

Research Article

Strong Convergence Theorem for Equilibrium Problems and Fixed Points of a Nonspreading Mapping in Hilbert Spaces

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Received 30 June 2010; Revised 10 October 2010; Accepted 13 December 2010

Academic Editor: Brailey Sims

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We introduce an iterative method for finding a common element of the set of solutions of equilibrium problems and the set of fixed points of a nonspreading mapping in a Hilbert space. Then, we prove a strong convergence theorem which is connected with the work of S. Takahashi and W. Takahashi (2007) and Iemoto and Takahashi (2009).

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, and let C be a closed convex subset of H . Let $F : C \times C \rightarrow \mathbb{R}$ be bifunction, where \mathbb{R} is the set of real numbers. The equilibrium problem for $F : C \times C \rightarrow \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \geq 0 \quad \forall y \in C. \quad (1.1)$$

The set of solution of (1.1) is denoted by $EP(F)$. Given a mapping $A : C \rightarrow H$, let $F(x, y) = \langle Ax, y - x \rangle$ for all $x, y \in C$. Then, $z \in EP(F)$ if and only if $\langle Az, y - z \rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1); see, for example, [1–9] and the references therein.

A mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and a mapping F is said to be *firmly nonexpansive* if $\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle$ for all $x, y \in C$. Let E be a smooth, strictly convex and reflexive Banach space, and let J be the

duality mapping of E and C a nonempty closed convex subset of E . A mapping $S : C \rightarrow C$ is said to be *nonspreading* if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x) \quad (1.2)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$ for all $x, y \in E$; see, for instance, Kohsaka and Takahashi [10]. In the case when E is a Hilbert space, we know that $\phi(x, y) = \|x - y\|^2$ for all $x, y \in E$. Then a nonspreading mapping $S : C \rightarrow C$ in a Hilbert space H is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2 \quad (1.3)$$

for all $x, y \in C$. Let $F(Q)$ be the set of fixed points of Q , and $F(Q)$ nonempty; a mapping $Q : C \rightarrow C$ is said to be *quasi-nonexpansive* if $\|Qx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(Q)$.

Remark 1.1. In a Hilbert space, we know that every firmly nonexpansive mapping is nonspreading and that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [10, 11].

In 1953, Mann [12] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad (1.4)$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Mann iteration has been extensively investigated for nonexpansive mappings. In an infinite-dimensional Hilbert space, Mann iteration can conclude only weak convergence (see [12, 13]). Fourteen years later, Halpern [14] introduced the following iterative scheme for approximating a fixed point of T :

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad (1.5)$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0, 1]$. Strong convergence of this type iterative sequence has been widely studied: Wittmann [15] discussed such a sequence in a Hilbert space.

On the other hand, Kohsaka and Takahashi [10] proved an existence theorem of fixed point for nonspreading mappings in a Banach space. Recently, Lemoto and Takahashi [16] studied the approximation theorem of common fixed points for a nonexpansive mapping T of C into itself and a nonspreading mapping S of C into itself in a Hilbert space. In particular, this result reduces to approximation fixed points of a nonspreading mapping S of C into itself in a Hilbert space by using iterative scheme

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n. \quad (1.6)$$

Some methods have been proposed to solve the equilibrium problem and fixed point problem of nonexpansive mapping: see, for instance, [1, 2, 6, 7, 17–20] and the references

therein. In 1997, Combettes and Hirstoaga [3] introduced an iterative scheme of finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Recently, S. Takahashi and W. Takahashi [8] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solution of equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space. Let $S : C \rightarrow H$ be a nonexpansive mapping. In 2008, Plubtieng and Punpaeng [7] introduced a new iterative sequence for finding a common element of the set of solution of equilibrium problems and the set of fixed points of a nonexpansive mapping in a Hilbert space which is the optimality condition for the minimization problem. Very recently, S. Takahashi and W. Takahashi [9] introduced an iterative method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space and then obtain that the sequence converges strongly to a common element of two sets.

In this paper, motivated by S. Takahashi and W. Takahashi [8] and Lemoto and Takahashi [16], we introduce an iterative sequence and prove a strong convergence theorem for finding solution of equilibrium problems and the set of fixed points of a nonspreading mapping in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space. When $\{x_n\}$ is a sequence in H , $x_n \rightharpoonup x$ implies that x_n converges weakly to x and $x_n \rightarrow x$ means the strong convergence. Let C be a nonempty closed convex subset of H . For every point $x \in H$, there exists a unique nearest point in C ; denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \forall y \in C. \quad (2.1)$$

P_C is called the metric projection of H onto C . We know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$,

$$z = P_C x \iff \langle x - z, z - y \rangle \geq 0 \quad \forall y \in C. \quad (2.2)$$

Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\begin{aligned} \langle x - P_C x, y - P_C y \rangle &\leq 0, \\ \|x - y\|^2 &\geq \|x - P_C x\|^2 + \|y - P_C y\|^2 \end{aligned} \quad (2.3)$$

for all $x \in H$, $y \in C$. We also know that H satisfies Opial's condition [21], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \quad (2.4)$$

holds for every $y \in H$ with $x \neq y$; see [21, 22] for more details.

The following lemmas will be useful for proving the convergence result of this paper.

Lemma 2.1 (see [23]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, one has*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha \|x\|^2 + \beta \|y\|^2 + \gamma \|z\|^2 - \alpha\beta \|x - y\|^2 - \alpha\gamma \|x - z\|^2 - \beta\gamma \|y - z\|^2. \quad (2.5)$$

Lemma 2.2 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H . Let S be a nonspreading mapping of C into itself. Then the following are equivalent.*

- (1) *There exists $x \in C$ such that $\{S^n x\}$ is bounded;*
- (2) *$F(S)$ is nonempty.*

Lemma 2.3 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H . Let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

Lemma 2.4. *Let H be a real Hilbert space. Then for all $x, y \in H$,*

- (1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle$;
- (2) $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, x \rangle$.

Lemma 2.5 (see [24]). *Let $\{a_n\}, \{b_n\} \subset [0, \infty)$, and let $\{c_n\} \subset [0, 1)$ be sequences of real numbers such that*

$$\begin{aligned} a_{n+1} &\leq (1 - c_n)a_n + b_n, \text{ for all } n \in \mathbb{N}, \\ \sum_{n=1}^{\infty} c_n &= \infty \text{ and } \sum_{n=1}^{\infty} b_n < \infty. \end{aligned}$$

Then, $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.6 (see [16]). *Let H be a Hilbert space, C a closed convex subset of H , and $S : C \rightarrow C$ a nonspreading mapping with $F(S) \neq \emptyset$. Then S is demiclosed, that is, $x_n \rightarrow u$ and $x_n - Sx_n \rightarrow 0$ imply $u \in F(S)$.*

Lemma 2.7 (see [16]). *Let H be a Hilbert space, C a nonempty closed convex subset of a real Hilbert space H , and let S be a nonspreading mapping of C into itself, and let $A = I - S$. Then*

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay \rangle + \frac{1}{2}(\|Ax\|^2 + \|Ay\|^2). \quad (2.6)$$

Lemma 2.8 (see [25]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0, \quad (2.7)$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that

- (1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (2) $\limsup_{n \rightarrow \infty} (\delta_n / \alpha_n) \leq 0$ or $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

For solving the equilibrium problems for a bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [26].

Lemma 2.9 (see [26]). *Let C be a nonempty closed convex subset of H , and let F be a bifunction of $C \times C$ into \mathbb{R} satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C. \quad (2.8)$$

The following lemma was also given in [4].

Lemma 2.10 (see [4]). *Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\} \quad (2.9)$$

for all $z \in H$. Then, the following hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$;
- (3) $F(T_r) = EP(F)$;
- (4) $EP(F)$ is closed and convex.

Lemma 2.11 (see [27]). *Let (Γ_n) be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $(\Gamma_{n_j})_{j \geq 0}$ of (Γ_n) which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $(\tau(n))_{n \geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}. \quad (2.10)$$

Then $(\tau(n))_{n \geq n_0}$ is a nondecreasing sequence verifying $\lim_{n \rightarrow \infty} \tau(n) = \infty$, and the following properties are satisfied for all $n \geq n_0$:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \Gamma_n \leq \Gamma_{\tau(n)+1}. \quad (2.11)$$

3. Main Result

In this section, we prove a strong convergence theorem for finding a common element of the set of fixed points of a nonspreading mapping and the set of solutions of the equilibrium problems.

Theorem 3.1. Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be a bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4), and let S be a nonspreading mapping of C into itself such that $F(S) \cap EP(F) \neq \emptyset$. Let $u \in C$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) u_n], \end{aligned} \quad (3.1)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ and $\{r_n\} \in (0, \infty)$ satisfy

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < a \leq \beta_n \leq b < 1, \\ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| &< \infty, \quad \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty, \\ \liminf_{n \rightarrow \infty} r_n &> 0, \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} u$.

Proof. Let $p \in F(S) \cap EP(F)$. From $u_n = T_{r_n} x_n$, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \leq \|x_n - p\| \quad (3.2)$$

for all $n \in \mathbb{N}$. Put $y_n = \alpha_n u + (1 - \alpha_n) u_n$. We divide the proof into several steps.

Step 1. We claim that the sequences $\{x_n\}, \{u_n\}, \{y_n\}$, and $\{S y_n\}$ are bounded. First, we note that

$$\begin{aligned} \|S y_n - p\| &\leq \|y_n - p\| \\ &= \|\alpha_n u + (1 - \alpha_n) u_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|u_n - p\| \\ &\leq \alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|, \end{aligned} \quad (3.3)$$

and so

$$\begin{aligned} \|x_{n+1} - p\| &= \|\beta_n x_n + (1 - \beta_n) S y_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|S y_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\ &= \beta_n \|x_n - p\| + (1 - \beta_n) \|\alpha_n u + (1 - \alpha_n) u_n - p\| \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) (\alpha_n \|u - p\| + (1 - \alpha_n) \|u_n - p\|) \\ &\leq \beta_n \|x_n - p\| + (1 - \beta_n) (\alpha_n \|u - p\| + (1 - \alpha_n) \|x_n - p\|) \\ &= (1 - \alpha_n (1 - \beta_n)) \|x_n - p\| + \alpha_n (1 - \beta_n) \|u - p\|. \end{aligned} \quad (3.4)$$

Putting $M = \max\{\|x_n - p\|, \|u - p\|\}$, we note that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - p\| \leq M$. Assume that $\|x_k - p\| \leq M$ for all $k \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|x_{k+1} - p\| &\leq (1 - \alpha_k(1 - \beta_k))\|x_k - p\| + \alpha_k(1 - \beta_k)\|u - p\| \\ &\leq (1 - \alpha_k(1 - \beta_k))M + \alpha_k(1 - \beta_k)M \\ &= M. \end{aligned} \tag{3.5}$$

By induction, we obtain that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. So, $\{x_n\}$ is bound. Hence, $\{u_n\}$, $\{y_n\}$, and $\{Sy_n\}$ are also bounded.

Step 2. Put $t_n = \beta_n y_n + (1 - \beta_n)Sy_n$. We claim that $\|x_{n+1} - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. We note that

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|(\beta_n x_n + (1 - \beta_n)Sy_n) - (\beta_{n-1}x_{n-1} + (1 - \beta_{n-1})Sy_{n-1})\| \\ &= \|\beta_n x_n - \beta_n x_{n-1} + \beta_n x_{n-1} - \beta_{n-1}x_{n-1} + (1 - \beta_n)Sy_n - (1 - \beta_n)Sy_{n-1} \\ &\quad + (1 - \beta_n)Sy_{n-1} - (1 - \beta_{n-1})Sy_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) \|Sy_n - Sy_{n-1}\| \\ &\quad + |(1 - \beta_n) - (1 - \beta_{n-1})| \|Sy_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) \|y_n - y_{n-1}\| + |\beta_{n-1} - \beta_n| \|Sy_{n-1}\| \\ &= \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) \\ &\quad \times \|\alpha_n u + (1 - \alpha_n)u_n - \alpha_{n-1}u - (1 - \alpha_{n-1})u_{n-1}\| + |\beta_n - \beta_{n-1}| \|Sy_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) \\ &\quad \times [\|\alpha_n u - \alpha_{n-1}u\| + \|(1 - \alpha_n)u_n - (1 - \alpha_{n-1})u_{n-1}\|] + |\beta_n - \beta_{n-1}| \|Sy_{n-1}\| \\ &= \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| \|u\| \\ &\quad + (1 - \beta_n) \|(1 - \alpha_n)u_n - (1 - \alpha_n)u_{n-1} + (1 - \alpha_n)u_{n-1} - (1 - \alpha_{n-1})u_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|Sy_{n-1}\| \\ &\leq \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| \|u\| \\ &\quad + (1 - \beta_n) (1 - \alpha_n) \|u_n - u_{n-1}\| + (1 - \beta_n) |(1 - \alpha_n) - (1 - \alpha_{n-1})| \|u_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|Sy_{n-1}\| \\ &= \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| \|x_{n-1}\| + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| \|u\| \\ &\quad + (1 - \beta_n) (1 - \alpha_n) \|u_n - u_{n-1}\| + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| \|u_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|Sy_{n-1}\| \\ &= \beta_n \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}| K_1 + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| K_1 \\ &\quad + (1 - \beta_n) (1 - \alpha_n) \|u_n - u_{n-1}\| + (1 - \beta_n) |\alpha_n - \alpha_{n-1}| K_1 + |\beta_n - \beta_{n-1}| K_1, \end{aligned} \tag{3.6}$$

where $K_1 = \sup\{\|x_n\| + \|Sy_n\| + \|u\| + \|u_{n-1}\| : n \in \mathbb{N}\}$. On the other hand, from $u_n = T_{r_n}x_n$ and $u_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad (3.7)$$

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \geq 0 \quad (3.8)$$

for all $y \in C$. Putting $y = u_{n+1}$ in (3.7) and $y = u_n$ in (3.8), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \frac{1}{r_n} \langle u_{n+1} - u_n, u_n - x_n \rangle &\geq 0, \\ F(u_{n+1}, u_n) + \frac{1}{r_{n+1}} \langle u_n - u_{n+1}, u_{n+1} - x_{n+1} \rangle &\geq 0. \end{aligned} \quad (3.9)$$

So, from (A2), we note that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \geq 0, \quad (3.10)$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \quad (3.11)$$

Without loss of generality, let us assume that there exists a real number d such that $r_n > d > 0$ for all $n \in \mathbb{N}$. Thus, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \quad (3.12)$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{d} |r_{n+1} - r_n| L, \end{aligned} \quad (3.13)$$

where $L = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. So, from (3.6), we note that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \beta_n \|x_n - x_{n-1}\| + 2|\beta_n - \beta_{n-1}|K_1 + 2(1 - \beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1 - \beta_n)(1 - \alpha_n) \left(\|x_n - x_{n-1}\| + \frac{1}{d}|r_n - r_{n-1}|L \right) \\
&= (\beta_n + (1 - \beta_n)(1 - \alpha_n)) \|x_n - x_{n-1}\| + 2|\beta_n - \beta_{n-1}|K_1 + 2(1 - \beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1 - \beta_n)(1 - \alpha_n) \frac{1}{d}|r_n - r_{n-1}|L \\
&= (1 - (1 - \beta_n)\alpha_n) \|x_n - x_{n-1}\| + 2|\beta_n - \beta_{n-1}|K_1 + 2(1 - \beta_n)|\alpha_n - \alpha_{n-1}|K_1 \\
&\quad + (1 - \beta_n)(1 - \alpha_n) \frac{L}{d}|r_n - r_{n-1}|.
\end{aligned} \tag{3.14}$$

By Lemma 2.5, we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0 \tag{3.15}$$

for $p \in F(S) \cup EP(F)$. We note from $u_n = T_{r_n}x_n$ that

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}x_n - T_{r_n}p\|^2 \leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p \rangle = \langle u_n - p, x_n - p \rangle \\
&= \frac{1}{2} (\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2),
\end{aligned} \tag{3.16}$$

and hence

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.17}$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n x_n + (1 - \beta_n)S y_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|S y_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\
&= \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|\alpha_n u + (1 - \alpha_n)u_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + \alpha_n (1 - \beta_n) \|u - p\|^2 + (1 - \beta_n)(1 - \alpha_n) \|u_n - p\|^2 \\
&\leq \beta_n \|x_n - p\|^2 + \alpha_n (1 - \beta_n) \|u - p\|^2 + (1 - \beta_n)(1 - \alpha_n) (\|x_n - p\|^2 - \|x_n - u_n\|^2) \\
&= (1 - (1 - \beta_n)\alpha_n) \|x_n - p\|^2 + \alpha_n (1 - \beta_n) \|u - p\|^2 + (1 - \beta_n)(1 - \alpha_n) \|x_n - u_n\|^2,
\end{aligned} \tag{3.18}$$

and hence

$$\begin{aligned}
(1 - \beta_n)(1 - \alpha_n)\|x_n - u_n\|^2 &\leq \alpha_n(1 - \beta_n)\|u - p\|^2 - \alpha_n(1 - \beta_n)\|x_n - p\|^2 \\
&\quad + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&= \alpha_n(1 - \beta_n)\|u - p\|^2 - \alpha_n(1 - \beta_n)\|x_n - p\|^2 \\
&\quad + (\|x_n - p\| - \|x_{n+1} - p\|)(\|x_n - p\| + \|x_{n+1} - p\|) \\
&\leq \alpha_n(1 - \beta_n)\|u - p\|^2 - \alpha_n(1 - \beta_n)\|x_n - p\|^2 \\
&\quad + \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|).
\end{aligned} \tag{3.19}$$

So, we have $\|x_n - u_n\| \rightarrow 0$. Indeed, since $y_n = \alpha_n u + (1 - \alpha_n)u_n$, it follows that

$$\begin{aligned}
\lim_{n \rightarrow \infty} \|x_n - y_n\| &= \lim_{n \rightarrow \infty} \|x_n - (\alpha_n u + (1 - \alpha_n)u_n)\| \\
&= \lim_{n \rightarrow \infty} \|(\alpha_n + (1 - \alpha_n))x_n - (\alpha_n u + (1 - \alpha_n)u_n)\| \\
&\leq \lim_{n \rightarrow \infty} [\alpha_n \|x_n - u\| + (1 - \alpha_n)\|x_n - u_n\|] \\
&= \lim_{n \rightarrow \infty} \alpha_n \|x_n - u\| + \lim_{n \rightarrow \infty} (1 - \alpha_n)\|x_n - u_n\| \\
&= 0.
\end{aligned} \tag{3.20}$$

Then, we note that

$$\begin{aligned}
\|x_{n+1} - t_n\| &= \|(\beta_n x_n + (1 - \beta_n)S y_n) - (\beta_n y_n + (1 - \beta_n)S y_n)\| \\
&= \|\beta_n(x_n - y_n) + (1 - \beta_n)(S y_n - S y_n)\| \\
&= \beta_n \|x_n - y_n\|.
\end{aligned} \tag{3.21}$$

Since, $0 < a \leq \beta_n \leq b < 1$ and $\|x_n - y_n\| \rightarrow 0$, it follows that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - t_n\| = 0. \tag{3.22}$$

Step 3. Put $A = I - S$. From $Ap = 0$, it follows by Lemma 2.7 that

$$\begin{aligned}
\|t_n - p\|^2 &= \|(\beta_n y_n + (1 - \beta_n)S y_n) - p\|^2 \\
&= \|(y_n - p) - (1 - \beta_n)(y_n - S y_n)\|^2 \\
&= \|(y_n - p) - (1 - \beta_n)A y_n\|^2 \\
&= \|(y_n - p)\|^2 - 2(1 - \beta_n)\langle y_n - p, A y_n - Ap \rangle + (1 - \beta_n)^2 \|A y_n\|^2
\end{aligned}$$

$$\begin{aligned}
&\leq \|(y_n - p)\|^2 - 2(1 - \beta_n) \left\{ \|Ay_n - Ap\|^2 - \frac{1}{2} (\|Ay_n\|^2 + \|Ap\|^2) \right\} \\
&\quad + (1 - \beta_n)^2 \|Ay_n\|^2 \\
&= \|\alpha_n(u - p) + (1 - \alpha_n)(u_n - p)\|^2 - 2(1 - \beta_n) \|Ay_n\|^2 \\
&\quad + (1 - \beta_n) \|Ay_n\|^2 + (1 - \beta_n)^2 \|Ay_n\|^2 \\
&\leq \alpha_n \|(u - p)\|^2 + (1 - \alpha_n) \|(u_n - p)\|^2 - \beta_n(1 - \beta_n) \|Ay_n\|^2 \\
&\leq \alpha_n \|(u - p)\|^2 + (1 - \alpha_n) \|(x_n - p)\|^2 - \beta_n(1 - \beta_n) \|Ay_n\|^2 \\
&\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 - \beta_n(1 - \beta_n) \|Ay_n\|^2.
\end{aligned} \tag{3.23}$$

Since $0 < a \leq \beta_n \leq b < 1$, we have $\beta_n(1 - \beta_n) \geq a(1 - b) := K_2$. Therefore, by (3.23), we obtain

$$\begin{aligned}
K_2 \|y_n - Sy_n\|^2 &= K_2 \|Ay_n\|^2 \\
&\leq \alpha_n \|u - p\|^2 + \|x_n - p\|^2 - \|t_n - p\|^2 \\
&\leq \alpha_n M^2 + \|x_n - p\|^2 - \|t_n - p\|^2 \\
&= \alpha_n M^2 + \|x_n - p\|^2 - \|(t_n - x_{n+1}) + (x_{n+1} - p)\|^2 \\
&= \alpha_n M^2 + \|x_n - p\|^2 - \|t_n - x_{n+1}\|^2 - 2\langle t_n - x_{n+1}, x_{n+1} - p \rangle - \|x_{n+1} - p\|^2 \\
&\leq \alpha_n M^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - 2\langle t_n - x_{n+1}, x_{n+1} - p \rangle.
\end{aligned} \tag{3.24}$$

Step 4. Putting $z = P_{F(S) \cap EP(F)}u$, we claim that the sequence $\{x_n\}$ converges strongly to $z = P_{F(S) \cap EP(F)}u$. Indeed, we discuss two possible cases.

Case 1. Assume that there exists n_0 such that the sequence $\{\|x_n - p\|\}$ is a nonincreasing sequence for all $n \geq n_0$. Then we have $\|x_{n+1} - p\| \leq \|x_n - p\|$ (for $n \geq n_0$), and hence $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. Therefore

$$\lim_{n \rightarrow \infty} \|x_n - p\| = \lim_{n \rightarrow \infty} \|x_{n+1} - p\|. \tag{3.25}$$

By (3.22), (3.24), and (3.25), we get

$$\|y_n - Sy_n\| \longrightarrow 0. \tag{3.26}$$

Let $\{y_{n_i}\}$ be a subsequence of $\{y_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u - z, y_n - z \rangle = \lim_{n \rightarrow \infty} \langle u - z, y_{n_i} - z \rangle. \tag{3.27}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ which converges weakly to w . Without loss of generality, we can assume that $y_{n_i} \rightharpoonup w$. Since C is closed and convex, we note that C is weakly closed. So, we have $w \in C$. Since $\|Sy_n - y_n\| \rightarrow 0$, it follows by Lemma 2.6 that $w \in F(S)$. From (3.27) and the property of metric projection, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle u - z, y_n - z \rangle &= \lim_{n \rightarrow \infty} \langle u - z, y_{n_i} - z \rangle \\ &= \langle u - z, w - z \rangle \leq 0. \end{aligned} \quad (3.28)$$

Finally, we prove that $x_n \rightarrow z$. In fact, since $y_n - z = \alpha_n(u - z) + (1 - \alpha_n)(u_n - z)$, it follows that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(\beta_n x_n + (1 - \beta_n)Sy_n) - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|Sy_n - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \|y_n - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n) \left[(1 - \alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n \langle u - z, y_n - z \rangle \right] \\ &\leq \beta_n \|x_n - z\|^2 + (1 - \beta_n)(1 - \alpha_n) \|x_n - z\|^2 + 2\alpha_n(1 - \beta_n) \langle u - z, y_n - z \rangle \\ &= (1 - \alpha_n(1 - \beta_n)) \|x_n - z\|^2 + 2\alpha_n(1 - \beta_n) \langle u - z, y_n - z \rangle. \end{aligned} \quad (3.29)$$

By (3.28) and $\sum_{n=1}^{\infty} \alpha_n = \infty$, we immediately deduce by Lemma 2.8 that $x_n \rightarrow z$.

Case 2. Assume that for all $n \in \mathbb{N}$, there exists $m \geq n$ such that $\|x_m - p\| < \|x_{m+1} - p\|$. Put $a_m =: \|x_m - p\|$ for all $m \in \mathbb{N}$. Thus, it follows that there exists a subsequence $(a_{n_k})_{k \geq 1}$ of $(a_n)_{n \geq 1}$ such that $a_{n_k} < a_{n_{k+1}}$ for all $k \in \mathbb{N}$. Let $\varphi : \mathbb{N}_1 \rightarrow \mathbb{N}$ be a mapping defined by

$$\varphi(n) = \max\{k \leq n : a_k \leq a_{k+1}\}, \quad (3.30)$$

where $\mathbb{N}_1 = \{n \in \mathbb{N} : n \geq n_1\}$. By Lemma 2.11, we note that $\varphi(n)$ is a nondecreasing sequence such that $\varphi(n) \rightarrow \infty$ as $n \rightarrow \infty$ and that the following properties are satisfied by all numbers $n \geq n_1$:

$$a_{\varphi(n)} \leq a_{\varphi(n)+1}, \quad a_n \leq a_{\varphi(n)+1}. \quad (3.31)$$

From (3.24), we have

$$\begin{aligned} K_2 \|y_{\varphi(n)} - Sy_{\varphi(n)}\|^2 &\leq \alpha_{\varphi(n)} M^2 + \|x_{\varphi(n)} - p\|^2 - \|x_{\varphi(n)+1} - p\|^2 \\ &\quad - 2 \langle t_{\varphi(n)} - x_{\varphi(n)+1}, x_{\varphi(n)+1} - p \rangle \\ &\leq \alpha_{\varphi(n)} M^2 - 2 \langle t_{\varphi(n)} - x_{\varphi(n)+1}, x_{\varphi(n)+1} - p \rangle. \end{aligned} \quad (3.32)$$

This implies that

$$\|\mathbf{y}_{\varphi(n)} - S\mathbf{y}_{\varphi(n)}\| \longrightarrow 0. \quad (3.33)$$

Take a subsequence $\{\mathbf{y}_{\varphi(n)_i}\}$ of $\{\mathbf{y}_{\varphi(n)}\}$ such that

$$\limsup_{n \rightarrow \infty} \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle = \lim_{n \rightarrow \infty} \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)_i} - \mathbf{z} \rangle. \quad (3.34)$$

From the boundedness of $\{\mathbf{y}_{\varphi(n)_i}\}$, we can assume that $\mathbf{y}_{\varphi(n)_i} \rightharpoonup \mathbf{v}$. Since C is closed and convex, it follows that C is weakly closed. So, we have $\mathbf{v} \in C$. Since $\|S\mathbf{y}_{\varphi(n)} - \mathbf{y}_{\varphi(n)}\| \rightarrow 0$, it follows by Lemma 2.6 that $\mathbf{v} \in F(S)$. From (3.34) and the property of metric projection, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle &= \lim_{n \rightarrow \infty} \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)_i} - \mathbf{z} \rangle \\ &= \langle \mathbf{u} - \mathbf{z}, \mathbf{v} - \mathbf{z} \rangle \\ &\leq 0. \end{aligned} \quad (3.35)$$

By the same argument as (3.29) in Case 1, we conclude immediately that, for all $n \geq 1$,

$$\begin{aligned} 0 &\leq \|\mathbf{x}_{\varphi(n)+1} - \mathbf{z}\|^2 - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\leq \beta_{\varphi(n)} \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 + (1 - \beta_{\varphi(n)}) \|S\mathbf{y}_{\varphi(n)} - \mathbf{z}\|^2 - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\leq \beta_{\varphi(n)} \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 + (1 - \beta_{\varphi(n)}) \|\mathbf{y}_{\varphi(n)} - \mathbf{z}\|^2 - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\leq \beta_{\varphi(n)} \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 + (1 - \beta_{\varphi(n)}) \\ &\quad \times \left[(1 - \alpha_{\varphi(n)})^2 \|\mathbf{u}_{\varphi(n)} - \mathbf{z}\|^2 + 2\alpha_{\varphi(n)} \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle \right] - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\leq \beta_{\varphi(n)} \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 + (1 - \beta_{\varphi(n)}) (1 - \alpha_{\varphi(n)}) \|\mathbf{u}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\quad + 2\alpha_{\varphi(n)} (1 - \beta_{\varphi(n)}) \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\leq \beta_{\varphi(n)} \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 + (1 - \beta_{\varphi(n)}) (1 - \alpha_{\varphi(n)}) \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &\quad + 2\alpha_{\varphi(n)} (1 - \beta_{\varphi(n)}) \langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \\ &= \alpha_{\varphi(n)} (1 - \beta_{\varphi(n)}) \left[2\langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \right] \\ &\leq 2\langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle - \|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2, \end{aligned} \quad (3.36)$$

which implies that

$$\|\mathbf{x}_{\varphi(n)} - \mathbf{z}\|^2 \leq 2\langle \mathbf{u} - \mathbf{z}, \mathbf{y}_{\varphi(n)} - \mathbf{z} \rangle. \quad (3.37)$$

By (3.35), we have

$$\lim_{n \rightarrow \infty} \|x_{\varphi(n)} - z\| = 0, \quad (3.38)$$

and hence

$$\lim_{n \rightarrow \infty} \|x_{\varphi(n)+1} - z\| = \lim_{n \rightarrow \infty} \|x_{\varphi(n)} - z\| = 0. \quad (3.39)$$

Since $\|x_n - z\| = a_n \leq a_{\varphi(n)} = \|x_{\varphi(n)} - z\|$ for all $n \geq n_1$, we have

$$\lim_{n \rightarrow \infty} \|x_n - z\| = 0. \quad (3.40)$$

This completes the proof. \square

As direct consequences of Theorem 3.1, we obtain corollaries.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be a bifunctions from $C \times C \rightarrow \mathbb{R}$ satisfying (A1)–(A4), and let S be a firmly nonexpansive mapping of C into itself such that $F(S) \cap EP(F) \neq \emptyset$. Let $u \in C$, and let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 \in C$ and*

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) u_n], \end{aligned} \quad (3.41)$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$ and $\{r_n\} \in (0, \infty)$ satisfy

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n &= 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < a \leq \beta_n \leq b < 1, \\ \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| &< \infty, \quad \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty, \\ \liminf_{n \rightarrow \infty} r_n &> 0, \quad \text{and} \quad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty. \end{aligned}$$

Then $\{x_n\}$ converges strongly to $z \in F(S) \cap EP(F)$, where $z = P_{F(S) \cap EP(F)} u$.

Acknowledgments

The authors would like to thank the referees for the insightful comments and suggestions. Moreover, the authors gratefully acknowledge the Thailand Research Fund Master Research Grants (TRF-MAG, MRG-WII515S029) for funding this paper.

References

- [1] S.-S. Chang, H. W. Joseph Lee, and C. K. Chan, "A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization," *Nonlinear Analysis Theory: Methods & Applications*, vol. 70, no. 9, pp. 3307–3319, 2009.

- [2] V. Colao, G. Marino, and H.-K. Xu, "An iterative method for finding common solutions of equilibrium and fixed point problems," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 1, pp. 340–352, 2008.
- [3] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.
- [4] P. L. Combettes and S. A. Hirstoaga, "Equilibrium programming in Hilbert spaces," *Journal of Nonlinear and Convex Analysis*, vol. 63, pp. 123–145, 1994.
- [5] J.-W. Peng, Y. Wang, D. S. Shyu, and J.-C. Yao, "Common solutions of an iterative scheme for variational inclusions, equilibrium problems, and fixed point problems," *Journal of Inequalities and Applications*, vol. 2008, Article ID 720371, 15 pages, 2008.
- [6] S. Plubtieng and R. Punpaeng, "A new iterative method for equilibrium problems and fixed point problems of nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 197, no. 2, pp. 548–558, 2008.
- [7] S. Plubtieng and R. Punpaeng, "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455–469, 2007.
- [8] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.
- [9] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.
- [10] F. Kohsaka and W. Takahashi, "Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces," *Archiv der Mathematik*, vol. 91, no. 2, pp. 166–177, 2008.
- [11] S. Plubtieng and K. Sombut, "Weak convergence theorems for a system of mixed equilibrium problems and nonspreading mappings in a Hilbert space," *Journal of Inequalities and Applications*, vol. 2010, Article ID 246237, 12 pages, 2010.
- [12] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, pp. 506–510, 1953.
- [13] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.
- [14] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.
- [15] R. Wittmann, "Approximation of fixed points of nonexpansive mappings," *Archiv der Mathematik*, vol. 58, no. 5, pp. 486–491, 1992.
- [16] S. Iemoto and W. Takahashi, "Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. e2082–e2089, 2009.
- [17] S. Plubtieng and W. Sriprad, "A viscosity approximation method for finding common solutions of variational inclusions, equilibrium problems, and fixed point problems in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 567147, 20 pages, 2009.
- [18] S. Plubtieng and W. Sriprad, "Hybrid methods for equilibrium problems and fixed points problems of a countable family of relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 962628, 17 pages, 2010.
- [19] S. Plubtieng and T. Thammathiwat, "A viscosity approximation method for equilibrium problems, fixed point problems of nonexpansive mappings and a general system of variational inequalities," *Journal of Global Optimization*, vol. 46, no. 3, pp. 447–464, 2010.
- [20] S. Plubtieng and T. Thammathiwat, "A viscosity approximation method for finding a common solution of fixed points and equilibrium problems in Hilbert spaces," *Journal of Global Optimization*. In press.
- [21] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.
- [22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.
- [23] M. O. Osilike and D. I. Igbokwe, "Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations," *Computers & Mathematics with Applications*, vol. 40, no. 4-5, pp. 559–567, 2000.

- [24] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.
- [25] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.
- [26] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *Mathematics Students*, vol. 6, pp. 117–136, 2005.
- [27] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.