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Research Article

Normality of Composite Analytic Functions and Sharing an Analytic Function

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A result of Hinchliffe (2003) is extended to transcendental entire function, and an alternative proof is given in this paper. Our main result is as follows: let $\alpha(z)$ be an analytic function, $\mathcal{F}$ a family of analytic functions in a domain D, and $H(z)$ a transcendental entire function. If $H \circ f(z)$ and $H \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z), g(z) \in \mathcal{F}$, and one of the following conditions holds: (1) $H(z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$; (2) $\alpha(z)$ is nonconstant, and there exists $z_0 \in D$ such that $H(z) - \alpha(z_0) := (z - \beta_0)^p Q(z)$ has only one distinct zero β_0, and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for each $f(z) \in \mathcal{F}$, where $Q(\beta_0) \neq 0$; (3) there exists a $z_0 \in D$ such that $H(z) - \alpha(z_0)$ has no zero, and $\alpha(z)$ is nonconstant, then $\mathcal{F}$ is normal in D.

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the whole complex plane $\mathbf{C}$, and let a be a finite complex value or function. We say that f and g share a CM (or IM) provided that $f - a$ and $g - a$ have the same zeros counting (or ignoring) multiplicity. It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory

$$T(r, f), m(r, f), N(r, f)\overline{N}(r, f), \ldots \tag{1.1}$$

([1] or [2]). We denote by $S(r, f)$ any function satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to \infty$, possibly outside of a set of finite measure.

A meromorphic function $\alpha(z)$ is called a small function related to $f(z)$ if $T(r,\alpha) = S(r,f)$.

In 1952, Rosenbloom [3] proved the following theorem.

Theorem A. *Let $P(z)$ be a polynomial of degree at least 2 and $f(z)$ a transcendental entire function. Then*

$$\lim_{r\to\infty} \inf \frac{N(r,1/[P(f)-z])}{T(r,f)} \geq 1. \tag{1.2}$$

Influenced from Bloch's principle ([1] or [4]), that is, there is a normal criterion corresponding to every Liouville-Picard type theorem, Fang and Yuan [5] proved a corresponding normality criterion for inequality (1.2).

Theorem B. *Let $\mathcal{F}$ be a family of analytic functions in a domain D and $P(z)$ a polynomial of degree at least 2. If $P(f(z)) \neq z$ for each $f(z) \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 1995, Zheng and Yang [6] proved the following result.

Theorem C. *Let $P(z)$ be a polynomial of degree p at least 2, $f(z)$ a transcendental entire function, and $\alpha(z)$ a nonconstant meromorphic function satisfying $T(r,\alpha) = S(r,f)$. Then,*

$$T(r,f) \leq \mu \overline{N}\left(r, \frac{1}{P(f)-\alpha(z)}\right) + S(r,f). \tag{1.3}$$

Here $\mu = 2/(p-1)$ if $P'(z)$ has only one zero; otherwise $\mu = 2$.

In 2000, Fang and Yuan [7] improved (1.3) and obtained the best possible k.

Theorem D. *Let $P(z)$ be a polynomial of degree p at least 2 and $f(z)$ a transcendental entire function, and $\alpha(z)$ a nonconstant meromorphic function satisfying $T(r,\alpha) = S(r,f)$. If $\alpha(z)$ is a constant, we also require that there exists a constant $A \neq \alpha$ such that $P(z) - A$ has a zero of multiplicity at least 2. Then*

$$T(r,f) \leq \mu \overline{N}\left(r, \frac{1}{P(f)-\alpha(z)}\right) + S(r,f). \tag{1.4}$$

Here $\mu = 1/(p-1)$ if $P'(z)$ has only one zero; otherwise $\mu = 1$.

The corresponding normal criterion below to Theorem D was obtained by Fang and Yuan [7].

Theorem E. *Let $\mathcal{F}$ be a family of analytic functions in a domain D and $P(z)$ a polynomial of degree at least 2. Suppose that $\alpha(z)$ is either a nonconstant analytic function or a constant function such that $P(z) - \alpha$ has at least two distinct zeros. If $P \circ f(z) \neq \alpha(z)$ for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

In 2003, Hinchliffe [8] proved the following theorem.

Theorem F. *Let $\alpha(z) = z$, $\mathcal{F}$ a family of analytic functions in a domain D, and $h(z)$ a transcendental meromorphic function. If $\widehat{\mathbf{C}} \setminus h(\mathbf{C}) = \emptyset$, $\{\infty\}$ or $\{\xi_1, \xi_2\}$, where $\{\xi_1, \xi_2\}$ are two distinct values in $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$, suppose that $h \circ f(z) \neq \alpha(z)$ for each $f \in \mathcal{F}$ and all $z \in D$. Then, $\mathcal{F}$ is normal in D.*

In 2004, Bergweiler [9] deals also with the case that $\alpha(z)$ is meromorphic in Theorem F and extended Theorem E as follows.

Theorem G. *Let $\alpha(z)$ be a nonconstant meromorphic function, $\mathcal{F}$ a family of analytic functions in a domain D, and $R(z)$ a rational function of degree at least 2. Suppose that $R \circ f(z) \neq \alpha(z)$ for each $f \in \mathcal{F}$ and all $z \in D$. Then, $\mathcal{F}$ is normal in D.*

Recently, Yuan et al. [10] generalized Theorem G in another manner and proved the following result.

Theorem H. *Let $\alpha(z)$ be a nonconstant meromorphic function, $\mathcal{F}$ a family of analytic functions in a domain D, and $R(z)$ a rational function of degree at least 2. If $R \circ f(z)$ and $R \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z)$, $g(z) \in \mathcal{F}$ and one of the following conditions holds:*

(1) *$R(z) - \alpha(z_0)$ has at least two distinct zeros or poles for any $z_0 \in D$;*

(2) *there exists $z_0 \in D$ such that $R(z) - \alpha(z_0) := P(z)/Q(z)$ has only one distinct zero (or pole) β_0 and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$ (or $k \neq lq$), for each $f(z) \in \mathcal{F}$, where $P(z)$ and $Q(z)$ are two of no common zero polynomials with degree p and q, respectively, and $\alpha(z_0) \in \mathbf{C} \cup \{\infty\}$.*

Then, $\mathcal{F}$ is normal in D.

In this paper, we improve Theorems E and F and obtain the main result Theorem 1.1 which is proved below in Section 3.

Theorem 1.1. *Let $\alpha(z)$ be an analytic function, $\mathcal{F}$ a family of analytic functions in a domain D, and $H(z)$ a transcendental entire function. If $H \circ f(z)$ and $H \circ g(z)$ share $\alpha(z)$ IM for each pair $f(z), g(z) \in \mathcal{F}$, and one of the following conditions holds:*

(1) *$H(z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$;*

(2) *$\alpha(z)$ is nonconstant, and there exists $z_0 \in D$ such that $H(z) - \alpha(z_0) := (z - \beta_0)^p Q(z)$ has only one distinct zero β_0 and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, for each $f(z) \in \mathcal{F}$, where $Q(\beta_0) \neq 0$;*

(3) *there exists a $z_0 \in D$ such that $H(z) - \alpha(z_0)$ has no zero, and $\alpha(z)$ is nonconstant.*

Then, $\mathcal{F}$ is normal in D.

2. Preliminary Lemmas

In order to prove our result, we need the following lemmas. Lemma 2.1 is an extending result of Zalcman [11] concerning normal families.

Lemma 2.1 (see [12]). *Let $\mathcal{F}$ be a family of functions on the unit disc. Then, $\mathcal{F}$ is not normal on the unit disc if and only if there exist*

(a) *a number $0 < r < 1$;*

(b) *points z_n with $|z_n| < r$;*

(c) *functions $f_n \in \mathcal{F}$;*

(d) *positive numbers $\rho_n \to 0$*

such that $g_n(\zeta) := f_n(z_n + \rho_n\zeta)$ *converges locally uniformly to a nonconstant meromorphic function* $g(\zeta)$*, which order is at most 2.*

Remark 2.2. $g(\zeta)$ is a nonconstant entire function if $\mathcal{F}$ is a family of analytic functions on the unit disc in Lemma 2.1.

The following Lemma 2.3 is very useful in the proof of our main theorem. We denote by $U(z_0, r)$ the open disc of radius r around z_0, that is, $U(z_0, r) := \{z \in \mathbf{C} : |z - z_0| < r\}$. $U^0(z_0, r) := \{z \in \mathbf{C} : 0 < |z - z_0| < r\}$.

Lemma 2.3 (see [13] or [14]). *Let* $\{f_n(z)\}$ *be a family of analytic functions in* $U(z_0, r)$*. Suppose that* $\{f_n(z)\}$ *is not normal at* z_0 *but is normal in* $U^0(z_0, r)$*. Then, there exists a subsequence* $\{f_{n_k}(z)\}$ *of* $\{f_n(z)\}$ *and a sequence of points* $\{z_{n_k}\}$ *tending to* z_0 *such that* $f_{n_k}(z_{n_k}) = 0$*, but* $\{f_{n_k}(z)\}$ *tending to infinity locally uniformly on* $U^0(z_0, r)$*.*

3. Proof of Theorem

Proof of Theorem 1.1. Without loss of generality, we assume that $D = \{z \in \mathbf{C}, |z| < 1\}$. Then, we consider three cases:

Case 1. $H(z) - \alpha(z_0)$ has at least two distinct zeros for any $z_0 \in D$

Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z = 0$.

Set $H(z) - \alpha(0)$ have two distinct zeros β_1 and β_2.

By Lemma 2.1, there exists a sequence of points $z_n \to 0$, $f_n \in \mathcal{F}$ and $\rho_n \to 0^+$ such that

$$F_n(\xi) := f_n(z_n + \rho_n\xi) \longrightarrow F(\xi) \tag{3.1}$$

uniformly on any compact subset of $\mathbf{C}$, where $F(\xi)$ is a nonconstant entire function.

Hence,

$$H \circ f_n(z_n + \rho_n\xi) - \alpha(z_n + \rho_n\xi) \longrightarrow H \circ F(\xi) - \alpha(0) \tag{3.2}$$

uniformly on any compact subset of $\mathbf{C}$.

We claim that $H \circ F(\xi) - \alpha(0)$ had at least two distinct zeros.

If $F(\xi)$ is a nonconstant polynomial, then both $F(\xi) - \beta_1$ and $F(\xi) - \beta_2$ have zeros. So $H \circ F(\xi) - \alpha(0)$ has at least two distinct zeros.

If $F(\xi)$ is a transcendental entire function, then either $F(\xi) - \beta_1$ or $F(\xi) - \beta_2$ has infinite zeros. Indeed, suppose that it is not true, then by Picard's theorem [2], we obtain that $F(\xi)$ is a polynomial, a contradiction.

Thus, the claim gives that there exist ξ_1 and ξ_2 such that

$$H \circ F(\xi_1) - \alpha(0) = 0; \quad H \circ F(\xi_2) - \alpha(0) = 0 \quad (\xi_1 \neq \xi_2). \tag{3.3}$$

We choose a positive number δ small enough such that $D_1 \cap D_2 = \varnothing$ and $F(\xi) - \alpha(0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2, where

$$D_1 = \{\xi \in \mathbf{C}; |\xi - \xi_1| < \delta\}, \qquad D_2 = \{\xi \in \mathbf{C}; |\xi - \xi_2| < \delta\}. \tag{3.4}$$

By hypothesis and Hurwitz's theorem [14], for sufficiently large n there exist points $\xi_{1n} \in D_1$, $\xi_{2n} \in D_2$ such that

$$\begin{aligned} H \circ f_n(z_n + \rho_n \xi_{1n}) - \alpha(z_n + \rho_n \xi_{1n}) &= 0, \\ H \circ f_n(z_n + \rho_n \xi_{2n}) - \alpha(z_n + \rho_n \xi_{2n}) &= 0. \end{aligned} \tag{3.5}$$

Note that $H \circ f_m(z)$ and $H \circ f_n(z)$ share $\alpha(z)$ IM; it follows that

$$\begin{aligned} H \circ f_m(z_n + \rho_n \xi_{1n}) - \alpha(z_n + \rho_n \xi_{1n}) &= 0, \\ H \circ f_m(z_n + \rho_n \xi_{2n}) - \alpha(z_n + \rho_n \xi_{2n}) &= 0. \end{aligned} \tag{3.6}$$

Taking $n \to \infty$, we obtain

$$H \circ f_m(0) - \alpha(0) = 0. \tag{3.7}$$

Since the zeros of

$$H \circ f_m(\xi) - \alpha(\xi) \tag{3.8}$$

have no accumulation points, we have

$$z_n + \rho_n \xi_{1n} = 0, \qquad z_n + \rho_n \xi_{2n} = 0, \tag{3.9}$$

or equivalently

$$\xi_{1n} = -\frac{z_n}{\rho_n}, \qquad \xi_{2n} = -\frac{z_n}{\rho_n}. \tag{3.10}$$

This contradicts with the facts that $\xi_{1n} \in D_1$, $\xi_{2n} \in D_2$, and $D_1 \cap D_2 = \varnothing$.

Case 2. $\alpha(z)$ is nonconstant, and there exists $z_0 \in D$ such that $H(z) - \alpha(z_0) := (z - \beta_0)^p Q(z)$ has only one distinct zero β_0, and suppose that the multiplicities l and k of zeros of $f(z) - \beta_0$ and $\alpha(z) - \alpha(z_0)$ at z_0, respectively, satisfy $k \neq lp$, possibly outside finite $f(z) \in \mathcal{F}$, where $Q(\beta_0) \neq 0$.

We shall prove that $\mathcal{F}$ is normal at $z_0 \in D$. Without loss of generality, we can assume that $z_0 = 0$.

By $\alpha(z)$ nonconstant and analytic, we see that there exists a neighborhood $U(0, r)$ such that

$$\alpha(z) \neq \alpha(0). \tag{3.11}$$

Hypothesis implies that $H(z) - \alpha(0)$ has only one zero β_0, that is, $H(\beta_0) = \alpha(0)$.

We claim that $\mathcal{F}$ is normal at $z_0 \in U^0(0, r)$ for small enough r. In fact, $H(z) - \alpha(z_0)$ has infinite zeros by Picard theorem. Hence, the conclusion of Case 1 tells us that this claim is true.

Next, we prove $\mathcal{F}$ is normal at $z = 0$. For any $\{f_n(z)\} \subset \mathcal{F}$, by the former claim, there exists a subsequence of $\{f_n(z)\}$, denoted $\{f_n(z)\}$ for the sake of simplicity, such that

$$f_n(z) \longrightarrow G(z), \tag{3.12}$$

uniformly on a punctured disc $U^0(0, r) \subset U$.

By hypothesis, we see that $\{H \circ f_n(z) - \alpha(z)\}$ is an analytic family in the disc $U(0, r)$.

If $\{f_n(z)\}$ is not normal at $z = 0$, then Lemma 2.3 gives that $G(z) = \infty$, on a punctured disc $U^0(0, r)$ and $f_n(z'_n) = 0$ for a sequence of points $z'_n \to 0$.

We claim that there exists a sequence of points $z_n \in U(0, r)$ $(z_n \to 0)$ such that $H \circ f_n(z_n) - \alpha(z_n) = 0$.

In fact we may find $\rho, \epsilon > 0$ such that $|H(z) - \alpha(0)| > \epsilon$ for $|z - \beta_0| = \rho$. Next, we choose δ with $0 < \delta < r$ such that $|\alpha(z) - \alpha(0)| < \epsilon$ for $|z| < \delta$.

Since $f_n(z) \to \infty$ on $U^0(0, r)$ and $f_n(z'_n) = 0$ for a sequence of points $z'_n \to 0$, we know that if n sufficiently large, then

$$\left|\left(f_n(z) - \beta\right) - f_n(z)\right| = |\beta| \le |\beta_0| + \rho < \left|f_n(z)\right| \tag{3.13}$$

for $|z| = \delta$ and $\beta \in U(\beta_0, \rho)$. For large n, we also have $|z'_n| < \delta$, and thus we deduce that from Rouché's theorem that $f_n(z)$ takes the value $\beta \in U(0, \delta)$, that is, we have $f_n(U(0, \delta)) \supset U(\beta, \rho)$ for large n. Since also $f_n(\partial U(0, \delta)) \cap U(\beta, \rho) = \emptyset$ for large n, we find a component U of $f_n^{-1}(U(\beta_0, \rho))$ contained in $U(0, \delta)$ for such n. Moreover, U is a Jordan domain, and $f_n : U \to U(\beta_0, \rho)$ is a proper map.

For $z \in \partial U$, we then have $f_n(z) \in \partial U(\beta_0, \rho)$, and thus $|H \circ f_n(z) - \alpha(0)| > \epsilon$. Hence

$$\left|H \circ f_n(z) - \alpha(z) - \left(H \circ f_n(z) - \alpha(0)\right)\right| = |\alpha(z) - \alpha(0)| < \epsilon < \left|H \circ f_n(z) - \alpha(0)\right| \tag{3.14}$$

for $z \in \partial U$. Now f_n, in particular, takes the value β_0 in U, say, $f_n(z''_n) = \beta_0$ with $z''_n \in U$. Hence, $H \circ f_n(z''_n) - \alpha(0) = 0$, and thus Rouché's theorem now shows that our claim holds.

By the similar argument as Case 1, we obtain that $z_n = 0$ for sufficiently large n. Because $H(z) - \alpha(0) = (z - \xi_0)^p H(z)$, we have

$$\begin{gathered} H \circ f_n(z) - \alpha(z) = \left(f_n(z) - \xi_0\right)^p H\left(f_n(z)\right) - (\alpha(z) - \alpha(0)), \\ \left(f_n(0) - \xi_0\right)^p H\left(f_n(0)\right) = H \circ f_n(0) - \alpha(0) = 0. \end{gathered} \tag{3.15}$$

Hence,

$$H \circ f_n(z) - \alpha(z) = z^k \left[z^{lp-k} h_n(z) - \beta(z) \right], \quad \text{if } lp > k;$$
$$H \circ f_n(z) - \alpha(z) = z^{lp} \left[h_n(z) - z^{k-lp} \beta(z) \right], \quad \text{if } lp < k, \tag{3.16}$$

where $h_n(z)$, $\beta(z)$ are analytic functions and $h_n(0) \neq 0$, $\beta(0) \neq 0$.

Set $H_n(z) := z^{lp-k} h_n(z) - \beta(z)$, if $lp > k$; or $H_n(z) := h_n(z) - z^{k-lp}\beta(z)$, if $lp < k$. Thus, $H_n(0) = -\beta(0) \neq 0$ or $H_n(0) = h_n(0) \neq 0$. Noting that $lp \neq k$, we see that $\{H_n(z)\}$ is an analytic family and normal in $U^0(0, r)$.

By the same argument as above, there exists a sequence of points $z_n^* \in U'$ such that $z_n^* \to 0$, and $H_n(z_n^*) = 0$. Obviously, $z_n^* \neq 0$ and

$$H \circ f_n(z_n^*) - \alpha(z_n^*) = z_n^* H_n(z_n^*) = 0. \tag{3.17}$$

Noting that $H \circ f_n(z)$ and $H \circ f_m(z)$ share $\alpha(z)$ IM, we obtain that

$$H \circ f_m(z_n^*) - \alpha(z_n^*) = 0 \tag{3.18}$$

for each m. That is, $z_n^* H_m(z_n^*) = 0$. Noting that $z_n^* \neq 0$, we deduce that $H_m(z_n^*) = 0$. Thus, taking $n \to \infty$, $H_m(0) = 0$, contradicting the hypothesis for $H_m(0)$.

Case 3. There exists a $z_0 \in D$ such that $H(z) - \alpha(z_0)$ has no zero, and $\alpha(z)$ is nonconstant.

Suppose that $\mathcal{F}$ is not normal in D. Without loss of generality, we assume that $\mathcal{F}$ is not normal at $z = 0$.

By Picard theorem and (3.11), we know that $H(z) - \alpha(z_0)$ has at least two distinct zeros at any $z_0 \in U^0(0, r)$ for small enough r. The result of Case 1 tell us that $\mathcal{F}$ is normal in $U^0(0, r)$.

Thus, for any $\{f_n(z)\} \subset \mathcal{F}$, by the former conclusion and Lemma 2.3, there exists a subsequence of $\{f_n(z)\}$, denoted by $\{f_n(z)\}$ for the sake of simplicity, such that

$$f_n(z) \longrightarrow \infty, \tag{3.19}$$

uniformly on a punctured disc $U^0(0, r) \subset U$ and $f_n(z_n') = 0$ for a sequence of points $z_n' \to 0$.

Obviously, $\{H \circ f_n(z) - \alpha(z)\}$ is an analytic normal family in the punctured disc $U^0(0, r)$ for small enough r. We consider two subcases.

Subcase 1 ($\{H \circ f_n(z) - \alpha(z)\}$ is not normal at $z = 0$). Using Lemma 2.3 for $\{H \circ f_n(z) - \alpha(z)\}$, we get that there exists a sequence of points $z_n \in U(0, r)$ such that $z_n \to 0$ and $H \circ f_n(z_n) - \alpha(z_n) = 0$.

Noting that $H \circ f_m(z)$ and $H \circ f_n(z)$ share $\alpha(z)$ IM, and $H(z) - \alpha(0)$ has no zero, it follows that $z_n \neq 0$ and $H \circ f_m(z_n) - \alpha(z_n) = 0$. Taking $n \to \infty$, we obtain $H \circ f_m(0) - \alpha(0) = 0$. A contradiction with the hypothesis that $H(z) - \alpha(0)$ has no zero.

Subcase 2 ($\{H \circ f_n(z) - \alpha(z)\}$ is normal at $z = 0$). Then, $\{(H \circ f_n(z) - \alpha(0))/(\alpha(z) - \alpha(0))\}$ is normal in $U^0(0, r)$, which tends to a limit function $h(z)$, which is either identically infinite or analytic in $U^0(0, r)$. Set

$$M_n := \min\{|f_n(z)| : |z| = r\}, \tag{3.20}$$

noting that $M_n \to \infty$ as $n \to \infty$. If n is large enough, we have $z'_n \in U(0, r)$, and hence $U(0, M_n) \subseteq f_n(U(0, r))$. Denote $\partial f_n(U(0, r))$ by Γ_n, and note that the Γ_n are closed curves, arbitrarily distant from and surrounding the origin.

Suppose that $h(z) \equiv \infty$ on $U^0(0, r)$. Since $h_n(z) := (H \circ f_n(z) - \alpha(0))/(\alpha(z) - \alpha(0)) \to \infty$ locally uniformly on $\partial U(0, r)$, there exists, for arbitrarily large positive M, an $n_0(M)$ such that, for $n \geq n_0$, $|h_n(z)| \geq M$ on $\partial U(0, r)$. Thus, we have $|H \circ f_n(z) - \alpha(0)| \geq M|\alpha(z) - \alpha(0)|$ on $\partial U(0, r)$. Hence, for large n, $H(z)$ is bounded away from $\alpha(0)$ on the curves Γ_n, and this contradicts Iversen's theorem [15].

On the other hand, suppose that $h(z)$ is analytic on $U^0(0, r)$. Then, there exists some constant L such that $|h(z)| \leq L$ on $\partial U(0, r)$, and so, for large n, $|h_n(z)| \leq 2L$ on $\partial U(0, r)$. Hence, $|H \circ f_n(z) - \alpha(0)| \leq 2L|\alpha(z) - \alpha(0)|$ on $\partial U(0, r)$. Again, $H(z)$ is therefore bounded away from ∞ of its omitted value on the curves Γ_n, contradicting Iversen's theorem.

Therefore $\mathcal{F}$ is normal in Case 3.

Theorem 1.1 is proved completely. □

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