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Research Article

Krasnosel'skii-Type Fixed-Set Results

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Some new Krasnosel'skii-type fixed-set theorems are proved for the sum $S + T$, where S is a multimap and T is a self-map. The common domain of S and T is not convex. A positive answer to Ok's question (2009) is provided. Applications to the theory of self-similarity are also given.

1. Introduction

The Krasnosel'skii fixed-point theorem [1] is a well-known principle that generalizes the Schauder fixed-point theorem and the Banach contraction principle as follows.

Krasnosel'skii Fixed-Point Theorem

Let M be a nonempty closed convex subset of a Banach space E, $S : M \to E$, and $T : M \to E$. Suppose that

(a) S is compact and continuous;

(b) T is a k-contraction;

(c) $Sx + Ty \in M$ for every $x, y \in M$.

Then there exists $x^* \in M$ such that $Sx^* + Tx^* = x^*$.

This theorem has been extensively used in differential and functional differential equations and was motivated by the observation that the inversion of a perturbed differential operator may yield the sum of a continuous compact map and a contraction map. Note that the conclusion of the theorem does not need to hold if the convexity of M is relaxed even if T is the zero operator. Ok [2] noticed that the Krasnosel'skii fixed-point theorem can be reformulated by relaxing or removing the convexity hypothesis of M and by allowing the fixed-point to be a fixed-set. For variants or extensions of Krasnosel'skii-type fixed-point results, see [3–9], and for other interesting results see [10–13].

In this paper, we prove several new Krasnosel'skii-type fixed-set theorems for the sum $S + T$, where S is a multimap and T is a self-map. The common domain of S and T is not convex. Our results extend, generalize, or improve several fixed-point and fixed-set results including that given by Ok [2]. A positive answer to Ok's question [2] is provided. Applications to the theory of self-similarity are also given.

2. Preliminaries

Let M be a nonempty subset of a metric space $X := (X, d)$, $E := (E, \|\cdot\|)$ a normed space, ∂M the boundary of M, $\operatorname{int} M$ the interior of M, $\operatorname{cl} M$ the closure of M, $2^X \setminus \{\emptyset\}$ the set all nonempty subsets of X, $\mathcal{B}(X)$ the set of nonempty bounded subsets of X, $\mathcal{CD}(X)$ the family of nonempty closed subsets of X, $\mathcal{K}(X)$ the family of nonempty compact subsets of X, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+ := [0, \infty)$. A map $\alpha_K : \mathcal{B}(M) \to \mathbb{R}_+$ is called the *Kuratoswki measure of noncompactness* on M if

$$\alpha_K(A) := \inf\left\{\epsilon > 0 : A \subseteq \bigcup_{i=1}^{n} A_i \text{ and } \operatorname{diam} A_i \leq \epsilon\right\}, \tag{2.1}$$

for every $A \in \mathcal{B}(M)$, where $\operatorname{diam} A_i$ denotes the diameter of A_i. Let $T : M \to X$ and $S : M \to 2^X \setminus \{\emptyset\}$. We write $S(M) := \cup\{S(x) : x \in M\}$. We say that (a) $x \in M$ is a *fixed point* of T if $x = Tx$, and the set of fixed points of T will be denoted by $F(T)$; (b) T is *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in M$; (c) T is *k-contraction* if $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in M$ and some $k \in (0, 1)$; (d) T is *α_K-condensing* if it is continuous and, for every $A \in \mathcal{B}(M)$ with $\alpha_K(A) > 0$, $T(A) \in \mathcal{B}(X)$ and $\alpha_K(T(A)) < \alpha_K(A)$; (e) T is 1-*set-contractive* if it is continuous and, for every $A \in \mathcal{B}(M)$, $T(A) \in \mathcal{B}(X)$, and $\alpha_K(T(A)) \leq \alpha_K(A)$; (f) S is *compact* if $\operatorname{cl} S(M)$ is a compact subset of X.

Definition 2.1. Let $T : M \to X$, and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be either "a nondecreasing map satisfying $\lim_{n\to\infty}\varphi^n(t) = 0$ for every $t > 0$" or "an upper semicontinuous map satisfying $\varphi(t) < t$ for every $t > 0$." One says that T is a φ-contraction if $d(Tx, Ty) \leq \varphi(d(x, y))$ for all $x, y \in M$.

Remark 2.2. A mapping $T : M \to X$ is said to be a φ-contraction in the sense of Garcia-Falset [6] if there exists a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying either "φ is continuous and $\varphi(t) < t$ for $t > 0$" or "there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\psi(0) = 0$ and nondecreasing such that $0 < \psi(r) \leq r - \varphi(r)$" for which the inequality $d(Tx, Ty) \leq \varphi(d(x, y))$ holds for all $x, y \in M$. Our definition for φ-contraction is different in some sense from that of Garcia-Falset.

Lemma 2.3 (see [2]). *Let M be a nonempty closed subset of a normed space E. If $T : M \to 2^M \setminus \{\emptyset\}$ is compact and continuous, then there exists a minimal $A \in \mathcal{K}(M)$ such that $A = \operatorname{cl}(T(A))$.*

Theorem 2.4 (see [14]). *Let M be a nonempty bounded closed convex subset of a Banach space E. Suppose that $T : M \to M$ is an α_K-condensing map. Then T has a fixed point in M.*

Theorem 2.5 (see [15–17]). *Let X be a complete metric space. If $T : X \to X$ is a φ-contraction, then T has a unique fixed point in X.*

Theorem 2.6 (see [14]). *Let M be a closed subset of a Banach space E such that* $\operatorname{int} M$ *is bounded, open, and containing the origin. Suppose that $T : M \to E$ is an α_K-condensing map satisfying $Tx \neq \mu x$ for all $x \in \partial M$ and $\mu > 1$. Then T has a fixed point in M.*

Theorem 2.7 (see [14]). *Let M be a closed subset of a Banach space E such that* $\operatorname{int} M$ *is bounded, open, and containing the origin. Suppose that $T : M \to E$ is a 1-set-contractive map satisfying $Tx \neq \mu x$ for all $x \in \partial M$ and $\mu > 1$. If $(I - T)(M)$ is closed, then T has a fixed point in M.*

3. Fixed-Set Results

We now reformulate the Krasnosel'skii fixed-point theorem by allowing the fixed-point to be a fixed-set and removing the convexity hypothesis of M. Under suitable conditions, we look for a nonempty compact subset A of M such that

$$S(A) + T(A) = A \tag{3.1}$$

or

$$(I - T)(A) = S(A). \tag{3.2}$$

Theorem 3.1. *Let M be a nonempty closed subset of a Banach space E, $S : M \to \mathcal{CD}(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) *T is α_K-condensing and $T(M)$ is a bounded subset of E;*

(c) *$S(M) + T(M) \subseteq M$.*

Then there exists $A \in \mathcal{K}(M)$ such that $S(A) + T(A) = A$.

Proof. Fix $y \in S(M) + T(M)$. Let $\mathcal{A}$ denote the set of closed subsets C of M for which $y \in C$ and $S(C) + T(C) \subseteq C$. Note that $\mathcal{A}$ is nonempty since $M \in \mathcal{A}$. Take $C_0 := \cap_{C \in \mathcal{A}} C$. As C_0 is closed, $y \in C_0$, and $S(C_0) + T(C_0) \subseteq C_0$, we have $C_0 \in \mathcal{A}$. Let $L =: \operatorname{cl}((S(C_0) + T(C_0)) \cup \{y\})$. Notice that $\operatorname{cl}((S(M) + T(M))$ is a bounded subset of M containing L. So L is a closed subset of C_0, $y \in L$, and

$$S(L) + T(L) \subseteq S(C_0) + T(C_0) \subseteq L. \tag{3.3}$$

This shows that $L = C_0 \in \mathcal{A}$ and $\mathcal{K}(L) \subseteq \mathcal{K}(M)$. Since L is a bounded subset of M and $\operatorname{cl} S(L)$ is compact, we have

$$\begin{aligned}
\alpha_K(L) &= \alpha_K\big(\operatorname{cl}\big((S(L) + T(L)) \cup \{y\}\big)\big) \\
&= \alpha_K(S(L) + T(L)) \\
&\leq \alpha_K(S(L)) + \alpha_K(T(L)) \\
&= \alpha_K(\operatorname{cl} S(L)) + \alpha_K(T(L)) = 0 + \alpha_K(T(L)).
\end{aligned} \tag{3.4}$$

As T is α_K-condensing, it follows that $\alpha_K(L) = 0$. Thus L is a compact subset of M. As the Vietoris topology and the Hausdorff metric topology coincide on $\mathcal{K}(L)$ [18, page 17 and page 41], $\mathcal{K}(L)$ is compact and hence closed. Define $F : \mathcal{K}(L) \to 2^M$ by $F(A) := S(A) + T(A)$. It follows that

$$F(A) = S(A) + T(A) \subseteq S(L) + T(L) \subseteq L \tag{3.5}$$

for every $A \in \mathcal{K}(L)$. Since T is continuous and S is compact-valued and continuous, both $S(A)$ and $T(A)$ are compact subsets of E and hence $F : \mathcal{K}(L) \to \mathcal{K}(L)$. Moreover, the maps $A \to S(A)$ and $A \to T(A)$ are continuous, so F is continuous. By Lemma 2.3, there exists $\mathcal{C} \in \mathcal{K}(\mathcal{K}(L))$ such that $\mathcal{C} = \mathrm{cl}(F(\mathcal{C})) = F(\mathcal{C})$ since $F(\mathcal{C})$ is compact and hence closed. Let $A := \cup_{C \in \mathcal{C}} C$. As $\mathcal{C} = F(\mathcal{C})$, we have

$$A = \bigcup_{C \in \mathcal{C}} F(C) = F\left(\bigcup_{C \in \mathcal{C}} C\right) = F(A) = S(A) + T(A). \tag{3.6}$$

However A is a compact subset of L [18, page 16], so $A \in \mathcal{K}(M)$. □

Corollary 3.2 (see [2, Theorem 2.4]). *Let M be a nonempty closed subset of a Banach space E, $S : M \to \mathcal{CD}(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) *T is compact and continuous;*

(c) *$S(M) + T(M) \subseteq M$.*

Then there exists $A \in \mathcal{K}(M)$ such that $S(A) + T(A) = A$.

In the following corollary, we assume that $\liminf_{t\to\infty}(t - \varphi(t)) > 0$ whenever φ is upper semicontinuous.

Corollary 3.3. *Let M be a nonempty closed subset of a Banach space E, $S : M \to \mathcal{CD}(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) *T is a φ-contraction and $T(M)$ is bounded;*

(c) *$S(M) + T(M) \subseteq M$.*

Then there exists $A \in \mathcal{K}(M)$ such that $S(A) + T(A) = A$.

Remark 3.4. The following statements are equivalent [19]:

(i) T is a φ-contraction, where φ is nondecreasing, right continuous such that $\varphi(t) < t$ for all $t > 0$ and $\lim_{t\to\infty}(t - \varphi(t)) > 0$;

(ii) T is a φ-contraction, where φ is upper semicontinuous such that $\varphi(t) < t$ for all $t > 0$ and $\liminf_{t\to\infty}(t - \varphi(t)) > 0$.

Note that Corollary 3.3 provides a positive answer to the following question of Ok [2]. *We do not know at present if the fixed-set can be taken to be a compact set in the statement of* [2, Corollary 3.3].

Theorem 3.5. *Let M be a nonempty closed subset of a normed space E, $S : M \to \mathcal{CD}(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) $\operatorname{cl} S(M) \subseteq (I - T)(M)$;

(c) *$(I - T)^{-1}$ is a continuous single-valued map on $S(M)$.*

Then

(i) *there exists a minimal $L \in \mathcal{K}(M)$ such that $(I - T)(L) = S(L)$ and $L \subseteq S(L) + T(L)$;*

(ii) *there exists a maximal $A \in 2^M$ such that $S(A) + T(A) = A$.*

Proof. Let $y \in M$. Then, by (b), there exists $A \subseteq M$ such that $Sy \subseteq (I - T)A$, and, as $(I - T)^{-1}$ is a single-valued map on $S(M)$,

$$\left((I - T)^{-1} \circ S\right)y = (I - T)^{-1}(Sy) \subseteq A \subseteq M. \tag{3.7}$$

So $(I - T)^{-1} \circ S : M \to 2^M \setminus \{\emptyset\}$. Note that S is compact-valued and $\operatorname{cl} S(M)$ is a compact subset of $(I - T)(M)$. The continuity of $(I - T)^{-1} \circ S$ follows from that of S and $(I - T)^{-1}$. Moreover, $(I - T)^{-1}(\operatorname{cl} S(M))$ is a compact subset of M, and hence $\operatorname{cl}((I - T)^{-1} \circ S(M))$ is a compact subset of M. By Lemma 2.3, there exists a minimal $L \in \mathcal{K}(M)$ such that $L = \operatorname{cl}((I - T)^{-1} \circ S(L))$. But, since $(I - T)^{-1}$ is continuous and S is compact-valued, $(I - T)^{-1} \circ S$ is compact-valued and maps compact sets to compact sets. Then $(I - T)^{-1} \circ S(L)$, is a compact subset of M, so $L = (I - T)^{-1} \circ S(L)$. Thus $(I - T)(\mathrm{L}) = S(L)$, and hence $L \subseteq S(L) + T(L)$.

Let

$$\mathcal{C} := \left\{C \in 2^M : C \subseteq S(C) + T(C)\right\} \tag{3.8}$$

and $A := \cup_{C \in \mathcal{C}} C$. Clearly A is nonempty since $L \in \mathcal{C}$. Then $A \subseteq S(A) + T(A)$. Take $y \in S(A) + T(A)$. It follows that

$$A \cup \{y\} \subseteq S(A) + T(A) \subseteq S(A \cup \{y\}) + T(A \cup \{y\}), \tag{3.9}$$

and hence $A \cup \{y\} \in \mathcal{C}$ and $y \in A$. Thus $S(A) + T(A) = A$. □

Theorem 3.6. *Let M be a nonempty closed subset of a normed space E, $S : M \to \mathcal{CD}(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) *T is a φ-contraction;*

(c) *if $(I - T)x_n \to y$, then (x_n) has a convergent subsequence;*

(d) $S(M) + T(M) \subseteq M$.

Then

(i) *there exists a minimal* $L \in \mathcal{K}(M)$ *such that* $(I - T)(L) = S(L)$ *and* $L \subseteq S(L) + T(L)$*;*

(ii) *there exists a maximal* $A \in 2^M$ *such that* $S(A) + T(A) = A$*.*

Proof. Let $z \in \operatorname{cl} S(M)$. By (b), (d), and the closeness of M, the map $x \to z + Tx$ is a φ-contraction from M into M. So, by Theorem 2.5, there exists a unique $x_0 \in M$ such that $x_0 = z + Tx_0$. Then $z = x_0 - Tx_0 \in (I - T)(M)$, and so $\operatorname{cl} S(M) \subseteq (I - T)(M)$. Since the map $\to z + Tx$ has a unique fixed-point, its fixed-point set $(I - T)^{-1} z$ is singleton. So $(I - T)^{-1} : \operatorname{cl} S(M) \to M$ is a single-valued map. To show that $(I - T)^{-1}$ is continuous, let (y_n) be a sequence in $\operatorname{cl} S(M)$ such that $y_n \to y \in (I - T)(M)$. Define $x_n := (I - T)^{-1} y_n$ and $x := (I - T)^{-1} y$. Then $(I - T)x_n = y_n$, and $(I - T)x = y$. We claim that (x_n) is convergent. First, notice that (x_n) is bounded; otherwise, (x_n) has a subsequence (x_{n_k}) such that $\|x_{n_k}\| \to \infty$. As $(I - T)x_{n_k} \to (I - T)x$, (c) implies that (x_{n_k}) has a convergent subsequence, a contradiction. Next, as $I - T$ is continuous and one-to-one, it follows from (c) that the sequence (x_n) converges to x. Therefore, $(I - T)^{-1}$ is continuous. Now the result follows from Theorem 3.5. □

In the following result, we assume that $\liminf_{t \to \infty}(t - \varphi(t)) > 0$ whenever φ is upper semicontinuous.

Theorem 3.7. *Let* M *be a nonempty compact subset of a Banach space* E*,* $S : M \to \mathcal{CD}(E)$*, and* $T : M \to E$*. Suppose that*

(a) S *is continuous;*

(b) T *is a* φ*-contraction;*

(c) $S(M) + T(M) \subseteq M$.

Then

(i) *there exists a minimal* $L \in \mathcal{K}(M)$ *such that* $(I - T)(L) = S(L)$ *and* $L \subseteq S(L) + T(L)$*;*

(ii) *there exists a maximal* $A \in 2^M$ *such that* $S(A) + T(A) = A$*.*

(iii) *there exists* $B \in \mathcal{K}(M)$ *such that* $S(B) + T(B) = B$*.*

Proof. Parts (i) and (ii) follow from Theorem 3.6. Part (iii) follows from Theorem 3.1. □

Theorem 3.8. *Let* M *be a closed subset of a Banach space* E *such that* $\operatorname{int} M$ *is bounded, open, and containing the origin,* $S : M \to \mathcal{CD}(E)$*, and* $T : M \to E$*. Suppose that*

(a) S *is compact and continuous;*

(b) T *is an* α_K*-condensing map satisfying* $\operatorname{cl} S(M) \cap (\mu I - T)(\partial M) = \emptyset$ *for all* $\mu > 1$*;*

(c) $(I - T)^{-1}$ *is a continuous single-valued map on* $S(M)$*;*

(d) $S(M) + T(M) \subseteq M$.

Then

(i) *there exists a minimal* $L \in \mathcal{K}(M)$ *such that* $(I - T)(L) = S(L)$ *and* $L \subseteq S(L) + T(L)$*;*

(ii) *there exists a maximal* $A \in 2^M$ *such that* $S(A) + T(A) = A$*.*

(iii) *there exists* $B \in \mathcal{K}(M)$ *such that* $S(B) + T(B) = B$*.*

Proof. Let $z \in \operatorname{cl} S(M)$. As T is α_K-condensing, part (d) and the closeness of M imply that the map $x \to z + Tx$ is an α_K-condensing self-map of M. Moreover, this map satisfies $z + Tx \neq \mu x$ for all $x \in \partial M$ and $\mu > 1$; otherwise, there are $x_0 \in \partial M$ and $\mu_0 > 1$ such that $z + Tx_0 = \mu_0 x_0$. This implies that

$$z = \mu_0 x_0 - Tx_0 = (\mu_0 I - T)x_0 \in (\mu_0 I - T)(\partial M) \tag{3.10}$$

which contradicts the second part of (b). It follows from Theorem 2.6 that there exists $v \in M$ such that $z + Tv = v$. Then $z = v - Tv \in (I - T)(M)$, and so $\operatorname{cl} S(M) \subseteq (I - T)(M)$. Now parts (i) and (ii) follow from Theorem 3.5. Part (iii) follows from Theorem 3.1. □

Theorem 3.9. *Let M be a closed subset of a Banach space E such that* $\operatorname{int} M$ *is bounded, open, and containing the origin, $S : M \to CD(E)$, and $T : M \to E$. Suppose that*

(a) *S is compact and continuous;*

(b) *T is a 1-set-contractive map satisfying* $\operatorname{cl} S(M) \cap (\mu I - T)(\partial M) = \emptyset$ *for all $\mu > 1$;*

(c) *$(I - T)(M)$ is closed, and $(I - T)^{-1}$ is a continuous single-valued map on $S(M)$;*

(d) *$S(M) + T(M) \subseteq M$.*

Then

(i) *there exists a minimal $L \in \mathcal{K}(M)$ such that $(I - T)(L) = S(L)$ and $L \subseteq S(L) + T(L)$;*

(ii) *there exists $A \in 2^M$ such that $S(A) + T(A) = A$.*

Proof. Let $z \in \operatorname{cl} S(M)$. As T is 1-set-contractive, part (d) and the closeness of M imply that the map $x \to z + Tx$ is a 1-set-contractive self-map of M. Moreover, this map satisfies $z + Tx \neq \mu x$ for all $x \in \partial M$ and $\mu > 1$; otherwise, there are $x_0 \in \partial M$ and $\mu_0 > 1$ such that $z + Tx_0 = \mu_0 x_0$. This implies that

$$z = \mu_0 x_0 - Tx_0 = (\mu_0 I - T)x_0 \in (\mu_0 I - T)(\partial M) \tag{3.11}$$

which contradicts the second part of (b). It follows from Theorem 2.7 that there exists $v \in M$ such that $z + Tv = v$. Then $z = v - Tv \in (I - T)(M)$, and so $\operatorname{cl} S(M) \subseteq (I - T)(M)$. Now the result follows from Theorem 3.5. □

Definition 3.10 (self-similar sets). Let M be a nonempty closed subset of a Banach space E. If $F_1, \ldots, F_n$ are finitely many self-maps of M, then the list $(M, \{F_1, \ldots, F_n\})$ is called an*iterated function system* (IFS). This IFS is continuous (resp., contraction, α_K-condensing, etc.) if each F_i is so. A nonempty subset A of M is said to be *self-similar with respect to* the IFS $(M, \{F_1, \ldots, F_n\})$ if

$$F_1(A) \cup \cdots \cup F_n(A) = A. \tag{3.12}$$

Remark 3.11. It is well known that there exists a unique compact self-similar set with respect to any contractive IFS; see [20].

Example 3.12. Consider an IFS $(M, \{F_1, \ldots, F_n, F_{n+1}\})$ such that

(a) $F_1 \cup \cdots \cup F_n$ is a compact and continuous multimap;

(b) $F_i(M) + F_{n+1}(M) \subseteq M$ for each $i = 1, 2, \ldots, n$.

Then the existence of a compact self-similar set with respect to the IFS $(M, \{F_1, \ldots, F_n\})$ is ensured by letting F_{n+1} to be zero in each of the following situations.

(i) Suppose that F_{n+1} is an α_K-condensing map such that $F_{n+1}(M)$ is bounded. Then Theorem 3.1 ensures the existence of a compact subset A of M such that

$$(F_1(A) \cup \cdots \cup F_n(A)) + F_{n+1}(A) = A. \tag{3.13}$$

(ii) Suppose that F_{n+1} is a φ-contraction satisfying condition (c) of Theorem 3.6. Then there exists a minimal compact subset L of M such that

$$(I - F_{n+1})(L) = F_1(L) \cup \cdots \cup F_n(L). \tag{3.14}$$

(iii) Suppose that M is a closed subset of a Banach space E such that $\operatorname{int} M$ is bounded, open, and containing the origin, F_{n+1} is an α_K-condensing map satisfying $\operatorname{cl}(F_1(M) \cup \cdots \cup F_n(M)) \cap (\mu I - F_{n+1})(\partial M) = \emptyset$ for all $\mu > 1$, and $(I - F_{n+1})^{-1}$ is a continuous single-valued map on $(F_1 \cup \cdots \cup F_n)(M)$. Then Theorem 3.8 ensures the existence of a minimal compact subset L of M such that

$$(I - F_{n+1})(L) = F_1(L) \cup \cdots \cup F_n(L). \tag{3.15}$$

(iv) Suppose that M is a closed subset of a Banach space E such that $\operatorname{int} M$ is bounded, open, and containing the origin, F_{n+1} is a 1-set-contractive map satisfying $\operatorname{cl}(F_1(M) \cup \cdots \cup F_n(M)) \cap (\mu I - F_{n+1})(\partial M) = \emptyset$ for all $\mu > 1$, $(I - F_{n+1})(M)$ is closed, and $(I - F_{n+1})^{-1}$ is a continuous single-valued map on $(F_1 \cup \cdots \cup F_n)(M)$. Then Theorem 3.9 ensures the existence of a minimal compact subset L of M such that

$$(I - F_{n+1})(L) = F_1(L) \cup \cdots \cup F_n(L). \tag{3.16}$$

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