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Research Article

Weak and Strong Convergence Theorems for Asymptotically Strict Pseudocontractive Mappings in the Intermediate Sense

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We study the convergence of Ishikawa iteration process for the class of asymptotically κ-strict pseudocontractive mappings in the intermediate sense which is not necessarily Lipschitzian. Weak convergence theorem is established. We also obtain a strong convergence theorem by using hybrid projection for this iteration process. Our results improve and extend the corresponding results announced by many others.

1. Introduction and Preliminaries

Throughout this paper, we always assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\rightharpoonup$ and $\to$ denote weak and strong convergence, respectively. $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, that is, $\omega_w(x_n) = \{x \in H : \exists x_{n_j} \rightharpoonup x\}$. Let C be a nonempty closed convex subset of H. It is well known that for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \tag{1.1}$$

for all $y \in C$. P_C is called the metric projection of H onto C. P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Let $T : C \to C$ be a mapping. In this paper, we denote the fixed point set of T by $F(T)$. Recall that T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$, such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall x, y \in C,\ \forall n \ge 1. \tag{1.3}$$

T is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, such that

$$\|T^n x - T^n y\| \le k_n\|x - y\|, \quad \forall x, y \in C,\ \forall n \ge 1. \tag{1.5}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as a generalization of the class of nonexpansive mappings. T is said to be asymptotically nonexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right) \le 0. \tag{1.6}$$

Observe that if we define

$$\tau_n = \max\left\{0, \sup_{x,y\in C} \left(\|T^n x - T^n y\| - \|x - y\|\right)\right\}, \tag{1.7}$$

then $\tau_n \to 0$ as $n \to \infty$. It follows that (1.6) is reduced to

$$\|T^n x - T^n y\| \le \|x - y\| + \tau_n, \quad \forall x, y \in C,\ \forall n \ge 1. \tag{1.8}$$

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck et al. [2]. It is known [3] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate sense contains properly the class of asymptotically nonexpansive mappings.

Recall that T is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0, 1)$, such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.9}$$

T is said to be an asymptotically κ-strict pseudocontraction with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\gamma_n \to 0$ as $n \to \infty$, such that

$$\left\|T^n x - T^n y\right\|^2 \leq (1+\gamma_n)\left\|x-y\right\|^2 + \kappa\left\|(I-T^n)x-(I-T^n)y\right\|^2, \quad \forall x,y \in C,\ n \geq 1. \tag{1.10}$$

The class of asymptotically κ-strict pseudocontractions was introduced by Qihou [4] in 1996 (see also [5]). Kim and Xu [6] studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{(\kappa + \sqrt{1+(1-\kappa)\gamma_n})/(1+\kappa) : n \in N\}$.

Recently, Sahu et al. [7] introduced a class of new mappings: asymptotically κ-strict pseudocontractive mappings in the intermediate sense. Recall that T is said to be an asymptotically κ-strict pseudocontraction in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\} \subset [0,\infty)$ with $\gamma_n \to 0$ as $n \to \infty$, such that

$$\limsup_{n\to\infty}\sup_{x,y\in C}\left(\left\|T^n x - T^n y\right\|^2 - (1+\gamma_n)\left\|x-y\right\|^2 - \kappa\left\|(I-T^n)x-(I-T^n)y\right\|^2\right) \leq 0. \tag{1.11}$$

Throughout this paper, we assume that

$$c_n = \max\left\{0, \sup_{x,y\in C}\left(\left\|T^n x - T^n y\right\|^2 - (1+\gamma_n)\left\|x-y\right\|^2 - \kappa\left\|(I-T^n)x-(I-T^n)y\right\|^2\right)\right\}. \tag{1.12}$$

It follows that $c_n \to 0$ as $n \to \infty$ and (1.11) is reduced to the relation

$$\left\|T^n x - T^n y\right\|^2 \leq (1+\gamma_n)\left\|x-y\right\|^2 + \kappa\left\|(I-T^n)x-(I-T^n)y\right\|^2 + c_n, \quad \forall x,y \in C. \tag{1.13}$$

They obtained a weak convergence theorem of modified Mann iterative processes for the class of mappings which is not necessarily Lipschitzian. Moreover, a strong convergence theorem was also established in a real Hilbert space by hybrid projection methods; see [7] for more details.

In this paper, we consider the problem of convergence of Ishikawa iterative processes for the class of asymptotically κ-strict pseudocontractive mappings in the intermediate sense.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 (see [8, 9]). *Let $\{\delta_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n, \quad \forall n \geq 1. \tag{1.14}$$

If $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty}\gamma_n < \infty$, then $\lim_{n\to\infty}\delta_n$ exists.

Lemma 1.2 (see [10]). *Let $\{x_n\}$ be a bounded sequence in a reflexive Banach space X. If $\omega_w(x_n) = \{x\}$, then $x_n \rightharpoonup x$.*

Lemma 1.3 (see [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if $\langle x - z, y - z\rangle \leq 0$, for all $y \in C$.*

Lemma 1.4 (see [11]). *For a real Hilbert space H, the following identities hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$,

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0,1]$, *for all* $x, y \in H$;

(iii) *(Opial condition) If $\{x_n\}$ is a sequence in H weakly convergent to z, then*

$$\limsup_{n\to\infty}\|x_n - y\|^2 = \limsup_{n\to\infty}\|x_n - z\|^2 + \|z - y\|^2, \quad \forall y \in H. \tag{1.15}$$

Lemma 1.5 (see [7]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-\kappa}\left(\kappa\|x - y\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x - y\|^2 + (1-\kappa)c_n}\right), \quad \forall x, y \in C,\ \forall n \in \mathbb{N}. \tag{1.16}$$

Lemma 1.6. *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $n \in \mathbb{N}$. If $\gamma_n < 1$, then*

$$\|T^n x - T^n y\| \leq \frac{1}{1-\kappa}\left(\left(\kappa + \sqrt{2-\kappa}\right)\|x - y\| + \sqrt{c_n}\right), \quad \forall x, y \in C. \tag{1.17}$$

Proof. If $\gamma_n < 1$, for $x, y \in C$, we obtain from Lemma 1.5 that

$$\begin{aligned}
\|T^n x - T^n y\| &\leq \frac{1}{1-\kappa}\left(\kappa\|x - y\| + \sqrt{(1 + (1-\kappa)\gamma_n)\|x - y\|^2 + (1-\kappa)c_n}\right) \\
&\leq \frac{1}{1-\kappa}\left(\kappa\|x - y\| + \sqrt{(2-\kappa)\|x - y\|^2 + c_n}\right) \\
&\leq \frac{1}{1-\kappa}\left\{\kappa\|x - y\| + \sqrt{\left(\sqrt{2-\kappa}\|x - y\| + \sqrt{c_n}\right)^2}\right\} \\
&= \frac{1}{1-\kappa}\left(\left(\kappa + \sqrt{2-\kappa}\right)\|x - y\| + \sqrt{c_n}\right).
\end{aligned} \tag{1.18}$$

□

Lemma 1.7 (see [7]). *Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$, then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 1.8 (see [7, Proposition 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - T^m x_n\| = 0$, then $(I-T)x = 0$.*

Lemma 1.9 (see [7]). *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense. Then $F(T)$ is closed and convex.*

2. Main Results

Theorem 2.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following Ishikawa iterative process:*

$$\begin{gathered} x_1 \in C, \\ y_n = \beta_n T^n x_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad \forall n \geq 1, \end{gathered} \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that the following restrictions are satisfied:

(i) $\sum_{n=1}^{\infty}\alpha_n c_n < \infty$ *and* $\sum_{n=1}^{\infty}((1+\gamma_n)^2 - 1) < \infty$,

(ii) $0 < a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and* $b \in (0, (-(1-\kappa)^2 + \sqrt{(1-\kappa)^4 + 2(\kappa+\sqrt{2-\kappa})^2(1-\kappa)^2})/2(\kappa+\sqrt{2-\kappa})^2)$.

Then the sequence $\{x_n\}$ given by (2.1) converges weakly to an element of $F(T)$.

Proof. Let $p \in F(T)$. From (1.13) and Lemma 1.4, we see that

$$\begin{aligned} \|y_n - p\|^2 &= \|\beta_n(T^n x_n - p) + (1-\beta_n)(x_n - p)\|^2 \\ &= \beta_n\|T^n x_n - p\|^2 + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\leq \beta_n\Big((1+\gamma_n)\|x_n - p\|^2 + \kappa\|x_n - T^n x_n\|^2 + c_n\Big) \\ &\quad + (1-\beta_n)\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\leq (1+\gamma_n)\|x_n - p\|^2 - \beta_n(1-\beta_n-\kappa)\|x_n - T^n x_n\|^2 + \beta_n c_n. \end{aligned} \tag{2.2}$$

Without loss of generality, we may assume that $\gamma_n < 1$ for all $n \in \mathbb{N}$. Since

$$\|x_n - y_n\|^2 = \|x_n - \beta_n T^n x_n - (1-\beta_n)x_n\|^2 = \beta_n^2 \|x_n - T^n x_n\|^2, \tag{2.3}$$

it follows from Lemma 1.6 that

$$\begin{aligned}
\|y_n - T^n y_n\|^2 &= \|\beta_n (T^n x_n - T^n y_n) + (1-\beta_n)(x_n - T^n y_n)\|^2 \\
&= \beta_n \|T^n x_n - T^n y_n\|^2 + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&\le \frac{\beta_n}{(1-\kappa)^2}\left(\left(\kappa + \sqrt{2-\kappa}\right)\|x_n - y_n\| + \sqrt{c_n}\right)^2 \\
&\quad + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\
&\le 2\beta_n^3\left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa}\right)^2 \|x_n - T^n x_n\|^2 + \frac{2\beta_n c_n}{(1-\kappa)^2} \\
&\quad + (1-\beta_n)\|x_n - T^n y_n\|^2 - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2.
\end{aligned} \tag{2.4}$$

By (2.2) and (2.4), we obtain that

$$\begin{aligned}
&\|T^n y_n - p\|^2 \\
&\quad \le (1+\gamma_n)\|y_n - p\|^2 + \kappa\|y_n - T^n y_n\|^2 + c_n \\
&\quad \le (1+\gamma_n)^2\|x_n - p\|^2 - \beta_n(1+\gamma_n)(1-\beta_n-\kappa)\|x_n - T^n x_n\|^2 \\
&\qquad + \beta_n(1+\gamma_n)c_n + 2\kappa\beta_n^3\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2\|x_n - T^n x_n\|^2 + \frac{2\kappa\beta_n c_n}{(1-\kappa)^2} \\
&\qquad + \kappa(1-\beta_n)\|x_n - T^n y_n\|^2 - \kappa\beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 + c_n \\
&\quad = (1+\gamma_n)^2\|x_n - p\|^2 - \beta_n\left[(1+\gamma_n)(1-\beta_n-\kappa) - 2\kappa\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 + \kappa(1-\beta_n)\right] \\
&\qquad \times \|x_n - T^n x_n\|^2 + \kappa(1-\beta_n)\|x_n - T^n y_n\|^2 + c_n M_1,
\end{aligned} \tag{2.5}$$

where $M_1 = \sup_{n\geq 1}\{\beta_n(1+\gamma_n) + 2\kappa\beta_n/(1-\kappa)^2 + 1\}$. It follows from (2.5) and $\alpha_n \leq \beta_n$ that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\alpha_n(T^n y_n - p) + (1-\alpha_n)(x_n - p)\|^2 \\
&\quad = \alpha_n\|T^n y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\quad \leq \alpha_n(1+\gamma_n)^2\|x_n - p\|^2 - \alpha_n\beta_n\left[(1+\gamma_n)(1-\beta_n-\kappa) - 2\kappa\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 + \kappa(1-\beta_n)\right] \\
&\qquad \times \|x_n - T^n x_n\|^2 + \alpha_n\kappa(1-\beta_n)\|x_n - T^n y_n\|^2 \\
&\qquad + \alpha_n c_n M_1 + (1-\alpha_n)\|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|T^n y_n - x_n\|^2 \\
&\quad \leq (1+\gamma_n)^2\|x_n - p\|^2 - \alpha_n\beta_n\left[(1+\gamma_n)(1-\beta_n) - \kappa\gamma_n - 2\kappa\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\beta_n\right] \\
&\qquad \times \|x_n - T^n x_n\|^2 - \alpha_n(1-\alpha_n-\kappa(1-\beta_n))\|x_n - T^n y_n\|^2 + \alpha_n c_n M_1 \\
&\quad \leq (1+\gamma_n)^2\|x_n - p\|^2 - \alpha_n\beta_n\left[(1+\gamma_n)(1-\beta_n) - \kappa\gamma_n - 2\kappa\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\beta_n\right] \\
&\qquad \times \|x_n - T^n x_n\|^2 + \alpha_n c_n M_1.
\end{aligned} \tag{2.6}$$

From the condition (ii) and $\gamma_n \to 0$, we see that there exists n_0 such that

$$\begin{aligned}
&(1+\gamma_n)(1-\beta_n) - \kappa\gamma_n - 2\kappa\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\beta_n \\
&\quad \geq 1 - \beta_n - \kappa\gamma_n - 2\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\beta_n \\
&\quad \geq 1 - 2\beta_n - \kappa\gamma_n - 2\beta_n^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 \\
&\quad \geq 1 - 2b - 2b^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\gamma_n \\
&\quad \geq \frac{1}{2}\left(1 - 2b - 2b^2\left(\frac{\kappa+\sqrt{2-\kappa}}{1-\kappa}\right)^2\right) > 0, \quad \forall n \geq n_0.
\end{aligned} \tag{2.7}$$

By (2.6), we have

$$\|x_{n+1} - p\|^2 \le (1+\gamma_n)^2 \|x_n - p\|^2 + \alpha_n c_n M_1, \quad \forall n \ge n_0. \tag{2.8}$$

In view of Lemma 1.1 and the condition (i), we obtain that $\lim_{n\to\infty}\|x_n - p\|$ exists. For any $n \ge n_0$, it is easy to see from (2.6) and (2.7) that

$$\begin{aligned} &\frac{a^2}{2}\left(1 - 2b - 2b^2\left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa}\right)^2\right)\|x_n - T^n x_n\|^2 \\ &\quad \le (1+\gamma_n)^2\|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n c_n M_1, \end{aligned} \tag{2.9}$$

which implies that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{2.10}$$

Note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \alpha_n \|T^n y_n - x_n\| \\ &\le \alpha_n \|T^n y_n - T^n x_n\| + \alpha_n \|T^n x_n - x_n\| \\ &\le \frac{\alpha_n}{1-\kappa}\left(\left(\kappa + \sqrt{2-\kappa}\right)\|x_n - y_n\| + \sqrt{c_n}\right) + \alpha_n\|T^n x_n - x_n\| \\ &= \frac{\alpha_n \beta_n}{1-\kappa}\left(\kappa + \sqrt{2-\kappa}\right)\|x_n - T^n x_n\| + \frac{\alpha_n\sqrt{c_n}}{1-\kappa} + \alpha_n\|T^n x_n - x_n\|. \end{aligned} \tag{2.11}$$

From (2.10), we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.12}$$

Since T is uniformly continuous, we obtain from (2.10), (2.12) and Lemma 1.7 that

$$\lim_{n\to\infty} \|x_n - T x_n\| = 0. \tag{2.13}$$

By the boundedness of $\{x_n\}$, there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$. Observe that T is uniformly continuous and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, for any $m \in \mathbb{N}$ we have $\|x_n - T^m x_n\| \to 0$ as $n \to \infty$. From Lemma 1.8, we see that $x \in F(T)$.

To complete the proof, it suffices to show that $\omega_w(\{x_n\})$ consists of exactly one point, namely, x. Suppose there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to some $z \in C$ and $z \neq x$. As in the case of x, we can also see that $z \in F(T)$. It follows that $\lim_{n\to\infty}\|x_n - x\|$ and $\lim_{n\to\infty}\|x_n - z\|$ exist. Since H satisfies the Opial condition, we have

$$\begin{aligned}\lim_{n\to\infty}\|x_n - x\| &= \lim_{k\to\infty}\|x_{n_k} - x\| < \lim_{k\to\infty}\|x_{n_k} - z\| = \lim_{n\to\infty}\|x_n - z\|,\\ \lim_{n\to\infty}\|x_n - z\| &= \lim_{j\to\infty}\left\|x_{n_j} - z\right\| < \lim_{j\to\infty}\left\|x_{n_j} - x\right\| = \lim_{n\to\infty}\|x_n - x\|,\end{aligned} \tag{2.14}$$

which is a contradiction. We see $x = z$ and hence $\omega_w(\{x_n\})$ is a singleton. Thus, $\{x_n\}$ converges weakly to x by Lemma 1.2. □

Corollary 2.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the following Ishikawa iterative process:*

$$\begin{aligned}&x_1 \in C,\\ &y_n = \beta_n T^n x_n + (1-\beta_n)x_n,\\ &x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \quad \forall n \geq 1,\end{aligned} \tag{2.15}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Assume that the following restrictions are satisfied:

(i) $\sum_{n=1}^{\infty}((1+\gamma_n)^2 - 1) < \infty$,

(ii) $0 < a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and* $b \in (0, (-(1-\kappa)^2 + \sqrt{(1-\kappa)^4 + 2(\kappa + \sqrt{2-\kappa})^2(1-\kappa)^2})/2(\kappa + \sqrt{2-\kappa})^2)$.

Then the sequence $\{x_n\}$ given by (2.15) *converges weakly to an element of $F(T)$.*

Next, we modify Ishikawa iterative process to get a strong convergence theorem.

Theorem 2.3. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $F(T) \neq \emptyset$ and bounded. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Ishikawa iterative process:*

$$\begin{aligned}&x_1 \in C,\\ &y_n = \beta_n T^n x_n + (1-\beta_n)x_n,\\ &z_n = \alpha_n T^n y_n + (1-\alpha_n)x_n,\\ &C_n = \left\{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n - \rho_n\|x_n - T^n x_n\|^2\right\},\\ &Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\},\\ &x_{n+1} = P_{C_n \cap Q_n} x_1,\end{aligned} \tag{2.16}$$

where $\theta_n = \alpha_n c_n M_1 + (2\gamma_n + \gamma_n^2)\Delta_n$, $M_1 = \sup_{n\geq 1}\{\beta_n(1+\gamma_n) + 2\kappa\beta_n/(1-\kappa)^2 + 1\}$, $\Delta_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty$ and $\rho_n = \alpha_n\beta_n[1 - 2\beta_n - \kappa\gamma_n - 2\beta_n^2((\kappa + \sqrt{2-\kappa})/(1-\kappa))^2]$ for each

$n \geq 1$. *Assume that the control sequences* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are chosen such that* $0 < a \leq \alpha_n \leq \beta_n \leq b$ *for some* $a > 0$ *and* $b \in (0, (-(1-\kappa)^2 + \sqrt{(1-\kappa)^4 + 2(\kappa + \sqrt{2-\kappa})^2(1-\kappa)^2})/2(\kappa + \sqrt{2-\kappa})^2)$. *Then the sequence* $\{x_n\}$ *given by* (2.16) *converges strongly to an element of* $F(T)$.

Proof. We break the proof into six steps.

Step 1 ($C_n \cap Q_n$ is closed and convex for each $n \geq 1$). It is obvious that Q_n is closed and convex and C_n is closed for each $n \geq 1$. Note that the defining inequality in C_n is equivalent to the inequality

$$2\langle x_n - z_n, z\rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n - \rho_n\|x_n - T^n x_n\|^2, \tag{2.17}$$

it is easy to see that C_n is convex for each $n \geq 1$. Hence, $C_n \cap Q_n$ is closed and convex for each $n \geq 1$.

Step 2 ($F(T) \subset C_n \cap Q_n$ for each $n \geq 1$). Let $p \in F(T)$. Following (2.6), (2.7) and the algorithm (2.16), we have

$$\begin{aligned}
\|z_n - p\|^2 &\leq (1+\gamma_n)^2\|x_n - p\|^2 \\
&\quad - \alpha_n\beta_n\left[(1+\gamma_n)(1-\beta_n) - \kappa\gamma_n - 2\kappa\beta_n^2\left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\beta_n\right] \\
&\quad \times \|x_n - T^n x_n\|^2 + \alpha_n c_n M_1 \\
&\leq (1+\gamma_n)^2\|x_n - p\|^2 - \alpha_n\beta_n\left[1 - 2\beta_n - 2\beta_n^2\left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa}\right)^2 - \kappa\gamma_n\right] \\
&\quad \times \|x_n - T^n x_n\|^2 + \alpha_n c_n M_1 \\
&= \|x_n - p\|^2 - \rho_n\|x_n - T^n x_n\|^2 + \alpha_n c_n M_1 + \left(2\gamma_n + \gamma_n^2\right)\|x_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \rho_n\|x_n - T^n x_n\|^2 + \theta_n,
\end{aligned} \tag{2.18}$$

where $\theta_n = \alpha_n c_n M_1 + (2\gamma_n + \gamma_n^2)\Delta_n$, $M_1 = \sup_{n\geq 1}\{\beta_n(1+\gamma_n) + 2\kappa\beta_n/(1-\kappa)^2 + 1\}$, $\Delta_n = \sup\{\|x_n - z\|^2 : z \in F(T)\} < \infty$ and $\rho_n = \alpha_n\beta_n[1 - 2\beta_n - \kappa\gamma_n - 2\beta_n^2((\kappa + \sqrt{2-\kappa})/(1-\kappa))^2]$ for each $n \geq 1$. Hence $p \in C_n$ for each $n \geq 1$.

Next, we show that $F(T) \subset Q_n$ for each $n \geq 1$. We prove this by induction. For $n = 1$, we have $F(T) \subset C = Q_1$. Assume that $F(T) \subset Q_n$ for some $n > 1$. Since x_{n+1} is the projection of x_1 onto $C_n \cap Q_n$, we have

$$\langle x_{n+1} - z, x_1 - x_{n+1}\rangle \geq 0, \qquad \forall z \in C_n \cap Q_n. \tag{2.19}$$

By the induction consumption, we know that $F(T) \subset C_n \cap Q_n$. In particular, for any $p \in F(T)$ we have

$$\langle x_{n+1} - p, x_1 - x_{n+1} \rangle \geq 0. \tag{2.20}$$

This implies that $p \in Q_{n+1}$. That is, $F(T) \subset Q_{n+1}$. By the principle of mathematical induction, we get $F(T) \subset Q_n$ and hence $F(T) \subset C_n \cap Q_n$ for all $n \geq 1$. This means that the iteration algorithm (2.16) is well defined.

Step 3 ($\lim_{n\to\infty}\|x_n - x_1\|$ exists and $\{x_n\}$ is bounded). In view of (2.16), we see that $x_n = P_{Q_n}x_1$ and $x_{n+1} = P_{C_n\cap Q_n}x_1 \in Q_n$. It follows that

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\| \tag{2.21}$$

for each $n \geq 1$. We, therefore, obtain that the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. Noticing that $F(T) \subset Q_n$ and $x_n = P_{Q_n}x_1$, we have

$$\|x_1 - x_n\| \leq \left\|x_1 - p\right\|, \quad \forall p \in F(T). \tag{2.22}$$

This shows that the sequence $\{\|x_n - x_1\|\}$ is bounded. Therefore, the limit of $\{\|x_n - x_1\|\}$ exists and $\{x_n\}$ is bounded.

Step 4 ($x_{n+1} - x_n \to 0$). Observe that $x_n = P_{Q_n}x_1$ and $x_{n+1} \in Q_n$ which imply

$$\langle x_{n+1} - x_n, x_1 - x_n \rangle \leq 0. \tag{2.23}$$

Using Lemma 1.4, we obtain

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_1) - (x_n - x_1)\|^2 \\
&= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\
&\leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.
\end{aligned} \tag{2.24}$$

Hence, we obtain that $x_{n+1} - x_n \to 0$ as $n \to \infty$.

Step 5 ($x_n - Tx_n \to 0$ as $n \to \infty$). In view of $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n - \rho_n\|x_n - T^n x_n\|^2. \tag{2.25}$$

On the other hand, we see that

$$\begin{aligned}
\|z_n - x_{n+1}\|^2 &= \|z_n - x_n + x_n - x_{n+1}\|^2 \\
&= \|z_n - x_n\|^2 + \|x_n - x_{n+1}\|^2 + 2\langle z_n - x_n, x_n - x_{n+1} \rangle.
\end{aligned} \tag{2.26}$$

Combing (2.25) and (2.26) and noting $z_n = \alpha_n T^n y_n + (1-\alpha_n)x_n$, we obtain that

$$\alpha_n^2 \left\| T^n y_n - x_n \right\|^2 + 2\left\langle \alpha_n (T^n y_n - x_n), x_n - x_{n+1} \right\rangle \leq \theta_n - \rho_n \|x_n - T^n x_n\|^2. \tag{2.27}$$

From the assumption and (2.7), we see that there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} &1 - 2\beta_n - \kappa\gamma_n - 2\beta_n^2 \left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa} \right)^2 \\ &\quad \geq \frac{1}{2}\left(1 - 2b - 2b^2 \left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa} \right)^2 \right) > 0, \quad \forall n \geq n_0. \end{aligned} \tag{2.28}$$

For any $n \geq n_0$, it follows from the definition of ρ_n and (2.27) that

$$\frac{a^2}{2}\left(1 - 2b - 2b^2 \left(\frac{\kappa + \sqrt{2-\kappa}}{1-\kappa} \right)^2 \right) \|x_n - T^n x_n\|^2 \leq \theta_n + 2\alpha_n \left\| T^n y_n - x_n \right\| \cdot \|x_n - x_{n+1}\|. \tag{2.29}$$

Noting that $\theta_n \to 0$ as $n \to \infty$ and Step 4, we obtain that

$$\lim_{n\to\infty} \|x_n - T^n x_n\| = 0. \tag{2.30}$$

It follows from Step 4, (2.30) and Lemma 1.7 that $x_n - Tx_n \to 0$ as $n \to \infty$.

Step 6 ($x_n \to x \in F(T)$ as $n \to \infty$, where $x = P_{F(T)}x_1$). Since H is reflexive and $\{x_n\}$ is bounded, we get that $\omega_w(\{x_n\})$ is nonempty. First, we show that $\omega_w(\{x_n\})$ is a singleton. Assume that $\{x_{n_i}\}$ is subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x \in C$. Observe that T is uniformly continuous and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, for any $m \in \mathbb{N}$ we have $\|x_n - T^m x_n\| \to 0$ as $n \to \infty$. From Lemma 1.8, we see that $x \in \omega_w(\{x_n\}) \subset F(T)$.

Since $x_{n+1} = P_{C_n \cap Q_n} x_1$, we obtain that

$$\|x_1 - x_{n+1}\| \leq \left\| x_1 - P_{F(T)}x_1 \right\|, \tag{2.31}$$

for each $n \geq 1$. Observe that $x_1 - x_{n_i} \rightharpoonup x_1 - x$ as $n \to \infty$. By the weak lower semicontinuity of norm, we have

$$\left\| x_1 - P_{F(T)}x_1 \right\| \leq \|x_1 - x\| \leq \liminf_{n\to\infty} \|x_1 - x_{n_i}\| \leq \limsup_{n\to\infty} \|x_1 - x_{n_i}\| \leq \left\| x_1 - P_{F(T)}x_1 \right\|. \tag{2.32}$$

This implies that

$$\left\| x_1 - P_{F(T)} x_1 \right\| = \| x_1 - x \|, \tag{2.33}$$

$$\lim_{n \to \infty} \| x_1 - x_{n_i} \| = \left\| x_1 - P_{F(T)} x_1 \right\|. \tag{2.34}$$

Hence $x = P_{F(T)} x_1$ by the uniqueness of the nearest point projection of x_1 onto $F(T)$. Since $\{x_{n_i}\}$ is an arbitrary weakly convergent subsequence, it follows that $\omega_w(\{x_n\}) = \{x\}$ and hence $x_n \rightharpoonup x$. It is easy to see as (2.34) that $\|x_1 - x_n\| \to \|x_1 - x\|$. Since H has the Kadec-Klee property, we obtain that $x_1 - x_n \to x_1 - x$, that is, $x_n \to x = P_{F(T)} x_1$ as $n \to \infty$. This completes the proof. □

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