1 \preccurlyeq x_2 \Longrightarrow F(x_1,y) \preccurlyeq F(x_2,y),\\ &y_1,y_2 \in X, \quad y_1 \preccurlyeq y_2 \Longrightarrow F(x,y_1) \succcurlyeq F(x,y_2). \end{aligned} \tag{1.3}$$

Kada et al. [1] gave an example to show that p is not symmetric in general. We denote by $M(X)$ and $M_1(X)$, respectively, the class of all w-distances on X and the class of all w-distances on X which are symmetric for comparable elements in X. Also in the sequel, we will consider that (x,y) and (u,v) are comparable with respect to ordering in $X \times X$ if $x \succcurlyeq u$ and $y \preccurlyeq v$.

2. Coupled Coincidence Point

In this section, we prove coincidence point results in the frame work of partially ordered metric spaces in terms of a w-distance.

Theorem 2.1. *Let (X,d) be a partially ordered metric space with a w-distance p and $g : X \to X$ a strictly monotone increasing mapping. Suppose that a mixed monotone mapping $F : X \times X \to X$ is w-continuous with respect to g such that*

$$p(F(x,y),F(u,v)) \leq a_1 p(gu,gx) + a_2 p(gv,gy), \tag{2.1}$$

for all $x,y,u,v \in X$ with $x \succcurlyeq u, y \preccurlyeq v$ or $x \preccurlyeq u, y \succcurlyeq v$ and $a_1 + a_2 < 1$. Let $F(X \times X) \subseteq g(X)$ and $p(y,x) = 0$ whenever $p(x,y) = 0$, for some $x,y \in \mathrm{cl}_\omega(F(X \times X))$. If $g(X)$ is complete and there exist $x_0,y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0,y_0)$ and $F(y_0,x_0) \preccurlyeq gy_0$, then F and g have a coupled coincidence point.

Proof. Let $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$ for some $x_1,y_1 \in X$; this can be done since $F(X \times X) \subseteq g(X)$. Following the same arguments, we obtain $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$. Put

$$\begin{aligned} &F^1(x_0,y_0) = gx_1, \qquad F^2(x_0,y_0) = F(x_1,y_1) = gx_2,\\ &F^2(y_0,x_0) = F(y_1,x_1) = gy_2. \end{aligned} \tag{2.2}$$

Similarly for all $n \in N$,

$$gx_{n+1} = F^{n+1}(x_0, y_0), \qquad gy_{n+1} = F^{n+1}(y_0, x_0). \tag{2.3}$$

Since g is strictly monotone increasing and F has the mixed monotone property, we have

$$gx_2 = F^2(x_0, y_0) = F(x_1, y_1) \succcurlyeq F(x_0, y_0) = gx_1, \qquad gy_2 \preccurlyeq gy_1. \tag{2.4}$$

Similarly

$$\begin{aligned} gx_0 &\preccurlyeq F(x_0, y_0) = gx_1 \preccurlyeq F^2(x_0, y_0) = gx_2 \preccurlyeq \cdots \\ &\preccurlyeq F^{n+1}(x_0, y_0) = gx_{n+1} \preccurlyeq \cdots, \\ gy_0 &\succcurlyeq F(y_0, x_0) = gy_1 \succeq F^2(y_0, x_0) = gy_2 \succcurlyeq \cdots \\ &\succcurlyeq F^{n+1}(y_0, x_0) \succcurlyeq \cdots. \end{aligned} \tag{2.5}$$

Now for all $n \geq 2$, using (2.1), we get

$$\begin{aligned} &p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) \\ &\quad = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\quad \leq a_1 p(gx_n, gx_{n-1}) + a_2 p(gy_n, gy_{n-1}) \\ &\quad = a_1\Big[p\Big(F^n(x_0, y_0), F^{n-1}(x_0, y_0)\Big)\Big] + a_2\Big[p\Big(F^n(y_0, x_0), F^{n-1}(y_0, x_0)\Big)\Big], \\ &p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) \\ &\quad \leq a_1\Big[p\Big(F^n(y_0, x_0), F^{n-1}(y_0, x_0)\Big)\Big] + a_2\Big[p\Big(F^n(x_0, y_0), F^{n-1}(x_0, y_0)\Big)\Big]. \end{aligned} \tag{2.6}$$

From (2.6),

$$\begin{aligned} &p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) + p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) \\ &\quad \leq h\Big[p\Big(F^n(x_0, y_0), F^{n-1}(x_0, y_0)\Big) + p\Big(F^n(y_0, x_0), F^{n-1}(y_0, x_0)\Big)\Big], \end{aligned} \tag{2.7}$$

where $h = a_1 + a_2$. Continuing, we conclude that

$$\begin{aligned} &p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) + p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) \\ &\quad \leq h^n(p(gx_1, gx_0)) + \ p(gy_1, gy_0)) = h^n \delta_1 \end{aligned} \tag{2.8}$$

if n is odd, where $\delta_1 = p(gx_1, gx_0) + p(gy_1, gy_0)$. Also,

$$
\begin{aligned}
&p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) + p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) \\
&\quad \le h^n(p(gx_0, gx_1) + p(gy_0, gy_1)) = h^n\delta_2
\end{aligned}
\tag{2.9}
$$

if n is even, where

$$
\delta_2 = p(gx_0, gx_1) + p(gy_0, gy_1). \tag{2.10}
$$

Let $\delta_n = p(F^n(x_0, y_0), F^{n+1}(x_0, y_0)) + p(F^n(y_0, x_0), F^{n+1}(y_0, x_0))$; then for every n in N we have

$$
\delta_n \le h^n\delta_0, \tag{2.11}
$$

where

$$
\delta_0 = \max\{\delta_1, \delta_2\}. \tag{2.12}
$$

Hence,

$$
p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) \longrightarrow 0, \quad p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.13}
$$

For $m > n$, we get

$$
\begin{aligned}
&p(F^n(x_0, y_0), F^m(x_0, y_0)) + p(F^n(y_0, x_0), F^m(y_0, x_0)) \\
&\quad \le p\Big(F^n(x_0, y_0), F^{n+1}(x_0, y_0)\Big) + p\Big(F^{n+1}(x_0, y_0), F^{n+2}(x_0, y_0)\Big) + \cdots \\
&\qquad + p\Big(F^{m-1}(x_0, y_0), F^m(x_0, y_0)\Big) \\
&\qquad + p\Big(F^n(y_0, x_0), F^{n+1}(y_0, x_0)\Big) + p\Big(F^{n+1}(y_0, x_0), F^{n+2}(y_0, x_0)\Big) + \cdots \\
&\qquad + p\Big(F^{m-1}(y_0, x_0), F^m(y_0, x_0)\Big) \\
&\quad = \delta_n + \delta_{n+1} + \cdots + \delta_{m-1} \le h^n\delta_0 + h^{n+1}\delta_0 + \cdots + h^{m-1}\delta_0 \le \frac{h^n}{1-h}\delta_0
\end{aligned}
\tag{2.14}
$$

which further implies that

$$
\begin{aligned}
p(F^n(x_0, y_0), F^m(x_0, y_0)) &\le \frac{h^n}{1-h}\delta_0 \\
p(F^n(y_0, x_0), F^m(y_0, x_0)) &\le \frac{h^n}{1-h}\delta_0.
\end{aligned}
\tag{2.15}
$$

Lemma 1.3(3) implies that $\{F^n(x_0, y_0)\} = \{gx_n\}$ and $\{F^n(y_0, x_0)\} = \{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $x, y \in X$ such that $gx_n \to gx$ and $gy_n \to gy$. Since $p(gx_n, \cdot)$ is lower semicontinuous, we have

$$p(F^n(x_0, y_0)), gx) \leq \liminf_{m \to \infty} p(gx_n, gx_m) \leq \frac{h^n}{1-h}\delta_0 \tag{2.16}$$

which implies that

$$p(F^n(x_0, y_0)), gx) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.17}$$

Similarly

$$p(F^n(y_0, x_0)), gy) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.18}$$

Let $\varepsilon > 0$ be given. Since F is w-continuous at (x, y) with respect to g, there exists $\delta > 0$ such that for each n

$$p(gx_n, gx) + p(gy_n, gy) < \delta \text{ implies that } p(F(x, y), F(x_n, y_n)) < \frac{\varepsilon}{2}. \tag{2.19}$$

Since $p(gx_n, gx) \to 0$ and $p(gy_n, gy) \to 0$, for $\gamma = \min(\varepsilon/2, \delta/2)$, there exists n_0 such that, for all $n \geq n_0$,

$$p(gx_n, gx) < \gamma, \qquad p(gy_n, gy) < \gamma. \tag{2.20}$$

Now,

$$\begin{aligned} p(F(x, y), gx) &\leq p\Big(F(x, y), F^{n_0+1}(x_0, y_0)\Big) + p\Big(F^{n_0+1}(x_0, y_0), gx\Big) \\ &= p(F(x, y), F(x_{n_0}, y_{n_0})) + p(gx_{n_0+1}, gx) \\ &< \frac{\varepsilon}{2} + \gamma = \varepsilon \end{aligned} \tag{2.21}$$

implies that $p(F(x, y), gx) = 0$. Since

$$\begin{aligned} p(F^n(x_0, y_0), F(x, y)) &\leq p(F^n(x_0, y_0), gx) + p(gx, F(x, y)) \\ &\leq \frac{h^n}{1-h}\delta_0, \end{aligned} \tag{2.22}$$

using Lemma 1.3(1), we obtain $F(x, y) = gx$. Similarly, we can prove that $F(y, x) = gy$. Hence (x, y) is coupled coincidence point of F and g. □

Theorem 2.2. *Let (X,d) be a partially ordered metric space with a w-distance p having the following properties.*

(1) *If $\{x_n\}$ is in X with $x_n \preccurlyeq x_{n+1}$ for all n and $x_n \to x$ for some $x \in X$, then $x_n \preccurlyeq x$ for all n.*

(2) *If $\{y_n\}$ is in X with $y_{n+1} \preccurlyeq y_n$ for all n and $y_n \to y$ for some $y \in X$, then $y \preccurlyeq y_n$ for all n.*

Let $F : X \times X \to X$ be a mixed monotone and $g : X \to X$ a strict monotone increasing mapping such that

$$p(F(x,y),F(u,v)) \le a_1 p(gu,gx) + a_2 p(gv,gy), \tag{2.23}$$

for all $x,y,u,v \in X$ with $x \succcurlyeq u, y \preccurlyeq v$ or $x \preccurlyeq u, y \succcurlyeq v$ and $a_1 + a_2 < 1$. Let $F(X\times X) \subseteq g(X)$ and $p(y,x) = 0$ whenever $p(x,y) = 0$, for some $x,y \in \mathrm{cl}_\omega(F(X\times X))$. If $g(X)$ is complete and there exist $x_0, y_0 \in X$ such that $gx_0 \preccurlyeq F(x_0,y_0)$ and $F(y_0,x_0) \preccurlyeq gy_0$, then F and g have a coupled coincidence point.

Proof. Construct two sequences $\{gx_n\} = \{F^n(x_0,y_0)\}$ and $\{gy_n\} = \{F^n(y_0,x_0)\}$ such that $gx_n \preccurlyeq gx_{n+1}$ and $gy_n \succcurlyeq gy_{n+1}$ for all n and $gx_n \to gx$ and $gy_n \to gy$ for some $x \in X$, as given in the proof of Theorem 2.1. Now, we need to show that $F(x,y) = gx$ and $F(y,x) = gy$. Let $\varepsilon > 0$. Since $p(F^n(x_0,y_0),gx) \to 0$ and $p(F^n(y_0,x_0),gy) \to 0$, there exists $n_1 \in N$ such that, for all $n \ge n_1$, we have

$$p(F^n(x_0,y_0),gx) < \frac{\varepsilon}{3}, \qquad p(F^n(y_0,x_0),gy) < \frac{\varepsilon}{3}. \tag{2.24}$$

Consider

$$\begin{aligned} p(F(x,y),gx) &\le p\Big(F(x,y),F^{n+1}(x_0,y_0)\Big) + p\Big(F^{n+1}(x_0,y_0),gx\Big) \\ &= p(F(x,y),F(x_n,y_n)) + p\Big(F^{n+1}(x_0,y_0),gx\Big) \\ &\le a_1 p(gx_n,gx) + a_2 p(gy_n,gy) + p\Big(F^{n+1}(x_0,y_0),gx\Big) \\ &= a_1 p(F^n(x_0,y_0),gx) + a_2 p(F^n(y_0,x_0),gy) + p\Big(F^{n+1}(x_0,y_0),gx\Big) \\ &< a_1\frac{\varepsilon}{3} + a_2\frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\ &< \varepsilon, \end{aligned} \tag{2.25}$$

which implies that $p(F(x,y),gx) = 0$. Also, from Theorem 2.1, we have

$$p(F^n(x_0,y_0),gx) \le \frac{h^n}{1-h}\delta_0. \tag{2.26}$$

Therefore,

$$\begin{aligned} p(F^n(x_0,y_0),F(x,y)) &\\ &\leq p(F^n(x_0,y_0),gx)+p(gx,F(x,y)) \\ &\leq \frac{h^n}{1-h}\delta_0 \end{aligned} \tag{2.27}$$

implies that $gx = F(x,y)$. Similarly, we can prove that $F(y,x) = gy$. Hence (x,y) is coupled coincidence point of F and g. □

3. Coupled Common Fixed Point

In this section, using the concept of w-compatible maps, we obtain a unique coupled common fixed point of two mappings.

Theorem 3.1. *Let all the hypotheses of Theorem 2.1 (resp., Theorem 2.2) hold with $a_1 + a_2 < 1/2$. If for every $(x,y),(x^*,y^*) \in X \times X$ there exists $(u,v) \in X \times X$ that is comparable to (x,y) and (x^*,y^*) with respect to ordering in $X \times X$, then there exists a unique coupled point of coincidence of F and g. Moreover if F and g are w-compatible, then F and g have a unique coupled common fixed point.*

Proof. Let (gx^*,gy^*) be another coupled coincidence point of F and g. We will discuss the following two cases.

Case 1. If (x,y) is comparable to (x^*,y^*) with respect to ordering in $X \times X$, then

$$\begin{aligned} p(gx,gx^*)+p(gy,gy^*) &\\ &= p(F(x,y),F(x^*,y^*))+p(F(y,x),F(y^*,x^*)) \\ &\leq a_1p(gx^*,gx)+a_2p(gy^*,gy)+a_1p(gy^*,gy)+a_2p(gx^*,gx) \\ &\leq (a_1+a_2)[p(gx,gx^*)+p(gy,gy^*)] \end{aligned} \tag{3.1}$$

implies that $p(gx,gx^*)+p(gy,gy^*) = 0$. Hence $p(gx,gx^*) = 0 = p(gy,gy^*)$. Also,

$$\begin{aligned} p(gx,gx)+p(gy,gy) &= p(F(x,x),F(x,x))+p(F(y,y),F(y,y)) \\ &\leq 2a_1p(gx,gx)+2a_2p(gy,gy) \end{aligned} \tag{3.2}$$

gives that $p(gx,gx) = 0 = p(gy,gy)$. The result follows using Lemma 1.3(1).

Case 2. If (x,y) is not comparable to (x^*,y^*), then there exists an upper bound or lower bound (u,v) of $(x,y),(x^*,y^*)$. Again since g is strictly monotone increasing mapping and F satisfies mixed monotone property, therefore, for all $n = 0,1,\ldots,(F^n(u,v),F^n(v,u))$ is comparable to $(F^n(x,y), F^n(y,x)) = (gx, gy)$ and $(F^n(y,x), F^n(x,y)) = (gy, gx)$. Following similar arguments to those given in the proof of Theorem 2.1, we obtain

$$\begin{aligned} p(gx, gx^*) + p(gy, gy^*) &= p(F^n(x,y), F^n(x^*,y^*)) + p(F^n(y,x), F^n(y^*,x^*)) \\ &\le \left[p(F^n(x,y), F^n(u,v)) + p(F^n(u,v), F^n(x^*,y^*))\right] \\ &\quad + \left[p(F^n(y,x), F^n(v,u)) + p(F^n(v,u), F^n(y^*,x^*))\right] \\ &= \left[p(F^n(x,y), F^n(u,v)) + p(F^n(y,x), F^n(v,u))\right] \\ &\quad + \left[p(F^n(u,v), F^n(x^*,y^*)) + p(F^n(v,u), F^n(y^*,x^*))\right] \\ &\le h^n\beta_0 + h^n\gamma_0, \end{aligned} \tag{3.3}$$

where $\beta_0 = \max\{p(gu, gx) + p(gv, gy), p(gx, gu) + p(gy, gv)\}$ and $\gamma_0 = \max\{p(gx^*, gu) + p(gy^*, gv), p(gu, gx^*) + p(gv, gy^*)\}$. On taking limit as $n \to \infty$ on both sides of (3.3), we have

$$p(gx, gx^*) + p(gy, gy^*) = 0 \tag{3.4}$$

and $p(gx, gx^*) = 0 = p(gy, gy^*)$. By the same lines as in Case 1, we prove that $p(gx, gx) = 0 = p(gy, gy)$. Again Lemma 1.3(1) implies that $gx = gx^*$ and $gy = gy^*$. Hence (gx, gy) is unique coupled point of coincidence of F and g. Note that if (gx, gy) is a coupled point of coincidence of F and g, then (gy, gx) are also a coupled points of coincidence of F and g. Then $gx = gy$ and therefore (gx, gx) is unique coupled point of coincidence of F and g. Let $u = gx$. Since F and g are w-compatible, we obtain

$$gu = g(gx) = g(F(x,x)) = F(gx, gx) = F(u,u). \tag{3.5}$$

Consequently $gu = gx$. Therefore $u = gu = F(u,u)$. Hence (u,u) is a coupled common fixed point of F and g.

□

Remark 3.2. If in addition to the hypothesis of Theorem 2.1 (resp., Theorem 2.2) we suppose that $p \in M_1(X)$, x_0 and y_0 are comparable, then $gx = gy$.

Proof. Recall that $gx_0 \preccurlyeq F(x_0, y_0)$. Now, if $x_0 \preccurlyeq y_0$, then $gx_0 \preccurlyeq gy_0$. We claim that, for all $n \in N$, $gx_n \preccurlyeq gy_n$. Since g is strictly monotone increasing and F satisfies mixed monotone property, we have

$$gx_1 = F(x_0, y_0) \preccurlyeq F(y_0, x_0) = gy_1. \tag{3.6}$$

Assuming that $gx_n \preccurlyeq gy_n$, since g is strictly monotone increasing, so $x_n \preccurlyeq y_n$. By the mixed monotone property of F, we have

$$gx_{n+1} = F^{n+1}(x_0, y_0) = F(x_n, y_n) \preccurlyeq F(y_n, x_n) = gy_{n+1}. \tag{3.7}$$

Therefore,

$$gx_n \preccurlyeq gy_n \quad \forall n. \tag{3.8}$$

Letting $\varepsilon > 0$, there exists an $n_0 \in N$ such that $p(gx, F^n(x_0, y_0)) < \varepsilon/4$ and $p(F^n(y_0, x_0), gy) < \varepsilon/4$ for all $n \geq n_0$. Now,

$$\begin{aligned} p(gx, gy) &\leq p\Big(gx, F^{n_0+1}(x_0, y_0)\Big) + \Big(F^{n_0+1}(x_0, y_0), gy\Big) \\ &\leq p\Big(gx, F^{n_0+1}(x_0, y_0)\Big) + p\Big(F^{n_0+1}(x_0, y_0), F^{n_0+1}(y_0, x_0)\Big) + \Big(F^{n_0+1}(y_0, x_0), gy\Big) \\ &< \frac{\varepsilon}{4} + hp(F^{n_0}(x_0, y_0), F^{n_0}(y_0, x_0)) + \frac{\varepsilon}{4} \\ &\leq \frac{\varepsilon}{2} + h[p(F^{n_0}(x_0, y_0), gx) + p(gx, gy) + (gy, F^{n_0}(y_0, x_0))] \\ &< \frac{\varepsilon}{2} + h\frac{\varepsilon}{4} + hp(gx, gy) + h\frac{\varepsilon}{4} \\ &< \varepsilon + hp(gx, gy) \end{aligned} \tag{3.9}$$

implies that $(1 - h)p(gx, gy) < \varepsilon$. Since $h < 1$, therefore $p(gx, gy) = 0$. Similarly we can prove that $p(gx, gx) = 0$. Hence by Lemma 1.3(1), we have $gx = gy$. Similarly, if $gx_0 \succcurlyeq gy_0$, we can show that $gx_n \succcurlyeq gy_n$ for each n and $gx = gy$. □

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