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Research Article

Measures of Noncircularity and Fixed Points of Contractive Multifunctions

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In analogy to the Eisenfeld-Lakshmikantham measure of nonconvexity and the Hausdorff measure of noncompactness, we introduce two mutually equivalent measures of noncircularity for Banach spaces satisfying a Cantor type property, and apply them to establish a fixed point theorem of Darbo type for multifunctions. Namely, we prove that every multifunction with closed values, defined on a closed set and contractive with respect to any one of these measures, has the origin as a fixed point.

1. Introduction

Let $(X, \|\cdot\|)$ be a Banach space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In what follows, we write $B_X = \{x \in X : \|x\| \leq 1\}$ for the closed unit ball of X. Denote by 2^X the collection of all subsets of X and consider

$$b(X) := \left\{C \in 2^X \setminus \{\emptyset\} : C \text{ bounded}\right\}. \tag{1.1}$$

For $C, D \in b(X)$, define their nonsymmetric Hausdorff distance by

$$h(C, D) := \sup_{c \in C} \inf_{d \in D} \|c - d\| \tag{1.2}$$

and their symmetric Hausdorff distance (or Hausdorff-Pompeiu distance) by

$$H(C,D) := \max\{h(C,D), h(D,C)\}. \tag{1.3}$$

This H is a pseudometric on $b(X)$, since

$$H(C,D) = H\left(\overline{C},D\right) = H\left(C,\overline{D}\right) = H\left(\overline{C},\overline{D}\right), \tag{1.4}$$

where $\overline{A}$ denotes the closure of $A \in 2^X$.

Around 1955, Darbo [1] ensured the existence of fixed points for so-called condensing operators on Banach spaces, a result which generalizes both Schauder fixed point theorem and Banach contractive mapping principle. More precisely, Darbo proved that if $M \in b(X)$ is closed and convex, κ is a measure of noncompactness, and $f : M \to M$ is continuous and κ-contractive, that is, $\kappa(f(A)) \le r\kappa(A)$ $(A \in b(M))$ for some $r \in]0,1[$, then f has a fixed point. Below we recall the axiomatic definition of a regular measure of noncompactness on X; we refer to [2] for details.

Definition 1.1. A function $\kappa : b(X) \to [0,\infty[$ will be called a regular measure of noncompactness if κ satisfies the following axioms, for $A, B \in b(X)$, and $\lambda \in \mathbb{K}$:

(1) $\kappa(A) = 0$ if, and only if, $\overline{A}$ is compact.

(2) $\kappa(\operatorname{co} A) = \kappa(A) = \kappa(\overline{A})$, where $\operatorname{co} A$ denotes the convex hull of A.

(3) (monotonicity) $A \subset B$ implies $\kappa(A) \le \kappa(B)$.

(4) (maximum property) $\kappa(A \cup B) = \max\{\kappa(A), \kappa(B)\}$.

(5) (homogeneity) $\kappa(\lambda A) = |\lambda|\kappa(A)$.

(6) (subadditivity) $\kappa(A + B) \le \kappa(A) + \kappa(B)$.

A regular measure of noncompactness κ possesses the following properties:

(1) $\kappa(A) \le \kappa(B_X)\delta(A)$, where

$$\delta(A) := \sup_{x,y\in A} \|x - y\| \tag{1.5}$$

is the diameter of $A \in b(X)$ (cf. [2, Theorem 3.2.1]).

(2) (Hausdorff continuity) $|\kappa(A) - \kappa(B)| \le \kappa(B_X)H(A,B)$ $(A, B \in b(X))$ [2, page 12].

(3) (Cantor property) If $\{A_n\}_{n=0}^{\infty} \subset b(X)$ is a decreasing sequence of closed sets with $\lim_{n\to\infty}\kappa(A_n) = 0$, then $A_\infty = \bigcap_{n=0}^{\infty} A_n \ne \emptyset$, and $\kappa(A_\infty) = 0$ [3, Lemma 2.1].

In Sections 2 and 3 of this paper we introduce two mutually equivalent measures of noncircularity, the kernel (that is, the class of sets which are mapped to 0) of any of them consisting of all those $C \in b(X)$ such that $\overline{C}$ is balanced. Recall that $A \in 2^X \setminus \{\emptyset\}$ is balanced provided that $\mu A \subset A$ for all $\mu \in \mathbb{K}$ with $|\mu| \le 1$. For example, in $\mathbb{R}$ the only bounded balanced sets are the open or closed intervals centered at the origin. Similarly, in $\mathbb{C}$ as a complex vector space the only bounded balanced sets are the open or closed disks centered at the origin, while in $\mathbb{R}^2$ as a real vector space there are many more bounded balanced sets, namely all those bounded sets which are symmetric with respect to the origin.

Denoting by γ either one of the two measures introduced, in Section 4 we prove a result of Darbo type for γ-contractive multimaps (see Section 4 for precise definitions). It is shown that the origin is a fixed point of every γ-contractive multimap F with closed values defined on a closed set $M \in b(X)$ such that $F(M) \subset M$.

2. The E-L Measure of Noncircularity

The definition of the Eisenfeld-Lakshmikantham measure of nonconvexity [4] motivates the following.

Definition 2.1. For $C \in b(X)$, set

$$\alpha(C) := H(\mathrm{ba}C, C) = h(\mathrm{ba}C, C), \tag{2.1}$$

where ba$C$ denotes the balanced hull of C, that is,

$$\mathrm{ba}C := \{\mu c : |\mu| \leq 1,\ c \in C\}. \tag{2.2}$$

By analogy with the Eisenfeld-Lakshmikantham measure of nonconvexity, we shall refer to α as the E-L measure of noncircularity.

Next we gather some properties of α which justify such a denomination. Their proofs are fairly direct, but we include them for the sake of completeness.

Proposition 2.2. *In the above notation, for $C, D \in b(X)$, and $\lambda \in \mathbb{K}$, the following hold:*

(1) $\alpha(C) = 0$ *if, and only if,* $\overline{C}$ *is balanced.*

(2) $\alpha(\mathrm{co}\, C) \leq \alpha(C) = \alpha(\overline{C})$.

(3) $\alpha(C \cup D) \leq \max\{\alpha(C), \alpha(D)\}$.

(4) $\alpha(\lambda C) = |\lambda| \alpha(C)$.

(5) $\alpha(C + D) \leq \alpha(C) + \alpha(D)$.

(6) $\alpha(C) \leq 2\|C\|$, *where*

$$\|C\| := \sup_{c \in C} \|c\| \tag{2.3}$$

is the norm of C. In particular, if $0 \in C$ *then* $\alpha(C) \leq 2\delta(C)$, *where*

$$\delta(C) := \sup_{x, y \in C} \|x - y\| \tag{2.4}$$

is the diameter of C.

(7) $|\alpha(C) - \alpha(D)| \leq 2H(C, D)$.

Proof. Let $\overline{\text{ba}}C$ denote the closed balanced hull of C. The identity

$$\overline{\text{ba}}\,\overline{C} = \overline{\text{ba}}C \tag{2.5}$$

holds. Indeed, $\overline{C} \subset \overline{\text{ba}}C$ implies $\overline{\text{ba}}\,\overline{C} \subset \overline{\text{ba}}C$. Conversely, $C \subset \overline{\text{ba}}\,\overline{C}$ implies $\overline{\text{ba}}C \subset \overline{\text{ba}}\,\overline{C}$.

(1) By definition, $\alpha(C) = H(\text{ba}C, C) = h(\text{ba}C, C) = 0$ if, and only if, $\text{ba}C \subset \overline{C}$ or, equivalently, $\overline{\text{ba}}C \subset \overline{C}$. This means that $\overline{\text{ba}}C = \overline{C}$, which by (2.5) occurs if, and only if, $\overline{C}$ is balanced.

(2) In view of (1.4) and (2.5),

$$\begin{aligned} \alpha\left(\overline{C}\right) &= H\left(\text{ba}\overline{C}, \overline{C}\right) = H\left(\overline{\text{ba}}\,\overline{C}, C\right) \\ &= H\left(\overline{\text{ba}}C, C\right) = H(\text{ba}C, C) = \alpha(C). \end{aligned} \tag{2.6}$$

It only remains to prove that $\alpha(\text{co}\,C) \leq \alpha(C)$. Suppose $\alpha(C) < \varepsilon$, so that $\text{ba}C \subset C + \varepsilon B_X$. The set $\text{co}\,C + \varepsilon B_X$ being convex, it follows that $\text{baco}C \subset \text{coba}C \subset \text{co}\,C + \varepsilon B_X$, whence $\alpha(\text{co}\,C) \leq \varepsilon$. From the arbitrariness of ε we conclude that $\alpha(\text{co}\,C) \leq \alpha(C)$.

(3) Assume $\max\{\alpha(C), \alpha(D)\} < \varepsilon$, that is, $\alpha(C) < \varepsilon$ and $\alpha(D) < \varepsilon$. Then $\text{ba}C \subset C + \varepsilon B_X$, $\text{ba}D \subset D + \varepsilon B_X$, and the fact that $\text{ba}C \cup \text{ba}D$ is a balanced set containing $C \cup D$, imply

$$\text{ba}(C \cup D) \subset \text{ba}C \cup \text{ba}D \subset (C \cup D) + \varepsilon B_X, \tag{2.7}$$

whence $\alpha(C \cup D) \leq \varepsilon$. The arbitrariness of ε yields $\alpha(C \cup D) \leq \max\{\alpha(C), \alpha(D)\}$.

(4) For $\lambda = 0$, this is obvious. Suppose $\lambda \neq 0$. If $|\lambda|\alpha(C) < \varepsilon$ then $\text{ba}C \subset C + (\varepsilon/|\lambda|)B_X = C + (\varepsilon/\lambda)B_X$, whence $\text{ba}\lambda C = \lambda\text{ba}C \subset \lambda C + \varepsilon B_X$. Thus $\alpha(\lambda C) \leq \varepsilon$, and from the arbitrariness of ε we infer that $\alpha(\lambda C) \leq |\lambda|\alpha(C)$. Conversely, assume $\alpha(\lambda C) < \varepsilon$. Then $\text{ba}\lambda C \subset \lambda C + \varepsilon B_X$, whence $\text{ba}C = (1/\lambda)\text{ba}\lambda C \subset C + (\varepsilon/\lambda)B_X = C + (\varepsilon/|\lambda|)B_X$. Therefore $\alpha(C) \leq \varepsilon/|\lambda|$, and from the arbitrariness of ε we conclude that $|\lambda|\alpha(C) \leq \alpha(\lambda C)$.

(5) Let $\alpha(C) + \alpha(D) < \varepsilon$ and choose $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon = \varepsilon_1 + \varepsilon_2$, $\alpha(C) < \varepsilon_1$ and $\alpha(D) < \varepsilon_2$. Then $\text{ba}C \subset C + \varepsilon_1 B_X$, $\text{ba}D \subset D + \varepsilon_2 B_X$ and the fact that $\text{ba}C + \text{ba}D$ is a balanced set containing $C + D$, imply $\text{ba}(C + D) \subset \text{ba}C + \text{ba}D \subset C + D + \varepsilon B_X$, so that $\alpha(C + D) \leq \varepsilon$. The arbitrariness of ε yields $\alpha(C + D) \leq \alpha(C) + \alpha(D)$.

(6) Pick $x = \mu u \in \text{ba}C$, with $|\mu| \leq 1$ and $u \in C$, and let $c \in C$. As

$$\|x - c\| = \|\mu u - c\| \leq |\mu|\|u\| + \|c\| \leq 2\|C\|, \tag{2.8}$$

we obtain

$$\alpha(C) = \sup_{x \in \text{ba}C} \inf_{c \in C} \|x - c\| \leq 2\|C\| \leq 2\delta(C), \tag{2.9}$$

where for the validity of the latter estimate we have assumed $0 \in C$.

(7) It is enough to show that

$$\alpha(C) \leq \alpha(D) + h(C, D) + h(D, C), \tag{2.10}$$

since then, by symmetry,

$$\alpha(D) \leq \alpha(C) + h(C, D) + h(D, C), \tag{2.11}$$

whence the desired result. Now

$$\begin{aligned} \alpha(C) &= H(\mathrm{ba}C, C) = h(\mathrm{ba}C, C) \\ &\leq h(\mathrm{ba}C, \mathrm{ba}D) + h(\mathrm{ba}D, D) + h(D, C) \\ &= h(\mathrm{ba}C, \mathrm{ba}D) + \alpha(D) + h(D, C). \end{aligned} \tag{2.12}$$

To complete the proof we will establish that $h(\mathrm{ba}C, \mathrm{ba}D) \leq h(C, D)$. Indeed, suppose $h(C, D) < \varepsilon$, and let $x = \mu c \in \mathrm{ba}C$, with $|\mu| \leq 1$ and $c \in C$. Then there exists $d \in D$ such that $\|c - d\| < \varepsilon$. Consequently, for $y = \mu d \in \mathrm{ba}D$ we have

$$\|x - y\| = \|\mu c - \mu d\| = |\mu| \|c - d\| < \varepsilon. \tag{2.13}$$

This means that $\mathrm{ba}C \subset \mathrm{ba}D + \varepsilon B_X$, so that $h(\mathrm{ba}C, \mathrm{ba}D) \leq \varepsilon$. From the arbitrariness of ε we conclude that $h(\mathrm{ba}C, \mathrm{ba}D) \leq h(C, D)$. □

Remark 2.3. The identity $\alpha(\mathrm{co}\, C) = \alpha(C)$ $(C \in b(X))$ may not hold, as can be seen by choosing $C = \{-1, 1\} \in 2^{\mathbb{R}}$. In fact, $\mathrm{co}\, C = [-1, 1]$ is balanced, while C is not. Therefore, $\alpha(\mathrm{co}\, C) = 0 < \alpha(C)$.

In general, the identity $\alpha(C \cup D) = \max\{\alpha(C), \alpha(D)\}$ $(C, D \in b(X))$ does not hold either. To show this, choose C and D, respectively, as the upper and lower closed half unit disks of the complex plane. Then $C \cup D$ equals the closed unit disk, which is balanced, while C, D are not. Thus, $\alpha(C \cup D) = 0 < \max\{\alpha(C), \alpha(D)\}$.

Note that α is not monotone: from $C, D \in b(X)$ and $C \subset D$, it does not necessarily follow that $\alpha(C) \leq \alpha(D)$. Otherwise, $\alpha(D) = 0$ would imply $\alpha(C) = 0$, which is plainly false since not every subset of a balanced set is balanced.

3. The Hausdorff Measure of Noncircularity

The following definition is motivated by that of the Hausdorff measure of noncompactness (cf. [2, Theorem 2.1]).

Definition 3.1. We define the Hausdorff measure of noncircularity of $C \in b(X)$ by

$$\beta(C) := H(C, bb(X)) = \inf_{B \in bb(X)} H(C, B), \tag{3.1}$$

where $bb(X)$ denotes the class of all balanced sets in $b(X)$.

In general, $\alpha(C) \neq \beta(C)$, as the next example shows.

Example 3.2. Let $C = \{1\} \in 2^{\mathbb{R}}$. Then ba$C = [-1, 1]$, and

$$\alpha(C) = \sup_{|x| \leq 1} |x - 1| = 2. \tag{3.2}$$

If $B_r = [-r, r]$ $(r \geq 0)$ is any closed bounded balanced set in $\mathbb{R}$, we have

$$h(C, B_r) = \inf_{|x| \leq r} |x - 1|, \qquad h(B_r, C) = \sup_{|x| \leq r} |x - 1|, \tag{3.3}$$

so that

$$H(C, B_r) = \max\{h(C, B_r), h(B_r, C)\} = h(B_r, C). \tag{3.4}$$

Since

$$h(B_r, C) = \sup_{|x| \leq r} |x - 1| = 1 + r, \tag{3.5}$$

we obtain

$$\beta(C) = \inf_{r \geq 0} H(C, B_r) = \inf_{r \geq 0} (1 + r) = 1. \tag{3.6}$$

Thus, $2\beta(C) = 2 = \alpha(C)$.

Next we compare the measures α and β and establish some properties for the latter. Again, most proofs derive directly from the definitions, but we include them for completeness.

Proposition 3.3. *In the above notation, for $C, D \in b(X)$, and $\lambda \in \mathbb{K}$, the following hold:*

(1) $\beta(C) \leq \alpha(C) \leq 2\beta(C)$, *and the estimates are sharp.*

(2) $\beta(C) = 0$ *if, and only if,* $\overline{C}$ *is balanced.*

(3) $\beta(\operatorname{co} C) \leq \beta(C) = \beta(\overline{C})$.

(4) $\beta(C \cup D) \leq \max\{\beta(C), \beta(D)\}$.

(5) $\beta(\lambda C) = |\lambda| \beta(C)$.

(6) $\beta(C + D) \leq \beta(C) + \beta(D)$.

(7) $\beta(C) \leq 2\|C\|$, *where*

$$\|C\| := \sup_{c \in C} \|c\| \tag{3.7}$$

is the norm of C. *In particular, if* $0 \in C$ *then* $\beta(C) \leq 2\delta(C)$, *where*

$$\delta(C) := \sup_{x,y\in C} \|x - y\| \tag{3.8}$$

is the diameter of C.

(8) $|\beta(C) - \beta(D)| \leq H(C, D)$.

Proof. (1) That $\beta(C) \leq \alpha(C)$ follows immediately from the definitions of β and α. Let $\varepsilon > 2\beta(C)$ and choose $B \in bb(X)$ satisfying $H(C, B) < \varepsilon/2$, so that $C \subset B + (\varepsilon/2)B_X$ and $B \subset C + (\varepsilon/2)B_X$. Then $\mathrm{ba}C \subset B + (\varepsilon/2)B_X$ and $B \subset \mathrm{ba}C + (\varepsilon/2)B_X$, thus proving that $H(\mathrm{ba}C, B) \leq \varepsilon/2$. Now

$$\alpha(C) = H(\mathrm{ba}C, C) \leq H(\mathrm{ba}C, B) + H(B, C) < \varepsilon, \tag{3.9}$$

and the arbitrariness of ε yields $\alpha(C) \leq 2\beta(C)$. Example 3.2 shows that this estimate is sharp. In order to exhibit a set $C \in 2^{\mathbb{R}}$ such that $\beta(C) = \alpha(C)$, let $C = \{-1, 1\}$. Then $\mathrm{ba}C = [-1, 1]$, and

$$\alpha(C) = \sup_{|x|\leq 1} \inf_{c\in C} |x - c| = 1. \tag{3.10}$$

On the other hand, let $B_r = [-r, r]$ $(r \geq 0)$ be any closed bounded balanced subset of $\mathbb{R}$. For a fixed $r \geq 0$, there holds

$$\begin{gathered} h(B_r, C) = \sup_{|x|\leq r} \inf_{c\in C} |x - c| = \begin{cases} 1, & r \leq 1 \\ \max\{1, r - 1\}, & r > 1, \end{cases} \\ h(C, B_r) = \sup_{c\in C} \inf_{|x|\leq r} |x - c| = \begin{cases} 1 - r, & r \leq 1 \\ 0, & r > 1. \end{cases} \end{gathered} \tag{3.11}$$

Therefore,

$$H(B_r, C) = \max\{h(B_r, C), h(C, B_r)\} = \begin{cases} 1, & r \leq 1 \\ \max\{1, r - 1\}, & r > 1, \end{cases} \tag{3.12}$$

so that

$$\beta(C) = \inf_{r\geq 0} H(B_r, C) = 1 = \alpha(C). \tag{3.13}$$

(2) Let $C \in b(X)$. As we just proved, $\beta(C) = 0$ if, and only if, $\alpha(C) = 0$. In view of Proposition 2.2, this occurs if, and only if, $\overline{C}$ is balanced.

(3) By (1.4), there holds

$$\beta(C) = \inf_{B\in bb(X)} H(C, B) = \inf_{B\in bb(X)} H\left(\overline{C}, B\right) = \beta\left(\overline{C}\right). \tag{3.14}$$

Now we only need to show that $\beta(\operatorname{co} C) \leq \beta(C)$. Assuming $\beta(C) < \varepsilon$, choose $B \in bb(X)$ for which $H(C, B) < \varepsilon$, so that

$$C \subset B + \varepsilon B_X, \qquad B \subset C + \varepsilon B_X. \tag{3.15}$$

The sum of convex sets being convex, we infer

$$\operatorname{co} C \subset \operatorname{co} B + \varepsilon B_X, \qquad \operatorname{co} B \subset \operatorname{co} C + \varepsilon B_X. \tag{3.16}$$

Since $\operatorname{co} B$ is balanced we obtain $\beta(\operatorname{co} C) \leq \varepsilon$ and, as ε is arbitrary, we conclude that $\beta(\operatorname{co} C) \leq \beta(C)$.

(4) Suppose $\max\{\beta(C), \beta(D)\} < \varepsilon$, that is, $\beta(C) < \varepsilon$ and $\beta(D) < \varepsilon$. Pick $B_1, B_2 \in bb(X)$ satisfying $H(C, B_1) < \varepsilon$ and $H(D, B_2) < \varepsilon$. Then

$$\begin{aligned} C &\subset B_1 + \varepsilon B_X, \qquad B_1 \subset C + \varepsilon B_X, \\ D &\subset B_2 + \varepsilon B_X, \qquad B_2 \subset D + \varepsilon B_X. \end{aligned} \tag{3.17}$$

Thus we get

$$C \cup D \subset (B_1 \cup B_2) + \varepsilon B_X, \qquad B_1 \cup B_2 \subset (C \cup D) + \varepsilon B_X, \tag{3.18}$$

whence $H(C \cup D, B_1 \cup B_2) \leq \varepsilon$ and, $B_1 \cup B_2$ being balanced, also $\beta(C \cup D) \leq \varepsilon$. From the arbitrariness of ε we conclude that $\beta(C \cup D) \leq \max\{\beta(C), \beta(D)\}$.

(5) If $\lambda = 0$, the property is obvious. Assume $\lambda \neq 0$. Given $\varepsilon > |\lambda|\beta(C)$, there exists $B \in bb(X)$ such that

$$\begin{aligned} C &\subset B + \left(\frac{\varepsilon}{|\lambda|}\right) B_X = B + \left(\frac{\varepsilon}{\lambda}\right) B_X, \\ B &\subset C + \left(\frac{\varepsilon}{|\lambda|}\right) B_X = C + \left(\frac{\varepsilon}{\lambda}\right) B_X. \end{aligned} \tag{3.19}$$

Then

$$\lambda C \subset \lambda B + \varepsilon B_X, \qquad \lambda B \subset \lambda C + \varepsilon B_X, \tag{3.20}$$

so that $H(\lambda C, \lambda B) \leq \varepsilon$. Since λB is balanced, it follows that $\beta(\lambda C) \leq \varepsilon$ and, ε being arbitrary, we obtain $\beta(\lambda C) \leq |\lambda|\beta(C)$. Conversely, let $\varepsilon > \beta(\lambda C)$. Then there exists $B \in bb(X)$ such that

$$\lambda C \subset B + \varepsilon B_X, \qquad B \subset \lambda C + \varepsilon B_X. \tag{3.21}$$

Hence,

$$C \subset \left(\frac{1}{\lambda}\right)B + \left(\frac{\varepsilon}{\lambda}\right)B_X = \left(\frac{1}{\lambda}\right)B + \left(\frac{\varepsilon}{|\lambda|}\right)B_X,$$
$$\left(\frac{1}{\lambda}\right)B \subset C + \left(\frac{\varepsilon}{\lambda}\right)B_X = C + \left(\frac{\varepsilon}{|\lambda|}\right)B_X. \tag{3.22}$$

Therefore, $H(C,(1/\lambda)B) \le \varepsilon/|\lambda|$. Since $(1/\lambda)B$ is balanced we conclude that $\beta(C) \le \varepsilon/|\lambda|$, or $|\lambda|\beta(C) \le \varepsilon$. The arbitrariness of ε finally yields $|\lambda|\beta(C) \le \beta(\lambda C)$.

(6) Let $\beta(C) + \beta(D) < \varepsilon$ and let $\varepsilon_1, \varepsilon_2 > 0$ satisfy $\varepsilon = \varepsilon_1 + \varepsilon_2$, $\beta(C) < \varepsilon_1$ and $\beta(D) < \varepsilon_2$. Choose $B_1, B_2 \in bb(X)$ such that $H(C, B_1) < \varepsilon_1$ and $H(D, B_2) < \varepsilon_2$. Then

$$C \subset B_1 + \varepsilon_1 B_X, \qquad B_1 \subset C + \varepsilon_1 B_X,$$
$$D \subset B_2 + \varepsilon_2 B_X, \qquad B_2 \subset D + \varepsilon_2 B_X. \tag{3.23}$$

Thus we obtain

$$C + D \subset B_1 + B_2 + \varepsilon B_X, \qquad B_1 + B_2 \subset C + D + \varepsilon B_X, \tag{3.24}$$

whence $H(C + D, B_1 + B_2) \le \varepsilon$ and, $B_1 + B_2$ being balanced, also $\beta(C + D) \le \varepsilon$. From the arbitrariness of ε we conclude that $\beta(C + D) \le \beta(C) + \beta(D)$.

(7) This follows from Proposition 2.2.

(8) For $B \in bb(X)$ there holds $H(C,B) \le H(C,D) + H(D,B)$, whence $\beta(C) \le H(C,D) + \beta(D)$. Therefore, $\beta(C) - \beta(D) \le H(C,D)$. By symmetry, $\beta(D) - \beta(C) \le H(C,D)$, thus yielding $|\beta(C) - \beta(D)| \le H(C,D)$, as claimed. □

Remark 3.4. By the same reasons as α, the measure β fails to be monotone and, in general, the identities $\beta(\operatorname{co} C) = \beta(C)$ and

$$\beta(C \cup D) = \max\{\beta(C), \beta(D)\} \tag{3.25}$$

do not hold (cf. Remark 2.3).

4. A Fixed Point Theorem for Multimaps

The study of fixed points for multivalued mappings was initiated by Kakutani [5] in 1941 in finite dimensional spaces and extended to infinite dimensional Banach spaces by Bohnenblust and Karlin [6] in 1950 and to locally convex spaces by Fan [7] in 1952. Since then, it has become a very active area of research, both from the theoretical point of view and in applications. In this section we use the previous theory to obtain a fixed point theorem for multifunctions in the Banach space X. We begin by recalling some definitions.

Definition 4.1. Let $M \in 2^X \setminus \{\emptyset\}$. A multimap or multifunction F from M to the class $2^Y \setminus \{\emptyset\}$ of all nonempty subsets of a given set Y, written $F : M \multimap Y$, is any map from M to $2^Y \setminus \{\emptyset\}$.

If F is a multifunction and $A \in 2^M$, then

$$F(A) := \bigcup_{x \in A} F(x). \tag{4.1}$$

Definition 4.2. Given $M \in 2^X \setminus \{\emptyset\}$, let $F : M \multimap X$, and let γ represent any of the two measures of noncircularity introduced above. A fixed point of F is a point $x \in M$ such that $x \in F(x)$. The multifunction F will be called

(i) a γ-contraction (of constant k), if

$$\gamma(F(B)) \leq k\gamma(B) \quad \left(B \in b(X) \cap 2^M\right) \tag{4.2}$$

for some $k \in]0,1[$;

(ii) a (γ, ϕ)-contraction, if

$$\gamma(F(B)) \leq \phi(\gamma(B)) \quad \left(B \in b(X) \cap 2^M\right), \tag{4.3}$$

where $\phi : [0,\infty[\to [0,\infty[$ is a comparison function, that is, ϕ is increasing, $\phi(0) = 0$, and $\phi^n(r) \to 0$ as $n \to \infty$ for each $r > 0$.

Note that a γ-contraction of constant k corresponds to a (γ, ϕ)-contraction with $\phi(r) = kr \ (r \geq 0)$.

In order to establish our main result, we prove a property of Cantor type for the E-L and Hausdorff measures of noncircularity.

Proposition 4.3. *Let* X *be a Banach space and* $\{A_k\}_{k=0}^{\infty} \subset b(X)$ *a decreasing sequence of closed sets such that* $\lim_{k\to\infty}\gamma(A_k) = 0$*, where* γ *denotes either* α *or* β*. Then the set*

$$A_\infty := \bigcap_{k=0}^{\infty} A_k \tag{4.4}$$

satisfies

$$A_\infty = \bigcap_{k=0}^{\infty} baA_k. \tag{4.5}$$

Hence A_∞ *belongs to* $b(X)$ *and is closed and balanced.*

Proof. By Proposition 3.3 we have $\lim_{k\to\infty}\alpha(A_k) = 0$ if, and only if, $\lim_{k\to\infty}\beta(A_k) = 0$. Thus for the proof it suffices to set $\gamma = \alpha$.

Since $A_k \subset \mathrm{ba}A_k \ (k \in \mathbb{N})$, necessarily

$$A_\infty = \bigcap_{k=0}^{\infty} A_k \subset \bigcap_{k=0}^{\infty} \mathrm{ba}A_k. \tag{4.6}$$

Conversely, let $x \in \bigcap_{k=0}^{\infty} \mathrm{ba} A_k$. As $\lim_{k\to\infty}\alpha(A_k) = 0$, to every $\varepsilon > 0$ there corresponds $N \in \mathbb{N}$ such that $n \in \mathbb{N}$, $n \geq N$ implies $\mathrm{ba} A_n \subset A_n + \varepsilon B_X$. This yields an increasing sequence $\{n_m\}_{m=1}^{\infty}$ of positive integers and vectors $a_{n_m} \in A_{n_m}$ which satisfy $\|x - a_{n_m}\| \leq 1/m$ $(m \in \mathbb{N},\ m \geq 1)$. Thus the sequence $\{a_{n_m}\}_{m=1}^{\infty}$ converges to x as $m \to \infty$. Moreover, since $a_{n_m} \in A_{n_m} \subset A_k$ $(m, k \in \mathbb{N},\ m \geq 1,\ n_m \geq k)$ and A_k is closed, we find that $x \in A_k$ $(k \in \mathbb{N})$. In other words, $x \in A_\infty$. This proves (4.5).

Note that $\emptyset \neq A_n \subset \mathrm{ba} A_n$ implies $0 \in \mathrm{ba} A_n$ $(n \in \mathbb{N})$, whence $0 \in A_\infty \neq \emptyset$. Since the intersection of closed, bounded and balanced sets preserves those properties, so does A_∞. □

Remark 4.4. In contrast to Proposition 4.3, the Eisenfeld-Lakshmikantham measure of nonconvexity does not necessarily satisfy a Cantor property. Indeed, in real, nonreflexive Banach spaces one can find a decreasing sequence $\{A_n\}_{n=1}^{\infty}$ of nonempty, closed, bounded, convex sets with empty intersection. To construct such a sequence, just take a unitary continuous linear functional f in a real, nonreflexive Banach space X which fails to be norm-attaining on the closed unit ball B_X of X (the existence of such an f is guaranteed by a classical, well-known theorem of James, cf. [8]), and define

$$A_n = \left\{ x \in B_X : f(x) \geq 1 - \frac{1}{n} \right\} \quad (n \in \mathbb{N},\ n \geq 1). \tag{4.7}$$

Now we are in a position to derive the announced result. Here, and in the sequel, γ will stand for any one of the measures of noncircularity α or β.

Theorem 4.5. *Let X be a Banach space, and let $M \in b(X)$ be closed. If $F : M \multimap M$ is a (γ, ϕ)-contraction with closed values, then $0 \in M$ and 0 is a fixed point of F.*

Proof. Our hypotheses imply

$$\begin{gathered} F^{n+1}(M) \subset F^n(M) \quad (n \in \mathbb{N}), \\ \lim_{n\to\infty} \gamma(F^n(M)) \leq \lim_{n\to\infty} \phi^n\big(\gamma(M)\big) = 0. \end{gathered} \tag{4.8}$$

Setting $A_n = \overline{F^n(M)}$ $(n \in \mathbb{N})$, from Propositions 2.2 and 3.3 we find that $\{A_n\}_{n=0}^{\infty} \subset b(X)$ is a decreasing sequence of closed sets with $\lim_{n\to\infty}\gamma(A_n) = 0$. Proposition 4.3 shows that A_∞ is a nonempty, balanced subset of M; in particular, $0 \in A_\infty \subset M$. Now, $\{0\}$ being balanced, we have

$$\gamma(F(0)) \leq \phi\big(\gamma(\{0\})\big) = 0, \tag{4.9}$$

whence $\gamma(F(0)) = 0$. This shows that the nonempty set $F(0) = \overline{F(0)}$ is balanced and forces $0 \in F(0)$, as asserted. □

Corollary 4.6. *Let X be a Banach space, and let $M \in b(X)$ be closed. If $F : M \multimap M$ is a γ-contraction with closed values, then $0 \in M$ and 0 is a fixed point of F.*

Proof. It suffices to apply Theorem 4.5, with $\phi(r) = kr$ $(r \geq 0)$, for $k \in]0, 1[$. □

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