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Research Article

On Equivalence of Some Iterations Convergence for Quasi-Contraction Maps in Convex Metric Spaces

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We show the equivalence of the convergence of Picard and Krasnoselskij, Mann, and Ishikawa iterations for the quasi-contraction mappings in convex metric spaces.

1. Introduction

Let (E, d) be a complete metric space and $I = [0, 1]$. Denote $E^2 = E \times E$, $I^2 = I \times I$. A continuous mapping $W : E^2 \times I^2 \to E$ is said to be a convex structure on E [1] if for all $u, z_1, z_2 \in E$, λ_1, $\lambda_2 \in I$ with $\lambda_1 + \lambda_2 = 1$ such that

$$d(u, W(z_1, z_2; \lambda_1, \lambda_2)) \leq \lambda_1 d(u, z_1) + \lambda_2 d(u, z_2); \tag{1.1}$$

$$W(z_1, z_2; 1, 0) = z_1, \qquad W(z_1, z_2; 0, 1) = z_2. \tag{1.2}$$

If (E, d) satisfies the conditions of convex structure, then (E, d) is called convex metric space that is denoted as (E, d, W).

In the following part, we will consider a few iteration sequences in convex metric space (E, d, W). Suppose that T is a self-map of E.

Picard iteration is as follows:

$$\forall p_0 \in E, \quad p_{n+1} = Tp_n = T^{n+1}p_0, \quad n \geq 0. \tag{1.3}$$

Krasnoselskij iteration is as follows:

$$\forall v_0 \in E, \quad v_{n+1} = W(v_n, Tv_n; 1-\lambda, \lambda), \quad n \geq 0, \tag{1.4}$$

where $\lambda \in [0,1]$.

Mann iteration is as follows:

$$\forall u_0 \in E, \quad u_{n+1} = W(u_n, Tu_n; 1-a_n, a_n), \quad n \geq 0, \tag{1.5}$$

where $a_n \in [0,1]$.

Ishikawa iteration is as follows:

$$\begin{gathered} \forall x_0 \in E, \\ x_{n+1} = W(x_n, Ty_n; 1-a_n, a_n), \quad n \geq 0, \\ y_n = W(x_n, Tx_n; 1-b_n, b_n), \quad n \geq 0, \end{gathered} \tag{1.6}$$

where $a_n, b_n \in [0,1]$ for all $n \geq 0$.

A mapping $T : E \to E$ is called contractive if there exists $L \in (0,1)$ such that

$$d(Tx, Ty) \leq Ld(x, y), \tag{1.7}$$

for all $x, y \in E$.

The map T is called Kannan mapping [2] if there exists $b \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)], \tag{1.8}$$

for all $x, y \in E$.

A similar definition of mapping is due to the work Chatterjea [3] (that is called Chatterjea mapping), if there exists $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)], \tag{1.9}$$

for all $x, y \in E$.

Combining above three definitions, Zamfirescu [4] showed the following result.

Theorem 1.1. *Let (E, d) be a complete metric space and $T : E \to E$ a mapping for which there exist the real numbers a, b, and c satisfying $a \in (0,1)$, $b, c \in (0, 1/2)$ such that, for any pair $x, y \in E$, at least one of the following conditions holds:*

(z1) $d(Tx, Ty) \leq ad(x, y)$;

(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$;

(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point, and the Picard iteration converges to fixed point. This class mapping is called Zamfirescu mapping.

In 1974, Ćirić [5] introduced one of the most general contraction mappings and obtained that the unique fixed point can be approximated by Picard iteration. This mapping is called quasi-contractive if there exists $\delta \in (0,1)$ such that

$$d(Tx,Ty) \le \delta \cdot \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}, \tag{1.10}$$

for any $x, y \in E$.

Clearly, every quasi-contraction mapping is the most general of above mappings.

Later on, in 1992, Xu [6] proved that Ishikawa iteration can also be used to approximate the fixed points of quasi-contraction mappings in real Banach spaces.

Theorem 1.2. *Let C be any nonempty closed convex subset of a Banach space X and $T : C \to C$ a quasi-contraction mapping. Suppose that $\alpha_n > 0$ for all n and $\sum \alpha_n = \infty$. Then the Ishikawa iteration sequence $\{x_n\}$ defined by (1)–(3) converges strongly to the unique fixed point x^* of T.*

In this paper, we will show the equivalence of the convergence of Picard and Krasnoselskij, Mann, and Ishikawa iterations for the quasi-contraction mappings in convex metric spaces.

Lemma 1.3. *Let $\{\rho_n\}_{n=0}^{\infty}$ be a nonnegative sequence which satisfies the following inequality*

$$\rho_{n+1} \le (1-\theta_n)\rho_n + \sigma_n, \quad n \ge 0, \tag{1.11}$$

where $\theta_n \in (0,1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, and $\sigma_n/\theta_n \to 0$ as $n \to \infty$. Then $\rho_n \to 0$ as $n \to \infty$, (see [7]).

2. Results for Quasi-Contraction Mappings

Theorem 2.1. *Let (E,d,W) be a convex metric space, $T : E \to E$ a quasi-contraction mapping with $F(T) \neq \emptyset$. Suppose that $\{p_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are defined by the iterative processes* (1.3) *and* (1.4), *respectively. Then, the following two assertions are equivalent:*

(i) *Picard iteration* (1.3) *converges strongly to the unique fixed point $q \in F(T)$;*

(ii) *Krasnoselskij iteration* (1.4) *converges strongly to the unique fixed point $q \in F(T)$.*

Proof. First, we show (i) ⇒ (ii), that is, $d(p_n,q) \to 0$ as $n \to \infty \Rightarrow d(v_n,q) \to$ as $n \to \infty$.

From (1.3), (1.4), and (1.1), we can get

$$\begin{aligned}
d(v_{n+1},p_{n+1}) &= d(W(v_n,Tv_n;1-\lambda,\lambda),Tp_n) \\
&\le (1-\lambda)d(v_n,Tp_n) + \lambda d(Tv_n,Tp_n) \\
&\le (1-\lambda)d(v_n,p_n) + (1-\lambda)d(p_n,Tp_n) + \lambda d(Tv_n,Tp_n) \\
&\le (1-\lambda)d(v_n,p_n) + \frac{1-\lambda}{1-\delta}d(p_n,q) + \lambda d(Tv_n,Tp_n).
\end{aligned} \tag{2.1}$$

Next, we consider $d(Tv_n, Tp_n)$. Using (1.10) with $x = p_n, \quad y = v_n$, to obtain

$$d(Tv_n, Tp_n) \le \delta \cdot \max\{d(v_n, p_n), d(v_n, Tv_n), d(p_n, Tp_n), d(v_n, Tp_n), d(p_n, Tv_n)\}. \tag{2.2}$$

Set

$$A_n = \{v_i\}_{i=0}^n \cup \{p_i\}_{i=0}^n \cup \{Tp_i\}_{i=0}^n \cup \{Tv_i\}_{i=0}^n, \qquad \gamma_n = \operatorname{diam}\{A_n\}. \tag{2.3}$$

Then $\{A_n\}$ is bounded. Without loss of generality, we let $\gamma_n > 0$ for each n. Indeed, we will show this conclusion from the some following cases.

Case 1. Let $\gamma_n = d(Tp_i, Tv_j)$ for some $0 \le i, j \le n$. Then, from (1.10) and the above γ_n, we have

$$\begin{aligned} \gamma_n &= d(Tp_i, Tv_j) \\ &\le \delta \cdot \max\{d(p_i, v_j), d(v_j, Tv_j), d(p_i, Tp_i), d(v_j, Tp_i), d(p_i, Tv_j)\} \\ &\le \delta\gamma_n < \gamma_n, \end{aligned} \tag{2.4}$$

and it leads to a contradiction. Thus, $\gamma_n \ne d(Tp_i, Tv_j)$. Similarity to $\gamma_n = d(Tp_i, Tp_j)$ or $\gamma_n = d(Tv_i, Tv_j)$ is also impossible.

Case 2. Let $\gamma_n = d(p_i, v_j)$ for some $0 \le i, j \le n$.

(i) If $j = 0$, then $\gamma_n = d(p_i, v_0)$.

(ii) If $j \ge 1, \quad i = 0$, then, from (1.4) and (1.1)

$$\begin{aligned} \gamma_n &= d(p_0, v_j) \\ &= d(p_0, W(v_{j-1}, Tv_{j-1}; 1-\lambda, \lambda)) \\ &\le (1-\lambda) d(p_0, v_{j-1}) + \lambda d(p_0, Tv_{j-1}) \\ &\le (1-\lambda) d(p_0, v_{j-1}) + \lambda\gamma_n, \end{aligned} \tag{2.5}$$

that is, $\gamma_n = d(p_0, v_{j-1})$. By induction on j, we can obtain $\gamma_n = d(p_0, v_0)$.

(iii) If $j \ge 1, \quad i \ge 1$, from (1.4) and (1.1)

$$\begin{aligned} \gamma_n &= d(p_i, v_j) \\ &= d(p_i, W(v_{j-1}, Tv_{j-1}; 1-\lambda, \lambda)) \\ &\le (1-\lambda) d(p_i, v_{j-1}) + \lambda d(p_i, Tv_{j-1}) \\ &\le (1-\lambda) d(p_i, v_{j-1}) + \lambda\gamma_n, \end{aligned} \tag{2.6}$$

it implies that $\gamma_n = d(p_i, v_{j-1})$. By induction on j, we can get $\gamma_n = d(p_i, v_0)$.

Case 3. Let $\gamma_n = d(v_i, v_j)$ for some $0 \le i, j \le n$. Without loss of generality, we set $0 \le i < j \le n$. Then, from (1.4), (1.1)

$$\begin{aligned}\gamma_n &= d(v_i, v_j)\\ &\le \mathrm{d}(v_i, W(v_{j-1}, Tv_{j-1}; 1-\lambda, \lambda))\\ &\le (1-\lambda)d(v_i, v_{j-1}) + \lambda d(v_i, Tv_{j-1})\\ &\le (1-\lambda)d(v_i, v_{j-1}) + \lambda\gamma_n,\end{aligned} \tag{2.7}$$

it implies that $\gamma_n = d(v_i, v_{j-1})$, and by induction on j, we may get $\gamma_n = d(v_i, v_i) = 0$, which is a contradiction.

Case 4. Let $\gamma_n = d(v_i, Tp_j)$ for some $0 \le i, j \le n$.

(i) If $i = 0$, then $\gamma_n = d(v_0, Tp_j)$.

(ii) If $i \ge 1$, from (1.4), (1.1), then

$$\begin{aligned}\gamma_n &= d(v_i, Tp_j)\\ &\le d(W(v_{i-1}, Tv_{i-1}; 1-\lambda, \lambda), Tp_j)\\ &\le (1-\lambda)d(v_{i-1}, Tp_j) + \lambda d(Tv_{i-1}, Tp_j)\\ &\le (1-\lambda)d(v_{i-1}, Tp_j)\\ &\quad + \lambda\delta\cdot\max\{d(v_{i-1}, p_j), d(v_{i-1}, Tv_{i-1}), d(p_j, Tp_j), d(v_{i-1}, Tp_j), d(p_j, Tv_{i-1})\}\\ &\le (1-\lambda)d(v_{i-1}, Tp_j) + \lambda\delta\gamma_n\\ &\le (1-\lambda)d(v_{i-1}, Tp_j) + \lambda\gamma_n,\end{aligned} \tag{2.8}$$

it implies that $\gamma_n = d(v_{i-1}, Tp_j)$ and by induction on i, then $\gamma_n = d(v_0, Tp_j)$.

Case 5. Let $\gamma_n = \mathrm{d}(p_i, Tv_j)$ for some $0 \le i, j \le n$.

(i) If $i = 0$, then $\gamma_n = d(p_0, Tv_j)$.

(ii) If $i \ge 1$, then, from (1.3) and (1.10)

$$\begin{aligned}\gamma_n &= d(p_i, Tv_j)\\ &\le d(Tp_{i-1}, Tv_j)\\ &\le \lambda\delta\cdot\max\{d(p_{i-1}, v_j), d(p_{i-1}, Tp_{i-1}), d(v_j, Tv_j), d(p_{i-1}, Tv_j), d(v_j, \mathrm{Tp}_{i-1})\}\\ &\le \lambda\delta\gamma_n,\end{aligned} \tag{2.9}$$

this is a contradiction.

Case 6. let $\gamma_n = d(v_i, Tv_j)$ for some $0 \le i, j \le n$.

(i) If $i = 0$, then $\gamma_n = d(v_0, Tv_j)$.

(ii) If $i \ge 1$, then, from (1.4) and (1.10)

$$\begin{aligned}
\gamma_n &= d(v_i, Tv_j) \\
&\le d\left(W(v_{i-1}, Tv_{i-1}; 1-\lambda, \lambda), Tv_j\right) \\
&\le (1-\lambda) d\left(v_{i-1}, Tv_j\right) + \lambda d\left(Tv_{i-1}, Tv_j\right) \\
&\le (1-\lambda) d\left(v_{i-1}, Tv_j\right) \\
&\quad + \lambda\delta \cdot \max\left\{d\left(v_{i-1}, v_j\right), d(v_{i-1}, Tv_{i-1}), d\left(v_j, Tv_j\right), d\left(v_{i-1}, Tv_j\right), d\left(v_j, Tv_{i-1}\right)\right\} \\
&\le (1-\lambda) d\left(v_{i-1}, Tv_j\right) + \lambda\delta\gamma_n,
\end{aligned} \tag{2.10}$$

it implies that $\gamma_n = d(v_0, Tv_j)$.

Case 7. Let $\gamma_n = d(p_i, p_j)$ for some $0 \le i, j \le n$.

(i) If $i = 0$, $j > 0$, then $\gamma_n = d(p_0, p_j)$.

(ii) If $i, j \ge 1$, then, from (1.3), (1.10)

$$\begin{aligned}
\gamma_n &= d\left(p_i, p_j\right) \\
&\le d\left(Tp_{i-1}, Tp_{j-1}\right) \\
&\le \delta \cdot \max\left\{d\left(p_{i-1}, p_{j-1}\right), d\left(p_{i-1}, Tp_{i-1}\right), d\left(p_{j-1}, Tp_{j-1}\right), d\left(p_{i-1}, Tp_{j-1}\right), d\left(\mathrm{p}_{j-1}, Tp_{i-1}\right)\right\} \\
&\le \delta\gamma_n,
\end{aligned} \tag{2.11}$$

it is a contradiction.

Case 8. let $\gamma_n = d(p_i, Tp_j)$ for some $0 \le i, j \le n$.

(i) If $i = 0$, then $\gamma_n = d(p_0, Tp_j)$.

(ii) If $i \ge 1$, then, from (1.3) and (1.10)

$$\begin{aligned}
\gamma_n &= d\left(p_i, Tp_j\right) \\
&\le d\left(Tp_{i-1}, Tp_j\right) \\
&\le \delta \cdot \max\left\{d\left(p_{i-1}, p_j\right), d\left(p_{i-1}, Tp_{i-1}\right), d\left(p_j, Tp_j\right), d\left(p_{i-1}, Tp_j\right), d\left(p_j, Tp_{i-1}\right)\right\} \\
&\le \delta\gamma_n,
\end{aligned} \tag{2.12}$$

which is a contradiction.

Set

$$\begin{aligned}\eta_n = \max\{&\max\{d(p_i, v_0) : 0 < i \le n\}, \max\{d(v_0, Tp_i) : 0 < i \le n\},\\ &\max\{d(v_0, Tv_i) : 0 < i \le n\}, \max\{d(p_0, p_i) : 0 < i \le n\},\\ &\max\{d(p_0, Tv_i) : 0 < i \le n\}, \max\{d(p_0, Tp_i) : 0 < i \le n\}, M\},\end{aligned} \tag{2.13}$$

where $M = \max\{d(p_0, v_0), d(v_0, Tp_0), d(v_0, Tv_0), d(p_0, Tv_0), d(p_0, Tp_0)\}$.

In view of the above cases, then $\gamma_n = \eta_n$, and we obtain that $\{\gamma_n\}$ is bounded. Indeed, suppose that $\gamma_n = d(p_i, v_0)$ for some $0 < i \le n$. Then,

$$\begin{aligned}\gamma_n &= d(p_i, v_0)\\ &\le d(p_i, Tv_0) + d(Tv_0, v_0)\\ &= d(Tp_{i-1}, Tv_0) + d(Tv_0, v_0)\\ &\le d(Tp_{i-1}, Tv_0) + d(Tv_0, v_0)\\ &\le \delta \cdot \max\{d(p_{i-1}, v_0), d(v_0, Tv_0), d(p_{i-1}, Tp_{i-1}), d(v_0, Tp_{i-1}), d(p_{i-1}, Tv_0)\}\\ &\quad + d(Tv_0, v_0)\\ &\le \delta\gamma_n + d(Tv_0, v_0),\end{aligned} \tag{2.14}$$

which implies that $\gamma_n \le (1/(1-\delta))d(Tv_0, v_0)$. Similarly, if $\gamma_n = d(v_0, Tp_i)$ or $\gamma_n = d(v_0, Tv_i)$, we also obtain $\gamma_n \le (1/(1-\delta))d(Tv_0, v_0)$.

On the other hand, suppose that $\gamma_n = d(p_0, p_i)$ for some $0 < i \le n$. Then,

$$\begin{aligned}\gamma_n &= d(p_0, p_i)\\ &\le d(p_0, Tp_0) + d(Tp_0, Tp_{i-1})\\ &\le d(p_0, Tp_0)\\ &\quad + \delta \cdot \max\{d(p_0, p_{i-1}), d(p_0, Tp_0), d(p_{i-1}, Tp_{i-1}), d(p_0, Tp_{i-1}), d(p_{i-1}, Tp_0)\}\\ &\le d(p_0, Tp_0) + \delta\gamma_n,\end{aligned} \tag{2.15}$$

which implies that $\gamma_n \le (1/(1-\delta))d(Tp_0, p_0)$. Similarly, if $\gamma_n = d(p_0, Tv_i)$ or $\gamma_n = d(p_0, Tp_i)$, we also obtain $\gamma_n \le (1/(1-\delta))d(Tp_0, p_0)$. Therefore, from the above results, we obtain that $\gamma_n \le (1/(1-\delta))M$, that is, $\{A_n\}$ is bounded.

For each $n \in \mathbb{N}$, define

$$B_n = \{v_i\}_{i\ge n} \cup \{p_i\}_{i\ge n} \cup \{Tp_i\}_{i\ge n} \cup \{Tv_i\}_{i\ge n}, \qquad R_n = \operatorname{diam}(B_n). \tag{2.16}$$

Then, using the same proof above, it can be shown that

$$R_n = \operatorname{diam}(B_n) = \max\{\sup\{d(p_i, v_n) : i \geq n\}, \sup\{d(v_n, Tp_i) : i \geq n\},$$
$$\sup\{d(p_n, Tv_i) : i \geq n\}, \sup\{d(v_n, Tv_i) : i \geq n\},$$
$$\sup\{d(p_n, p_i) : i > n\}, \sup\{d(p_n, Tp_i) : i \geq n\}\}. \tag{2.17}$$

If $R_n = \sup\{d(p_i, v_n) : i \geq n\}$, and using (1.1) and (1.4), then

$$\begin{aligned}
R_n &= \sup_{i\geq n} d(p_i, v_n) \\
&= \sup_{i\geq n} d(p_i, W(v_{n-1}, Tv_{n-1}; 1-\lambda, \lambda)) \\
&\leq \sup_{i\geq n}\{(1-\lambda)d(p_i, v_{n-1}) + \lambda d(p_i, Tv_{n-1})\} \\
&\leq \sup_{i\geq n}\{(1-\lambda)R_{n-1} + \lambda d(Tp_{i-1}, Tv_{n-1})\} \\
&\leq \sup_{i\geq n}\{(1-\lambda)R_{n-1} + \lambda\delta \\
&\qquad \cdot \max\{d(p_{i-1}, v_{n-1}), d(v_{n-1}, Tv_{n-1}), d(p_{i-1}, Tp_{i-1}), d(v_{n-1}, Tp_{i-1}), d(p_{i-1}, Tv_{n-1})\}\} \\
&\leq (1-\lambda)R_{n-1} + \lambda\delta R_{n-1} \\
&= (1-\lambda(1-\delta))R_{n-1} \\
&\leq \cdots \\
&\leq (1-\lambda(1-\delta))^n R_0 \\
&\longrightarrow 0
\end{aligned} \tag{2.18}$$

as $n \to \infty$. Since $d(Tv_n, Tp_n) \leq R_n$, hence $d(Tv_n, Tp_n) \to 0$ as $n \to \infty$. Similarly, if $R_n = \sup\{d(v_n, Tp_i) : i \geq n\}$ or $R_n = \sup\{d(v_n, Tv_i) : i \geq n\}$, $R_n = \sup\{d(p_n, Tv_i) : i \geq n\}, R_n = \sup\{d(v_n, Tv_i) : i \geq n\}, R_n = \sup\{d(p_n, p_i) : i > n\}, R_n = \sup\{d(p_n, Tp_i) : i \geq n\}$, we may obtain the similar results. Therefore, from (2.1), we get

$$d(v_{n+1}, p_{n+1}) \leq (1-\lambda)d(v_n, p_n) + \sigma_n, \tag{2.19}$$

where $\sigma_n = ((1-\lambda)/(1-\delta))d(p_n, q) + \lambda d(Tv_n, Tp_n)$.

In (2.19), set $\rho_n = d(v_n, p_n)$. Then (2.19) is as follows:

$$\rho_{n+1} \leq (1-\lambda)\rho_n + \sigma_n. \tag{2.20}$$

By Lemma 1.3, we have $d(v_n, p_n) \to 0$ as $n \to \infty$. From the inequality $0 \leq d(v_n, q) \leq d(v_n, p_n) + d(p_n, q)$, we have $\lim_{n\to\infty} d(v_n, q) = 0$.

Conversely, we will prove that (ii) $\Rightarrow$ (i). If $\lambda = 1$, then $v_{n+1} = W(v_n, Tv_n; 0, 1) = Tv_n$ is Picard iteration. □

Theorem 2.2. *Let $(E, d, W), T, F(T)$ be as in Theorem 2.1. Suppose that $\{u_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$ are defined by the iterative processes* (1.5) *and* (1.6)*, respectively, and $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$ are real sequences in $[0,1]$ such that $\sum_{n=0}^{\infty} a_n = \infty$. Then, the following two assertions are equivalent:*

(i) *Mann iteration* (1.5) *converges strongly to the unique fixed point $q \in F(T)$;*

(ii) *Ishikawa iteration* (1.6) *converges strongly to the unique fixed point $q \in F(T)$.*

Proof. If the Ishikawa iteration (1.6) converges strongly to q, then setting $b_n = 0$, for all $n \geq 0$, in (1.6), we can get the convergence of Mann iteration (1.5). Conversely, we will show that (i) $\Rightarrow$ (ii). Letting $\lim_{n\to\infty} d(u_n, q) = 0$, we want to prove $\lim_{n\to\infty} d(x_n, q) = 0$.

From (1.5) and (1.6),

$$\begin{aligned}
&d(x_{n+1}, u_{n+1}) \\
&\quad = d(W(x_n, Ty_n; 1-a_n, a_n), W(u_n, Tu_n; 1-a_n, a_n)) \\
&\quad \leq (1-a_n) d(x_n, W(u_n, Tu_n; 1-a_n, a_n)) + a_n d(Ty_n, W(u_n, Tu_n; 1-a_n, a_n)) \\
&\quad \leq (1-a_n)^2 d(x_n, u_n) + a_n(1-a_n) d(x_n, Tu_n) \\
&\qquad + (1-a_n) a_n d(Ty_n, u_n) + a_n^2 d(Ty_n, Tu_n) \\
&\quad \leq (1-a_n)^2 d(x_n, u_n) + a_n(1-a_n) d(x_n, u_n) + a_n(1-a_n) d(u_n, Tu_n) \\
&\qquad + (1-a_n) a_n d(Ty_n, Tu_n) + (1-a_n) a_n d(Tu_n, u_n) + a_n^2 d(Ty_n, Tu_n) \\
&\quad = (1-a_n) d(x_n, u_n) + 2a_n(1-a_n) d(u_n, Tu_n) + a_n d(Ty_n, Tu_n) \\
&\quad \leq (1-a_n) d(x_n, u_n) + 2a_n(1-a_n) d(u_n, q) \\
&\qquad + 2a_n(1-a_n) d(Tu_n, Tq) + a_n d(Ty_n, Tu_n) \\
&\quad \leq (1-a_n) d(x_n, u_n) + 2a_n \frac{1-a_n}{1-\delta} d(u_n, q) + a_n d(Ty_n, Tu_n).
\end{aligned} \tag{2.21}$$

Using (1.10) with $x = y_n$, $y = u_n$, to obtain

$$d(Ty_n, Tu_n) \leq \delta \cdot \max\{d(y_n, u_n), d(u_n, Tu_n), d(y_n, Ty_n), d(y_n, Tu_n), d(u_n, Ty_n)\}, \tag{2.22}$$

set

$$\begin{gathered}
A_{nn} = \{u_i\}_{i=0}^{n} \cup \{y_i\}_{i=0}^{n} \cup \{x_i\}_{i=0}^{n} \cup \{Tu_i\}_{i=0}^{n} \cup \{Ty_i\}_{i=0}^{n} \cup \{Tx_i\}_{i=0}^{n}, \\
\gamma_{nn} = \operatorname{diam}(A_{nn}).
\end{gathered} \tag{2.23}$$

Applying the similar proof methods of Theorem 2.1, we obtain that $\{A_{nn}\}$ is also bounded. The other proof is the same as that of Theorem 2.1 and is here omitted. □

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