# Research Article

# A Note on Geodesically Bounded $\mathbb{R}$ -Trees

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It is proved that a complete geodesically bounded  $\mathbb{R}$ -tree is the closed convex hull of the set of its extreme points. It is also noted that if X is a closed convex geodesically bounded subset of a complete  $\mathbb{R}$ -tree Y, and if a nonexpansive mapping  $T: X \to Y$  satisfies  $\inf\{d(x, T(x)) : x \in X\} = 0$ , then T has a fixed point. The latter result fails if T is only continuous.

## 1. Introduction

Recall that for a metric space (X,d), a geodesic path (or metric segment) joining x and y in X is a mapping c of a closed interval [0,l] into X such that c(0)=x, c(l)=y, and d(c(t),c(t'))=|t-t'| for each  $t,t'\in[0,l]$ . Thus c is an isometry and d(x,y)=l. An  $\mathbb R$ -tree (or metric tree) is a metric space X such that:

- (i) there is a unique geodesic path (denoted by [x, y]) joining each pair of points  $x, y \in X$ ;
- (ii) if  $[y,x] \cap [x,z] = \{x\}$ , then  $[y,x] \cup [x,z] = [y,z]$ . From (i) and (ii), it is easy to deduce that
- (iii) if  $x, y, z \in X$ , then  $[x, y] \cap [x, z] = [x, w]$  for some  $w \in X$ .

The concept of an  $\mathbb{R}$ -tree goes back to a 1977 article of Tits [1]. Complete  $\mathbb{R}$ -trees posses fascinating geometric and topological properties. Standard examples of  $\mathbb{R}$ -trees include the "radial" and "river" metrics on  $\mathbb{R}^2$ . For the radial metric, consider all rays emanating from the origin in  $\mathbb{R}^2$ . Define the radial distance  $d_r$  between  $x,y\in\mathbb{R}^2$  to be the usual distance if they are on the same ray; otherwise take

$$d_r(x,y) = d(x,0) + d(0,y). (1.1)$$

(Here d denotes the usual Euclidean distance and 0 denotes the origin.) For the river metric  $\rho$  on  $\mathbb{R}^2$ , if two points x, and y are on the same vertical line, define  $\rho(x,y) = d(x,y)$ . Otherwise define  $\rho(x,y) = |x_2| + |y_2| + |x_1 - y_1|$ , where  $x = (x_1, x_2)$  and  $y = (y_1, y_2)$ . More subtle examples of  $\mathbb{R}$ -trees also exist, for example, the real tree of Dress and Terhalle [2].

It is shown in [3] that  $\mathbb{R}$ -trees complete are hyperconvex metric spaces (a fact that also follows from Theorem B of [4] and the characterization of [5]). They are also CAT(0) spaces in the sense of Gromov (see, e.g., [6, page 167]). Moreover, complete and geodesically bounded  $\mathbb{R}$ -trees have the fixed point property for continuous maps. This fact is a consequences of a result of Young [7] (see also [8]), and it suggests that complete geodesically bounded  $\mathbb{R}$ -trees have properties that one often associates with compactness. The two observations below serve to affirm this.

### 2. A Krein-Milman Theorem

In [9] Niculescu proved that a nonempty compact convex subset X of a complete CAT(0) space (called a global NPC space in [9]) is the convex hull of the set of all its extreme points. Subsequently, in [10], Borkowski et al. proved (among other things) that compactness is not needed in the special case when X is a complete and bounded  $\mathbb{R}$ -tree. Here we show that in complete  $\mathbb{R}$ -trees even the boundedness assumption may be relaxed.

**Theorem 2.1.** Let X be a complete and geodesically bounded  $\mathbb{R}$ -tree. Then X is the convex hull of its set E of extreme points.

*Proof.* Let  $x \in E$ , and let  $z \in X \setminus E$ . We will show that z lies on a segment joining x to some other element of E. We proceed by transfinite induction. Let  $\Omega$  denote the set of all countable ordinals, let  $z_0 = z$ , let  $\alpha \in \Omega$ , and assume that for all  $\beta \in \Omega$  with  $\beta < \alpha, z_\beta$  has been defined so that the following condition holds:

(i) 
$$\mu < \gamma < \alpha \Rightarrow z_{\mu} \in [x, z_{\gamma}]$$
, and  $z_{\gamma} \notin E \Rightarrow z_{\mu} \neq z_{\gamma}$ .

There are two cases.

- (1)  $\alpha = \beta + 1$ . If  $z_{\beta} \in E$ , there is nothing to prove because  $z = z_0 \in [x, z_{\beta}]$ . Otherwise, there are elements  $a, b \in X$  such that  $z_{\beta}$  lies on the segment [a, b] and  $a \neq z_{\beta} \neq b$ . At least one of these points, say a, does not lie on the segment  $[z_{\beta}, x]$ . Set  $z_{\alpha} = a$ , and observe that  $z_{\beta}$  lies on the segment  $[z_{\alpha}, x]$ .
- (2)  $\alpha$  is a limit ordinal. Since X is geodesically bounded, it must be the case that  $\sum_{\beta<\alpha} d(z_{\beta}, z_{\beta+1}) < \infty$ . This implies that  $(z_{\beta})_{\beta<\alpha}$  is a Cauchy net. Since X is complete, it must converge to some  $z_{\alpha} \in X$ .

Therefore,  $z_{\alpha}$  is defined for all  $\alpha \in \Omega$ . Since X is geodesically bounded,  $\sum_{\beta \in \Omega} d(z_{\beta}, z_{\beta+1}) < \infty$ . But since  $\Omega$  is uncountable, it is not possible that  $d(z_{\beta}, z_{\beta+1}) > 0$  for each  $\beta$ . Hence this transfinite process must terminate, and  $z_{\beta} = z_{\beta+1}$  for some  $\beta \in \Omega$ . It now follows from (i) that  $z_{\beta} \in E$  and z lies on the segment  $[z_{\beta}, x]$ .

Remark 2.2. The above proof shows that in fact each point of X is on a segment joining any given extreme point to some other extreme point.

#### 3. A Fixed Point Theorem

It is known that if K is a bounded closed convex subset of a complete CAT(0) space Y, and if  $f: K \to Y$  is a nonexpansive mapping for which

$$\inf\{d(x, f(x)) : x \in K\} = 0, \tag{3.1}$$

then f has a fixed point (see [11, Theorem 21]; also [12, Corollary 3.8]). This fact carries over to  $\mathbb{R}$ -trees since  $\mathbb{R}$ -trees are also CAT(0) spaces. However, we note here that if Y is an  $\mathbb{R}$ -tree, then again boundedness of K can be replaced by the assumption that K is merely geodesically bounded. In fact, we prove the following. (In the following theorem, we assume T is nonexpansive relative to the Hausdorff metric on the bounded nonempty closed subsets of Y.)

**Theorem 3.1.** Suppose X is a closed convex and geodesically bounded subset of a complete  $\mathbb{R}$ -tree Y, and suppose  $T: X \to 2^Y$  is a nonexpansive mapping taking values in the family of nonempty bounded closed convex subsets of Y. Suppose also that  $\inf\{\operatorname{dist}(x,T(x)):x\in X\}=0$ . Then there is a point  $x\in X$  for which  $x\in T(x)$ .

We will need the following result in the proof of Theorem 3.1. (See [13, 14] for more general set-valued versions of this theorem.)

**Theorem 3.2.** Suppose X is a closed convex geodesically bounded subset of a complete  $\mathbb{R}$ -tree Y and suppose  $f: X \to Y$  is continuous. Then either f has a fixed point or there exists a point  $z \in X$  such that

$$0 < d(z, f(z)) = \inf\{d(x, f(z)) : x \in X\}. \tag{3.2}$$

*Proof of Theorem 3.1.* Since complete  $\mathbb{R}$ -trees are hyperconvex, by Corollary 1 of [15] the selection  $f:X\to Y$  defined by taking f(x) to be the point of T(x) which is nearest to x for each  $x\in X$  is a nonexpansive single-valued mapping. Now assume f does not have a fixed point. Then by Theorem 3.2 there exists  $z\in X$  such that

$$0 < d(z, f(z)) = \inf\{d(x, f(z)) : x \in X\}. \tag{3.3}$$

We assert that  $d(x, f(x)) \ge d(z, f(z))$  for each  $x \in X$ . Indeed let  $x \in X$ . By (iii) there exists  $w \in Y$  such that  $[z, f(z)] \cap [z, x] = [z, w]$ . But since X is convex  $[z, x] \subseteq X$ , so  $w \in [z, x]$  implies  $w \in X$ . Also  $w \in [z, f(z)]$ , so it follows from (3.3) that w = z. Thus  $[z, f(z)] \cap [z, x] = \{z\}$ , and the segment [x, f(z)] must pass through z. Therefore,

$$d(x,z) + d(z,f(z)) = d(x,f(z))$$

$$\leq d(x,f(x)) + d(f(x),f(z))$$

$$\leq d(x,f(x)) + d(x,z).$$
(3.4)

Thus  $\inf\{d(x, f(x)) : x \in X\} \ge d(z, f(z)) > 0$  – a contradiction. Therefore, there exists  $x \in X$  such that  $x = f(x) \in T(x)$ .

**Corollary 3.3.** Suppose X is a closed convex and geodesically bounded subset of a complete  $\mathbb{R}$ -tree Y, and suppose  $f: X \to Y$  is a nonexpansive mapping for which  $\inf\{d(x, f(x)) : x \in X\} = 0$ . Then f has a fixed point.

*Example 3.4.* In view of the fact that continuous self-maps of  $X \to X$  have fixed points, it is natural to ask whether Corollary 3.3 holds for continuous mappings. The answer is no, even when X is bounded. Let Y be the Euclidean plane  $\mathbb{R}^2$  with the radial metric. Let  $\{e_n\}$  be a sequence of distinct points on the unit circle, and let  $X = \bigcup_{n=1}^{\infty} [e_n, 0]$ . We now define a continuous fixed-point free map  $f: X \to Y$  for which  $\inf\{d(x, f(x)) : x \in X\} = 0$ . First move each point of the segment  $[0, e_1]$  to the right onto a segment  $[e_1, b]$  where  $b \neq e_1$  and  $[e_1, b]$  is on the ray which extends  $[0, e_1]$ . (Thus  $f([0, e_1]) = [e_1, b]$ .) For each  $n \geq 2$ , let  $a_n$  denote the point on the segment  $[e_n, 0]$  which has distance 1/n from  $e_n$ . It is now clearly possible to construct a continuous (even lipschitzian) fixed point-free map f (a shift) of the segment  $[e_n, 0]$  onto the segment  $[a_n, e_1], n \geq 2$ , for which  $f(e_n) = a_n$ . Thus  $d(e_n, f(e_n)) = 1/n$  for all n.

*Remark 3.5.* Corollary 3.3 for bounded *X* is also a consequence of Theorem 6 of [15].

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