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Research Article

Results on the Existence and Convergence of Best Proximity Points

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We first consider a cyclic φ-contraction map on a reflexive Banach space X and provide a positive answer to a question raised by Al-Thagafi and Shahzad on the existence of best proximity points for cyclic φ-contraction maps in reflexive Banach spaces in one of their works (2009). In the second part of the paper, we will discuss the existence of best proximity points in the framework of more general metric spaces. We obtain some new results on the existence of best proximity points in hyperconvex metric spaces as well as in ultrametric spaces.

1. Introduction

Let $X = (X, d)$ be a metric space, and let A, B be two subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. In [1] Kirk et al. proved the following interesting extension of the Banach contraction principle:

Theorem 1.1 (see [1]). *Let A and B be two nonempty closed subsets of a complete metric space X. Suppose that T is a cyclic map such that*

$$d(Tx, Ty) \le \alpha d(x, y) \tag{1.1}$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$. Then T has a unique fixed point in $A \cap B$.

Later on, Eldred and Veeramani [2] considered the class of cyclic contractions.

Definition 1.2 (see [2]). Let A and B be two nonempty subsets of a metric space X, and let $T : A \cup B \to A \cup B$, $T(A) \subseteq B$, and $T(B) \subseteq A$. We say that T is a cyclic contraction if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) d(A, B) \tag{1.2}$$

for some $\alpha \in (0, 1)$ and for all $x \in A, y \in B$, where

$$d(A, B) := \inf\{d(x, y) : x \in A, y \in B\}. \tag{1.3}$$

We recall that a point $x \in A \cup B$ is said to be a *best proximity point* for T provided that $d(x, Tx) = d(A, B)$.

In the case that X is a uniformly convex Banach space, Eldred and Veeramani established the following theorem.

Theorem 1.3 (see [2]). *Let A and B be two nonempty closed convex subsets of a uniformly convex Banach space X, and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = d(A, B)$.*

In 2009, Al-Thagafi and Shahzad introduced a new class of mappings, namely, the class of cyclic φ-contraction maps. This new class contains the class of cyclic contraction maps.

Definition 1.4 (see [3]). Let A and B be two nonempty subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. T is said to be a cyclic φ-contraction map if there exists a strictly increasing function $\varphi : [0, +\infty) \to [0, +\infty)$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)) \tag{1.4}$$

for all $x \in A$ and $y \in B$.

In [3] the authors were able to establish some existence and convergence results for these mappings. Moreover, they proved the existence of a best proximity point for a cyclic contraction map in a reflexive Banach space X (see [3, Theorems 10, 11]). In this way they answered a question raised by Eldred and Veeramani in the affirmative. We recall that Theorem 1.3 above was proved in the setting of a uniformly convex Banach space. The authors of [3] then asked if the result stands true if we assume that X is a reflexive Banach space, rather than being uniformly convex.

Al-Thagafi and N. Shahzad then stated it was interesting to ask whether Theorems 9 and 10 (resp., Theorems 11 and 12) held true for cyclic φ-contraction maps when the Banach space in question is only reflexive (resp., reflexive and strictly convex).

In this paper we first take up these questions. It turns out that under some conditions the answer is positive. In the last section we study the existence of best proximity points in spherically complete ultrametric spaces, as well as in hyperconvex metric spaces. More precisely, we will see that best proximity points exist for cyclic φ-contraction maps on hyperconvex metric spaces. We will also provide an existence theorem for a cyclic map which satisfies some contractive condition on an ultrametric space.

2. Cyclic φ-Contraction Maps

In this section we first provide a positive answer to the question raised by the authors of [3]. Then we present some consequences and applications. Among other things, is a common fixed point theorem for two maps. We will begin with the following lemma.

Lemma 2.1 (see [3, Lemma 1]). *Let A and B be two nonempty subsets of a metric space X and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then one has*

(a) $-\varphi(d(x,y)) + \varphi(d(A,B)) \leq 0$ *for all $x \in A$ and $y \in B$,*

(b) $d(Tx,Ty) \leq d(x,y)$ *for all $x \in A$ and $y \in B$,*

(c) $d(x_{n+2},x_{n+1}) \leq d(x_{n+1},x_n)$ *for all $n \geq 0$.*

Now we state and prove the following lemma which is key to the proof of the main result of this section.

Lemma 2.2. *Let A and B be two nonempty subsets of a metric space X, and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then the sequences $\{x_{2n}\}$, and $\{x_{2n+1}\}$ are bounded if either of the following conditions holds:*

(i) $\lim_{t\to+\infty}\varphi(t) = +\infty$,

(ii) $d(A,B) = 0$.

Proof. We first show that the sequence $\{d(T^2x_0, T^{2n+1}x_0)\}$ is bounded. Suppose the contrary. Then for every positive integer k, there exists $n_k \geq 1$ such that

$$d\left(T^2x_0, T^{2n_k+1}x_0\right) \geq k, \qquad d\left(T^2x_0, T^{2n_k-1}x_0\right) < k. \tag{2.1}$$

We note that

$$k \leq d\left(T^2x_0, T^{2n_k+1}x_0\right) \leq d\left(Tx_0, T^{2n_k}x_0\right) - \varphi\left(d\left(Tx_0, T^{2n_k}x_0\right)\right) + \varphi(d(A,B)). \tag{2.2}$$

According to Lemma 2.1, T is nonexpansive, so that (by the property of φ)

$$\begin{aligned} d\left(Tx_0, T^{2n_k}x_0\right) &\leq d\left(x_0, T^{2n_k-1}x_0\right) - \varphi\left(d\left(x_0, T^{2n_k-1}x_0\right)\right) + \varphi(d(A,B)) \\ &\leq d\left(x_0, T^{2n_k-1}x_0\right). \end{aligned} \tag{2.3}$$

Therefore

$$\begin{aligned} k &\leq d\left(x_0, T^{2n_k-1}x_0\right) - \varphi\left(d\left(Tx_0, T^{2n_k}x_0\right)\right) + \varphi(d(A,B)) \\ &\leq d\left(x_0, T^2x_0\right) + d\left(T^2x_0, T^{2n_k-1}x_0\right) - \varphi\left(d\left(Tx_0, T^{2n_k}x_0\right)\right) + \varphi(d(A,B)). \end{aligned} \tag{2.4}$$

But since φ is increasing, it follows that

$$\varphi\left(d\left(T^2x_0, T^{2n_k+1}x_0\right)\right) \leq \varphi\left(d\left(Tx_0, T^{2n_k}x_0\right)\right). \tag{2.5}$$

Thus

$$\begin{aligned} k &\leq d\left(x_0, T^2x_0\right) + d\left(T^2x_0, T^{2n_k-1}x_0\right) - \varphi\left(d\left(T^2x_0, T^{2n_k+1}x_0\right)\right) + \varphi(d(A,B)) \\ &\leq d\left(x_0, T^2x_0\right) + k - \varphi(k) + \varphi(d(A,B)). \end{aligned} \tag{2.6}$$

This implies that for every positive integer k we have

$$\varphi(k) < d\left(x_0, T^2x_0\right) + \varphi(d(A,B)), \tag{2.7}$$

contradicting the hypothesis that $\lim_{t\to\infty}\varphi(t) = \infty$.

We now assume that condition (ii) holds. It follows from (2.7) that

$$\varphi(k) - \varphi(d(A,B)) < d\left(x_0, T^2x_0\right) \leq d(x_0, Tx_0) + d\left(Tx_0, T^2x_0\right) \leq 2d(x_0, Tx_0). \tag{2.8}$$

Since (2.8) holds for all $x_0 \in A$, we conclude that

$$\varphi(k) - \varphi(d(A,B)) < 2d\left(T^{2n}x_0, T\left(T^{2n}x_0\right)\right) = 2d(x_{2n}, x_{2n+1}) \tag{2.9}$$

for all $n \geq 0$. Letting now $n \to \infty$ and using Theorem 3 of [3] we conclude that

$$\varphi(k) - \varphi(d(A,B)) \leq 2d(A,B) = 0, \tag{2.10}$$

which contradicts the fact that φ is strictly increasing.

This arguments show that the sequence $\{d(T^2x_0, T^{2n+1}x_0)\}$ is bounded. But since

$$d\left(T^{2n}x_0, T^2x_0\right) \leq d\left(T^{2n}x_0, T^{2n+1}x_0\right) + d\left(T^{2n+1}x_0, T^2x_0\right), \tag{2.11}$$

and that both terms on the right-hand side are bounded, we conclude that $\{T^{2n}x_0\}$ is bounded.

Similarly, by considering the sequence $\{d(T^3x_0, T^{2n}x_0)\}$ we can prove that the sequence $\{T^{2n+1}x_0\}$ is bounded. □

We now come to the first main result of this paper generalizing Theorem 9 of [3] to cyclic φ-contraction maps.

Theorem 2.3. *Let A and B be two nonempty weakly closed subsets of a reflexive Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map satisfying either of the following:*

(i) $\lim_{t\to\infty}\varphi(t) = \infty$,

(ii) $d(A,B) = 0$.

Then there exists $(x,y) \in A \times B$ such that $\|x-y\| = d(A,B)$.

Proof. Let $x_0 \in A$ be arbitrarily chosen. We define $x_{n+1} = Tx_n$. It follows from Lemma 2.2 that the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded in A and in B, respectively. Since X is reflexive, every bounded sequence in X has a weakly convergent subsequence. Assume that $x_{2n_k} \to x$ weakly. Since A is weakly closed, $x \in A$. Similarly, we may assume that there is a $y \in B$ such that $x_{2n_k+1} \to y$, weakly. Therefore $x_{2n_k} - x_{2n_k+1} \to x - y$, weakly. But according to a well-known fact in basic functional analysis, we have

$$\|x-y\| \le \liminf_{k\to\infty}\|x_{2n_k} - x_{2n_k+1}\| = d(A,B), \tag{2.12}$$

from which it follows that $\|x-y\| = d(A,B)$. □

Remark 2.4. If we assume that the function φ satisfies either of the conditions (i) or (ii) of Lemma 2.2, then all three theorems (Theorems 10, 11, and 12 of [3] can be generalized to cyclic φ-contraction maps. We omit the details.

The next theorem generalizes Theorem 1.1 to reflexive Banach spaces. Note that if $d(A,B) = 0$ and $\varphi(t) = (1-\alpha)t$ for some fixed $\alpha \in (0,1)$, then T will be a cyclic contraction map, because for all $x \in A$ and all $y \in B$ we have

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B)) = \alpha d(x,y). \tag{2.13}$$

Theorem 2.5. *Let A and B be two nonempty subsets of a reflexive Banach space X such that A is weakly closed. Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map which is weakly continuous on A. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \ge 0$. If $d(A,B) = 0$ then T has a unique fixed point $x \in A \cap B$ and $x_n \to x$.*

Proof. Since T is cyclic φ-contraction, and $d(A,B) = 0$, it follows from Lemma 2.2 that $\{x_{2n}\}$ is bounded in A. Therefore we can find a weak convergent subsequence, say $\{x_{2n_k}\}$, to a point $x \in A$. On the other hand, T is weakly continuous, so that $Tx_{2n_k} \to Tx$ weakly. It follows that

$$x_{2n_k+1} - x_{2n_k} \longrightarrow Tx - x, \quad \text{weakly}. \tag{2.14}$$

As in the proof of Theorem 2.3 we conclude that $Tx = x$. The proof of uniqueness part is a verbatim repetition of the proof of Theorem 6 in [3]. We omit the details. □

As an application of Theorem 2.5, we will prove a theorem on the existence and approximation of common fixed points for two maps.

Theorem 2.6. *Let A be a nonempty subset of a reflexive Banach space X and $f, g : A \to A$ be two maps such that $f(A)$ is weakly closed in X and $d(f(A), g(A)) = 0$. Let $T : f(A) \cup g(A) \to f(A) \cup g(A)$ be a cyclic φ-contraction map that satisfies this property that if there exist $a_1, a_2 \in A$ such that $f(a_1) = g(a_2)$, then T commutes with f, g in $f(a_1)$. Then f, g have a common fixed point in A. Moreover, if $a \in A$, $x_0 := f(a)$ and $x_{n+1} := Tx_n$ for each $n \geq 0$ then the sequence $\{x_n\}$ converges to a common fixed point of f, g.*

Proof. By Theorem 2.5 there exists a unique $x \in f(A)$ such that $Tx = x$. Since $x \in f(A)$, there exists $a_1 \in A$ such that $x = f(a_1)$ so that $T(fa_1) = fa_1$. Also there exists $a_2 \in A$ such that $fa_1 = ga_2$, so that $T(ga_2) = ga_2$. Now we have

$$T(f(fa_1)) = f(T(fa_1)) = f(fa_1). \tag{2.15}$$

That is, $f(fa_1)$ is a fixed point for T. Since the fixed point of T is unique, we must have $f(f(a_1)) = fa_1$. Therefore fa_1 is a fixed point of f. Similarly we can show that ga_2 is a fixed point of g. Consequently fa_1 is a common fixed point for f, g. According to Theorem 2.5 the sequence $\{x_n\}$ converges to fa_1. □

Example 2.7. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|$. Let $A = [0, 1/2]$ and define $f, g : A \to A$ with $f(x) = x^2$ and $g(x) = x^3$. Also consider $T : f(A) \cup g(A) \to f(A) \cup g(A)$ by $T(x) = x/2$. Then T is cyclic contraction and satisfies the conditions of Theorem 2.6. Therefore f, g have a common fixed point. It is clear that this common fixed point is $x = 0$.

3. Cyclic φ-Contraction Maps in Metric Spaces

In this section we discuss the existence of best proximity points for cyclic φ-contraction maps in metric spaces. Indeed we prove two existence theorems on best proximity points in hyperconvex spaces, as well as in ultrametric spaces.

Lemma 3.1. *Let A, B be two nonempty subsets of a metric space X, and Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. If there exists $x \in A$ such that $T^2x = x$, then T has a best proximity point.*

Proof. Since $T^2x = x$, then $y := Tx$ is fixed point for T^2. Therefore we have

$$d(x, y) = d\left(T^2x, T^2y\right) \leq d(x, y) - 2\varphi(d(Tx, Ty)) + 2\varphi(d(A, B)). \tag{3.1}$$

Thus $\varphi(d(Tx, Ty)) \leq \varphi(d(A, B))$. Since φ is strictly increasing, we conclude that

$$d(A, B) = d(Tx, Ty) = d(Tx, x). \tag{3.2}$$

□

In the following definition we will use the notation $\chi(D)$ for the Kuratowski measure of noncompactness of a given set D. For more information see the book written by Khamsi and Kirk [4].

Definition 3.2. Let K be a subset of a metric space X. A mapping $T : K \to K$ is said to be condensing if T is bounded and continuous, moreover $\chi(T(D)) < \chi(D)$, for every bounded subset D of K for which $\chi(D) > 0$.

Definition 3.3 (see [4]). A metric space X is called hyperconvex if for any indexed class of closed balls $B(x_i; r_i), i \in I$, of X which satisfy

$$d(x_i, x_j) \le r_i + r_j, \quad i, j \in I, \tag{3.3}$$

it is necessarily the case that $\bigcap_{i\in I} B(x_i; r_i) \ne \emptyset$.

We recall that for a given set X, the notation $\mathcal{A}(X)$ denotes the family of all admissible subsets of X, that is, the family of subsets of X that can be written as the intersection of a family of closed balls centered at points of X. For further information on the subject we refer the reader to [4]. We now state and prove the first main result of this section.

Theorem 3.4. *Let X be a hyperconvex metric space, and A, B be two nonempty subsets of X such that $A \in \mathcal{A}(X)$. Suppose $T : A \cup B \to A \cup B$ is a cyclic φ-contraction map. Put $T_1 = T|_A$ and $T_2 = T|_B$. If $T_2T_1 : A \to A$ is a condensing map then T has a best proximity point.*

Proof. Since X is a hyperconvex metric space, and since $A \in \mathcal{A}(X)$, it follows from Proposition 4.5 of [5] that A is a hyperconvex metric space too. On the other hand, $T_2T_1 : A \to A$ is a condensing map, thus by Theorem 7.13 of [5], T_2T_1 or T^2 has a fixed point. It now follows from Lemma 3.1 that T has a best proximity point. □

Definition 3.5. A metric space X is an ultrametric space if, in addition to the usual metric axioms, the following property holds for each $x, y, z \in X$:

$$d(x, z) \le \max\{d(x, y), d(y, z)\}. \tag{3.4}$$

For example if X is a discrete metric space then X is an ultrametric space. Ultrametric spaces arise in the study of non-Archimedean analysis, and in particular in the study of Banach space over non-Archimedean valuation fields (see [4]).

Remark 3.6. It is immediate from Definition 3.5 that if $B(a; r_1)$ and $B(b; r_2)$ are two closed balls in an ultrametric space, with $r_1 \le r_2$, then either $B(a; r_1) \cap B(b; r_2) = \emptyset$ or $B(a; r_1) \subseteq B(b; r_2)$. In particular if $a \in B(b; r_2)$, then $\mathrm{B}(a; r_1) \subseteq B(b; r_2)$.

Definition 3.7. An ultrametric space X is said to be *spherically complete* if every chain of closed balls in X has nonempty intersection.

As a consequence of Remark 3.6, the admissible sets $\mathcal{A}(X)$ of X coincide with the closed balls of X. Here we state and prove the second main result of this section.

Theorem 3.8. *Suppose X is a spherically complete ultrametric space and A, B are two nonempty subsets of X such that $A \in \mathcal{A}(X)$. Let $T : A \cup B \to A \cup B$ be a cyclic map which satisfies the* *following condition:*

$$d(Tx,Ty) \leq \alpha \max\{d(Tx,x),d(Ty,y),d(x,y)\} + (1-\alpha)d(A,B) \tag{3.5}$$

for each $x \in A, y \in B$ and for some $\alpha \in (0,1)$. Then T has a best proximity point.

Proof. Let $x_0 \in A$ and define $x_{n+1} := Tx_n$ for $n \geq 0$. Put $r_n = d(x_n, x_{n+1})$. By Theorem 2 of [6], $r_n \to d(A,B)$. Now if there exits $N \geq 1$ such that $r_{N-1} \leq r_N$, then

$$\begin{aligned} r_N &= d(x_N, x_{N+1}) = d(Tx_{N-1}, Tx_N) \\ &\leq \alpha \max\{d(Tx_{N-1}, x_{N-1}), d(Tx_N, x_N), d(x_{N-1}, x_N)\} + (1-\alpha)d(A,B) \\ &= \alpha d(x_N, x_{N+1}) + (1-\alpha)d(A,B). \end{aligned} \tag{3.6}$$

Therefore $d(x_N, Tx_N) = d(A,B)$. This argument shows that T has a best proximity point.

Now let for all $n \geq 1$, we have $r_n < r_{n-1}$. Thus

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &= \max\{r_{2n}, r_{2n+1}\} = r_{2n}. \end{aligned} \tag{3.7}$$

Then $x_{2n+2} \in B(x_{2n}; r_{2n})$ (all balls are assumed to be closed). Now by Remark 3.6 we have

$$B(x_{2(n+1)}; r_{2(n+1)}) \subseteq B(x_{2n}; r_{2n}). \tag{3.8}$$

This shows that $\{B(x_{2n}; r_{2n})\}_{n\geq 1}$ is a descending chain of closed balls in X; in particular, each two members of this chain intersect. It is rather obvious that each member of this chain also intersects A (because $x_{2n} \in A$). Since $A \in \mathcal{A}(X)$ and X is a spherically complete ultrametric space, then A itself is a closed ball (see [4, page 114]). Now each two elements of the family consisting of A and $\{B(x_{2n}; r_{2n})\}_{n\geq 1}$ intersects. Therefore if we set $\mathcal{F} = A \cup \bigcup_{n\geq 1} B(x_{2n}; r_{2n})$, according to [4, page 115], there exists a point $a \in A$ which belongs to $\bigcap_{n\geq 1} B(x_{2n}; r_{2n})$ as well. Therefore

$$\begin{aligned} d(a, Ta) &\leq \max\{d(a, x_{2n}), d(Tx_{2n-1}, Ta)\} \\ &\leq \max\{r_{2n}, d(Tx_{2n-1}, Ta)\}. \end{aligned} \tag{3.9}$$

But for the second term we have

$$\begin{aligned} d(Tx_{2n-1}, Ta) &\leq \alpha \max\{d(Tx_{2n-1}, x_{2n-1}), d(Ta, a), d(x_{2n-1}, a)\} + (1-\alpha)d(A,B) \\ &\leq \alpha \max\{r_{2n-1}, d(Ta, a), r_{2n-1}\} + (1-\alpha)d(A,B) \\ &= \alpha \max\{r_{2n-1}, d(Ta, a)\} + (1-\alpha)d(A,B), \end{aligned} \tag{3.10}$$

because

$$d(x_{2n-1}, a) \le \max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, a)\} \le \max\{r_{2n-1}, r_{2n}\} = r_{2n-1}. \tag{3.11}$$

It now follows that

$$d(a, Ta) \le \max\{r_{2n}, \alpha \max\{r_{2n-1}, d(Ta, a)\} + (1-\alpha)d(A,B)\}. \tag{3.12}$$

Since the above relation holds for all $n \ge 1$ then we have

$$\begin{aligned} d(a, Ta) &\le \max\{d(A,B), \alpha d(Ta, a) + (1-\alpha)d(A,B)\} \\ &= \alpha d(Ta, a) + (1-\alpha)d(A,B). \end{aligned} \tag{3.13}$$

Therefore $d(Ta, a) = d(A,B)$, which means that T has a best proximity point. □

In the following example we will see that the condition that X is spherically complete is necessary.

Example 3.9. Let X $:= \{1 + 1/n : n \ge 1\}$ and define a metric d on X by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{x,y\} & \text{if } x \ne y. \end{cases} \tag{3.14}$$

It is clear that (X, d) is a complete ultrametric space (see [5]). Set

$$A := \left\{1 + \frac{1}{2n} : n \ge 1\right\}, \qquad B := \left\{1 + \frac{1}{2n-1} : n \ge 1\right\}; \tag{3.15}$$

and define the mapping $T : A \cup B \to A \cup B$ by $T(1 + 1/n) = 1 + 1/3(n+1)$. It is easy to see that T is cyclic and $d(A,B) = 1$. It is not difficult to see that T satisfies the relation (3.5) of the previous theorem for $\alpha = 1/2$, but T has no best proximity point. To see this, assume that

$$d\left(1 + \frac{1}{n}, T\left(1 + \frac{1}{n}\right)\right) = \max\left\{1 + \frac{1}{n}, 1 + \frac{1}{3(n+1)}\right\} = d(A,B) \tag{3.16}$$

for some $n \ge 1$. Thus $1 + 1/n = 1$ which is impossible. We claim that the ultrametric space $X = (X, d)$ is not spherically complete.

Consider the family of closed balls $\{B(1 + 1/(4n); 1 + 1/(2n))\}_{n \ge 1}$ in X. Since

$$d\left(1 + \frac{1}{4(n+1)}, 1 + \frac{1}{4n}\right) < 1 + \frac{1}{2n}, \tag{3.17}$$

it follows from Remark 3.6 that

$$B\left(1+\frac{1}{4n};1+\frac{1}{2n}\right)\supseteq B\left(1+\frac{1}{4(n+1)};1+\frac{1}{2(n+1)}\right). \tag{3.18}$$

Therefore this family is a chain of closed balls in X. Now let

$$1+\frac{1}{m}\in\bigcap_{n\geq 1}B\left(1+\frac{1}{4n};1+\frac{1}{2n}\right) \tag{3.19}$$

for some $m\geq 1$. This implies that for all $n\geq 1$ we have

$$\max\left\{1+\frac{1}{m},1+\frac{1}{4n}\right\}\leq 1+\frac{1}{2n} \tag{3.20}$$

which is a contradiction.

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