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Research Article

Remarks on Recent Fixed Point Theorems

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Coincidence and fixed point theorems for a new class of contractive, nonexpansive and hybrid contractions are proved. Applications regarding the existence of common solutions of certain functional equations are also discussed.

1. Introduction

The following remarkable generalization of the classical Banach contraction theorem, due to Suzuki [1], has led to some important contribution in metric fixed point theory (see, e.g., [1–8]).

Theorem 1.1. *Let (X, d) be a complete metric space and $S : X \to X$. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2} & \text{if } \dfrac{\sqrt{5}-1}{2} \le r \le \dfrac{1}{2}, \\ \dfrac{1}{1+r} & \text{if } \dfrac{1}{2} \le r < 1. \end{cases} \tag{1.1}$$

Assume that there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Sx) \le d(x, y) \quad \text{implies } d(Sx, Sy) \le rd(x, y) \tag{SC}$$

for all x, $y \in X$. Then S has a unique fixed point. A map satisfying condition (SC) *is called Suzuki contraction and the above theorem as the Suzuki contraction theorem (see [9]).*

Following Theorem 1.1, Edelstein's theorem for contractive maps has been generalized in [7] (cf. Theorem 2.1). Fixed point theorems for nonexpansive maps due to Browder [10, 11] and Göhde [12] have been generalized in [6] (cf. Theorem 3.1 below). Theorem 1.1 and Nadler's multivalued contraction theorem have been generalized by Kikkawa and Suzuki [2] (cf. Theorem 4.1 below). Further, Theorem 4.1 has been generalized by Moţ and Petruşel [3], Dhompongsa and Yingtaweesittikul [4], Singh and Mishra [9] and others. Combining the ideas of Suzuki [6, 7], Goebel [13] and Naimpally et al. [14], first we generalize Theorems 2.1 and 3.1 to a wider class of maps on an arbitrary nonempty set with values in a metric (resp. Banach) space. Using the notion of IT-commuting maps due to Itoh and Takahashi [15], we obtain generalizations of multivalued fixed point theorems due to Reich [16], Iseki [17], Kikkawa and Suzuki [2], Moţ and Petruşel [3], Dhompongsa and Yingtaweesittikul [4] and others to the case of Suzuki generalized hybrid contraction (cf. Theorem 4.1). Various examples presented in Section 5 demonstrate the generality of the assumptions used in our results. An experimental approach regarding the sequence of Jungck iterates [18] for the new class of contractive and nonexpansive maps is also discussed, which leads to a conjecture. Finally, we deduce the existence of a common solution for the Suzuki class of functional equations under much weaker conditions than those in [19–21].

2. Contractive Maps

The following result of Suzuki [7] generalizes the well-known fixed point theorem of Edelstein [22].

Theorem 2.1. *Let (X, d) be a compact metric space and $S : X \to X$. Assume that*

$$\frac{1}{2}d(x, Sx) < d(x, y) \quad \text{implies } d(Sx, Sy) < d(x, y) \tag{2.1}$$

for $x, y \in X$. Then S has a unique fixed point.

Throughout this paper Y will denote an arbitrary nonempty set. As a generalization of the results of Goebel [13], Edelstein [22] and Naimpally et al. [14, Corollary 3], we extend Theorem 2.1 for a pair of Suzuki contractive maps $S, T : Y \to X$ (cf. (2.2) and (2.3)), wherein (X, d) is a metric space.

Theorem 2.2. *Let $S, T : Y \to X$ be such that $S(Y) \subseteq T(Y)$ and $T(Y)$ is a compact subspace of X. Assume that for $x, y \in Y$,*

$$\frac{1}{2}d(Tx, Sx) < d(Tx, Ty) \tag{2.2}$$

implies

$$d(Sx, Sy) < d(Tx, Ty), \tag{2.3}$$

and $Tx = Ty$ implies $Sx = Sy$. Then S and T have a coincidence, that is, there exists $z \in Y$ such that $Sz = Tz$. Further, if $Y = X$, then S and T have a unique common fixed point provided that S and T commute at z.

Proof. Define $F : T(Y) \to T(Y)$ by $Fa = S(T^{-1}a))$ for each $a \in T(Y)$. To see that F is well defined, observe by $S(Y) \subseteq T(Y)$ that for $x \in T^{-1}(a)$,

$$Fa = Sx, \quad Fa \subseteq T(Y). \tag{2.4}$$

Take $x, y \in T^{-1}a$ such that $b = Sx,\ c = Sy$. Then, since $Tx = Ty$, we have $b = c$. Therefore F is well-defined.

Now, for $a \neq b$ and $a, b \in T(Y),\ T^{-1}a \cap T^{-1}b = \phi$. So, for distinct $a, b \in T(Y)$, we suppose $(1/2)d(a, Fa) < d(a, b)$. Then for $x \in T^{-1}a$ and $y \in T^{-1}b$, we have

$$\frac{1}{2}d(Tx, Sx) = \frac{1}{2}d(a, Fa) < d(a, b) = d(Tx, Ty). \tag{2.5}$$

This inequality implies that $d(Sx, Sy) < d(Tx, Ty)$. So $d(Fa, Fb) < d(a, b)$. Therefore, by Theorem 2.1, F has a unique fixed point w. Then for any $z \in T^{-1}w, Sz = Fw = w = Tz$. So, z is a coincidence point of S and T. If S and T are commuting at z, then $Sz = Tz \Rightarrow SSz = STz = TSz = TTz$ and $Sw = Tw$. If $Sz \neq SSz$, then $(1/2)d(Tz, Sz) = 0 < d(Tz, TTz) = d(Tz, TSz)$, and this implies that $d(w, Sw) = d(Sz, SSz) < d(Tz, TSz) = d(w, Sw)$, a contradiction. So, w is a common fixed point of S and T.

We conclude the proof by showing the unicity of the common fixed point. Suppose that $v \neq w$ is another common fixed point of S and T. Since $(1/2)d(Tw, Sw) = 0 < d(Tw, Tv)$, we have $d(w, v) = d(Sw, Sv) < d(Tw, Tv) = (w, v)$, a contradiction. Hence $v = w$. □

3. Nonexpansive Maps

A self-map S of a metric space X is nonexpansive if $d(Sx, Sy) \leq d(x, y)$ for all $x, y \in X$. The theory of nonexpansive maps is exciting and plays a vital role in nonlinear analysis and applications (see, e.g., [10–12, 23–28]). Recently, Suzuki [6] obtained the following theorem that generalizes the results of Browder [10, 11] and Göhde [12].

Theorem 3.1. *Let C be a convex subset of a Banach space E and $S : C \to C$. Assume that*

$$\frac{1}{2}\|x - Sx\| \leq \|x - y\| \quad \textit{implies} \ \|Sx - Sy\| \leq \|x - y\| \tag{3.1}$$

for all $x, y \in C$. Assume further that one of the following holds:

(i) *C is compact;*

(ii) *C is weakly compact and E has the Opial property;*

(iii) *C is weakly compact and E is uniformly convex in every direction (UCED).*

Then S has a fixed point.

For definitions and details of the Opial property [25], uniform convexity and UCED, one may refer to Goebel [23], Goebel and Kirk [24], Prus [26], Suzuki [6] and Takahashi [27, 28].

Now we present the following extension of Theorem 3.1 for a pair of Suzuki nonexpansive maps (cf. (3.2)).

Theorem 3.2. *Let E be a Banach space and $S,T : Y \to E$ such that $S(Y) \subseteq T(Y)$ and $T(Y)$ is a convex subset of E. Assume that for $x, y \in Y$,*

$$\frac{1}{2}\|Tx - Sx\| \leq \|Tx - Ty\| \text{ implies } \|Sx - Sy\| \leq \|Tx - Ty\|, \tag{3.2}$$

and $Tx = Ty$ implies $Sx = Sy$. Assume further that either of the following holds:

(i) *$T(Y)$ is compact;*

(ii) *$T(Y)$ is weakly compact and E has the Opial property;*

(iii) *$T(Y)$ is weakly compact and E is UCED.*

Then S and T have a coincidence.

Proof. As in the proof of Theorem 2.1, letting $Fa = S(T^{-1}a)$ for $a \in T(Y)$, it suffices to show that $F : T(Y) \to T(Y)$ has a fixed point.

Let $a, b \in T(Y)$ such that $(1/2)\|a - Fa\| \leq \|a - b\|$. Then for $x \in T^{-1}a$ and $y \in T^{-1}b$, we have $(1/2)\|Tx - Sx\| = (1/2)\|a - Fa\| \leq \|a - b\| = \|Tx - Ty\|$. By (3.2), we obtain $\|Sx - Sy\| \leq \|Tx - Ty\|$, and thus $\|Fa - Fb\| \leq \|a - b\|$. So, by Theorem 3.1, F has a fixed point. □

4. Multivalued Contractions

In all that follows, let CB(X) (resp. CL(X)) denote the family of all nonempty closed bounded (resp. closed) subsets of X. Let H denote the Hausdorff metric induced by the metric d of the metric space X. For any subsets A, B of X, $d(A, B)$ denotes the gap between the subsets A and B, while $\rho(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ and

$$BN(X) = \{A : \phi \neq A \subseteq X \text{ and the diameter of } A \text{ is finite}\}. \tag{4.1}$$

The following result of Kikkawa and Suzuki [2] is a generalization of Nadler [29].

Theorem 4.1. *Let (X, d) be a complete metric space and $P : X \to CB(X)$. Define a strictly decreasing function η from $[0, 1)$ onto $(1/2, 1]$ by*

$$\eta(r) = \frac{1}{1 + r}. \tag{4.2}$$

Assume that there exists $r \in [0, 1)$ such that

$$\eta(r)d(x, Px) \leq d(x, y) \quad \text{implies } H(Px, Py) \leq rd(x, y) \tag{KSMC}$$

for all $x, y \in X$. Then P has a fixed point, that is, there exists $z \in X$ such that $z \in Pz$.

Theorem 4.1 has further been generalized by Moţ and Petruşel [3], Dhompongsa and Yingtaweesittikul [4] and Singh and Mishra [9].

For $a, b, c, e, f \in [0,1)$, let β and γ be defined by

$$\beta = \frac{1-b-c}{1+a}, \qquad \gamma = \frac{1-c-e}{1+a}. \tag{4.3}$$

For a metric space X, we consider $P : Y \to CL(X)$ and $T : Y \to X$ satisfying

$$\gamma d(Tx, Px) \le d(Tx, Ty) \text{ implies } H(Px, Py) \le M(x, y) \tag{4.4}$$

for all $x,\ y \in Y$, where

$$M(x, y) = ad(Tx, Ty) + bd(Tx, Px) + cd(Ty, Py) + ed(Tx, Py) + fd(Ty, Px) \tag{4.5}$$

and $a + b + c + e + f < 1$.

We remark that $\beta, \gamma \in (1/2, 1]$. As regards the generality of condition (4.4), we offer the following remarks when $Y = X$ and T is the identity map on X.

Remarks 4.2. (i) The Kikkawa-Suzuki multivalued contraction (KSMC) is (4.4) with $a = r$ and $b = c = e = f = 0$.

(ii) Generalizing the (KSMC), Moţ and Petrusel [3] have studied (4.4) with $e = f = 0$ and $\gamma = \beta$.

(iii) Dhompongsa and Yingtaweesittikul [4] have discussed (4.4) when

$$M(x, y) = \gamma \cdot \max\{d(x, y), d(x, Px), d(y, Py)\} \tag{4.6}$$

with $\gamma = \theta(r)$ and some additional requirement.

(iv) Condition (4.4) includes a few important conditions for single-valued and multivalued maps due to Reich [16, 30], Hardy and Rogers [31], and Iseki [17] (see also condition (16) in Rhoades [32]).

By virtue of the symmetry in x and y in the expression $M(x, y)$, it is appropriate to consider (4.4) when $b = c$ and $e = f$ as follows:

$$\beta d(Tx, Px) \le d(Tx, Ty) \quad \text{implies } H(Px, Py) \le m(x, y) \tag{KSG}$$

for all $x, y \in Y$, where

$$m(x, y) = ad(Tx, Ty) + b[d(Tx, Px) + d(Ty, Py)] + c[d(Tx, Py) + d(Ty, Px)], \tag{4.7}$$

and $a + 2b + 2c < 1$.

In all that follows, we consider the nontrivial case $0 < a + 2b + 2c$.

The condition (KSG) will be called Kikkawa-Suzuki generalized hybrid contraction for the maps P and T. Following Itoh and Takahashi [15] (see also Singh and Mishra [33]), maps $P : X \to CL(X)$ and $T : X \to X$ are IT-commuting at $z \in X$ if $TPz \subseteq PTz$. We remark that IT-commuting maps are more general than commuting maps, weakly commuting maps and weakly compatible maps at a point. For details, one may refer to [33].

Theorem 4.3. *Let X be a metric space. Assume that the pair of maps $P : Y \to CL(X)$ and $T : Y \to X$ is Kikkawa-Suzuki generalized hybrid contraction such that $P(Y) \subseteq T(Y)$, and $P(Y)$ or $T(Y)$ is a complete subspace of X. Then P and T have a coincidence point, that is, there exists $z \in Y$ such that $Tz \in Pz$. Further, if $Y = X$, then P and T have a common fixed point provided that P and T are IT-commuting at z and Tz is a fixed point of T.*

Proof. Let $q = (a + 2b + 2c)^{-1/2}$. Pick $x_0 \in Y$. Following Singh and Kulshrestha [34] and Rhoades et al. [35], we construct two sequences $\{x_n\} \subseteq Y$ and $\{y_n = Tx_n\} \subseteq T(Y)$ in the following manner. Since $P(Y) \subseteq T(Y)$, we choose an element $x_1 \in Y$ such that $Tx_1 \in Px_0$. Analogously, choose $Tx_2 \in Px_1$ such that

$$d(Tx_1, Tx_2) \leq qH(Px_0, Px_1). \tag{4.8}$$

In general, we have sequences $\{x_n\}$ and $\{Tx_n\}$ such that $Tx_{n+1} \in Px_n$, $n = 0, 1, \ldots, q > 1$ and

$$d(Tx_{n+1}, Tx_{n+2}) \leq qH(Px_n, Px_{n+1}), \quad n = 0, 1, \ldots. \tag{4.9}$$

Since $\beta < 1$, we see that $\beta d(Tx_n, Px_n) \leq d(Tx_n, Tx_{n+1})$. Therefore by the assumption (KSG),

$$\begin{aligned} d(y_{n+1}, y_{n+2}) &\leq qH(Px_n, Px_{n+1}) \\ &\leq q\big[ad(y_n, y_{n+1}) + bd(y_n, Px_n) + bd(y_{n+1}, Px_{n+1}) \\ &\qquad + c\{d(y_n, Px_{n+1}) + d(y_{n+1}, Px_n)\}\big] \\ &\leq q\big[(a+b)d(y_n, y_{n+1}) + bd(y_{n+1}, y_{n+2}) + c\{d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})\}\big], \end{aligned} \tag{4.10}$$

yielding

$$d(y_{n+1}, y_{n+2}) \leq \lambda d(y_n, y_{n+1}), \tag{4.11}$$

where $\lambda = q(a+b+c)/(1 - q(b+c)) < 1$. So the sequence $\{y_n\}$ is Cauchy. If $T(Y)$ is complete, then it has a limit in $T(Y)$. If $P(Y)$ is complete, then the limit is still in $T(Y)$ as $P(Y) \subseteq T(Y)$. Call the limit w. Let $z \in T^{-1}w$. Then $z \in Y$ and $Tz = w$. Now as in [2], we show that

$$d(Tz, Px) \leq \frac{a+b+c}{\beta(1+a)} d(Tz, Tx) \tag{4.12}$$

for any $Tx \in T(Y) - \{Tz\}$. Since $y_n \to Tz$, there exists a positive integer n_0 such that

$$d(Tz, Tx_n) \leq \frac{1}{3} d(Tz, Tx) \quad \forall n \geq n_0. \tag{4.13}$$

Therefore for any $n \geq n_0$,

$$\begin{aligned}\beta d(Tx_n, Px_n) &\leq d(Tx_n, Tx_{n+1}) \\ &\leq d(Tx_n, Tz) + d(Tx_{n+1}, Tz) \\ &\leq \frac{2}{3} d(Tz, Tx) = d(Tz, Tx) - \frac{1}{3} d(Tz, Tx) \\ &\leq d(Tz, Tx) - d(Tz, Tx_n) \leq d(Tx_n, Tx).\end{aligned} \tag{4.14}$$

Hence by the assumption (KSG),

$$\begin{aligned}d(y_{n+1}, Px) &\leq H(Px_n, Px) \leq m(x_n, x) \\ &\leq ad(y_n, Tx) + b\left[d(y_n, y_{n+1}) + d(Tx, Px)\right] + c\left[d(y_n, Px) + d(Tx, y_{n+1})\right].\end{aligned} \tag{4.15}$$

Making $n \to \infty$, we have

$$d(Tz, Px) \leq \left(\frac{a+b+c}{1-b-c}\right) d(Tz, Tx). \tag{4.16}$$

This yields (4.12), $Tx \neq Tz$. Next we show that

$$H(Px, Pz) \leq m(x, z) \tag{4.17}$$

for any $x \in Y$. If $x = z$, then it holds trivially. So we take $x \neq z$ such that $Tx \neq Tz$. We can do so since, without any loss of generality, we take the map T nonconstant. By (4.12),

$$\begin{aligned}d(Tx, Px) &\leq d(Tx, Tz) + d(Tz, Px) \\ &\leq d(Tx, Tz) + \left(\frac{a+b+c}{1-b-c}\right) d(Tz, Tx).\end{aligned} \tag{4.18}$$

Hence $\beta d(Tx, Px) \leq d(Tx, Tz)$. This implies (4.17). Therefore

$$\begin{aligned}d(y_{n+1}, Pz) &\leq H(Px_n, Pz) \leq m(x_n, z) \\ &\leq ad(y_n, Tz) + b\left[d(y_n, y_{n+1}) + d(Tz, Pz)\right] + c\left[d(y_n, Pz) + d(Tz, y_{n+1})\right].\end{aligned} \tag{4.19}$$

Making $n \to \infty$, this yields $(1 - b - c)d(Tz, Pz) \leq 0$, and $Tz \in Pz$.

Further, if $Y = X$, $TTz = Tz$, and P and T are IT-commuting at z, then $Tz \in Pz$ implies that $TTz \in TPz \subseteq PTz$. This proves that Tz is a fixed point of P. □

We remark that the assumption that T has a fixed point in Theorem 4.3 is essential. Indeed, in general, a pair of continuous commuting maps on the space need not have a common fixed point (see, e.g., [14, 33]).

Corollary 4.4. *Let* X *be a complete metric space and* $P : X \to CL(X)$*. Assume there exist* $a, b, c \in [0,1)$ *such that*

$$\beta d(x,Px) \leq d(x,y) \quad \text{implies } H(Px,Py) \leq N(P;x,y) \tag{4.20}$$

for all $x, y \in X$*, where*

$$N(P;x,y) = ad(x,y) + b[d(x,Px) + d(y,Py)] + c[d(x,Py) + d(y,Px)] \tag{4.21}$$

and $a + 2b + 2c < 1$.

Proof. It comes from Theorem 4.1 when $Y = X$ and T is the identity map. □

The following two results are the extensions of Suzuki contraction theorem. Corollary 4.5 also generalizes the results of Kikkawa and Suzuki [2, Theorem 2], Jungck [18] and Dhompongsa and Yingtaweesittikul [4, Theorem 3.4(v)].

Corollary 4.5. *Let* $g, T : Y \to X$ *be such that* $g(Y) \subseteq T(Y)$ *and* $g(Y)$ *or* $T(Y)$ *is a complete subspace of* X*. Assume that there exist* $a, b, c \in [0,1)$ *such that*

$$\beta d(Tx,gx) \leq d(Tx,Ty) \tag{4.22}$$

implies

$$d(gx,gy) \leq ad(Tx,Ty) + b[d(Tx,gx) + d(Ty,gy)] + c[d(Tx,gy) + d(Ty,gx)] \tag{4.23}$$

for all $x, y \in X$*, where* $a + 2b + 2c < 1$*. Then* g *and* T *have a coincidence point* $z \in Y$*. Further, if* $Y = X$ *and* g*,* T *commute at* z*, then* g *and* T *have a unique common fixed point.*

Proof. Set $Px = \{gx\}$ for every $x \in Y$. Then it comes from Theorem 4.1 that there exists $z \in Y$ such that $gz = Tz$. Further, if $Y = X$ and g, T commute at z, then $ggz = gTz = Tgz$. Also, $\beta d(Tz,gz) = 0 \leq d(Tz,Tgz)$ and this implies

$$\begin{aligned} d(gz,ggz) &\leq ad(Tz,Tgz) + b[d(Tz,gz) + d(Tgz,ggz)] + c[d(Tz,ggz) + d(Tgz,gz)] \\ &= (a+2c)d(gz,ggz). \end{aligned} \tag{4.24}$$

This proves that gz is a common fixed point of g and T. The uniqueness of the common fixed point follows easily. □

Corollary 4.6. *Let* X *be a complete metric space and* $g : X \to X$*. Assume that there exist* $a, b, c \in [0,1)$ *such that* $\beta d(x,gx) \leq d(x,y)$ *implies* $d(gx,gy) \leq N(g;x,y)$ *for all* $x, y \in X$*, where* $a + 2b + 2c < 1$*. Then* g *has a unique fixed point.*

Proof. It follows from Corollary 4.5 if $Y = X$ and T is the identity map on X. □

Theorem 4.7. *Let X be a metric space, and let $P : Y \to BN(X)$ and $T : Y \to X$ be such that $P(Y) \subseteq T(Y)$ and $T(Y)$ is a complete subspace of X. Assume that there exist $a,\ b,\ c \in [0,1)$ such that*

$$\beta\rho(Tx, Px) \leq d(Tx, Ty) \tag{4.25}$$

implies

$$\rho(Px, Py) \leq ad(Tx, Ty) + b[\rho(Tx, Px) + \rho(Ty, Py)] + c[d(Tx, Py) + d(Ty, Px)] \tag{4.26}$$

for all $x,\ y \in Y$, where $a + 2b + 2c < 1$. Then the maps T and P have a coincidence.

Proof. It may be completed following Reich [30] and Ćirić [36] and using Corollary 4.5. However, for the sake of completeness, we give an outline of the same. Let $t = a+2b+2c$. For $p \in (0,1)$; define a single-valued map $g : Y \to X$ as follows. For each $x \in Y$, let gx be a point of Px such that $d(Tx, gx) \geq t^p\rho(Tx, Px)$. Notice that $f(Y) = \bigcup\{fx \in Px\} \subseteq P(Y) \subseteq T(Y)$. Since $gx \in Px, d(Tx, gx) \leq \rho(Tx, Px)$. So (4.25) gives $\beta d(Tx, gx) \leq \beta\rho(Tx, Px) \leq d(Tx, Ty)$, and this implies condition (4.26). Therefore

$$\begin{aligned} d(gx, gy) &\leq \rho(Px, Py) \\ &\leq t^{-p}\{at^p d(Tx, Ty) + bt^p[\rho(Tx, Px) + \rho(Ty, Py)] + ct^p[d(Tx, Py) + d(Ty, Px)]\} \\ &\leq t^{-p}\{ad(Tx, Ty) + b[d(Tx, gx) + d(Ty, gy)] + c[d(Tx, gy) + d(Ty, gx)]\}. \end{aligned} \tag{4.27}$$

So, taking $a' = at^{-p}$, $b' = bt^{-p}$, $c' = ct^{-p}$ and $\beta' = (1 - b' - c')/(1 + a')$, we see that $\beta' d(Tx, gx) \leq \beta d(Tx, gx) \leq d(Tx, Ty)$ implies

$$d(gx, gy) \leq a'd(Tx, Ty) + b'[d(Tx, gx) + d(Ty, gy)] + c'[d(Tx, gy) + d(Ty, gx)], \tag{4.28}$$

where $a' + 2b' + 2c' = at^{-p} + 2bt^{-p} + 2ct^{-p} = t^{1-p} < 1$. Hence, by virtue of Corollary 4.5, g and T have a coincidence at $z \in Y$. Evidently $Tz = gz$ implies $Tz \in Pz$. □

Theorem 4.8. *Let X be a complete metric space and $P : Y \to BN(X)$. Assume that there exist $a, b, c \in [0,1)$ such that $\beta\rho(x, Px) \leq d(x, y)$ implies*

$$\rho(Px, Py) \leq ad(x, y) + b[\rho(x, Px) + \rho(y, Py)] + c[d(x, Py) + d(y, Px)] \tag{4.29}$$

for all $x, y \in X$, where $a + 2b + 2c < 1$. Then P has a unique fixed point.

Proof. It may be completed, as above, using Corollary 4.6. □

5. Examples and Discussion

The following example shows that the Suzuki contractive condition (cf. (2.2) and (2.3)) for a pair of maps is indeed more useful than condition (2.1) for a map on a metric space. In all the examples of this section, spaces are endowed with the usual metric.

Example 5.1. Let $X = [0, 11/10]$ and let $S, T : X \to X$ be defined by,

$$Sx = \begin{cases} 0 & \text{if } 0 \le x \le \dfrac{1}{2}, \\ \dfrac{1}{2} & \text{if } \dfrac{1}{2} < x \le \dfrac{11}{10}, \end{cases}$$
$$Tx = \begin{cases} 0 & \text{if } x = 0, \\ \dfrac{1}{2} & \text{if } 0 < x \le \dfrac{1}{2}, \\ \dfrac{11}{10} & \text{if } \dfrac{1}{2} < x \le \dfrac{11}{10}. \end{cases} \tag{5.1}$$

Then assumption (2.1) of Theorem 2.1 is not satisfied for the map S (take, e.g., $x = 25/100$, $y = 51/100$). However, S and T satisfy all the assumptions of Theorem 2.2. Notice that $Sx = Sy$ whenever $Tx = Ty$ for any $x, y \in Y$. Moreover, $S(X) = \{0, 1/2\} \subset \{0, 1/2, 11/10\} = T(X)$. So, Theorem 2.2 guarantees the existence of a coincidence point, namely, 0 which is the unique common fixed point of S and T.

Sequence of Iterates

For maps S and T studied in Theorems 2.2 and 3.2, a sequence of iterates may be constructed following Jungck [18]. For any $x_0 \in Y$, choose an $x_1 \in Y$ such that $Tx_1 = Sx_0$. We can do this since $S(Y) \subseteq T(Y)$. Now choose $x_2 \in Y$ such that $Tx_2 = Sx_1$. Continuing this process, we choose $x_{n+1} \in Y$ such that $Tx_{n+1} = Sx_n$, $n = 0, 1, 2, \ldots$. For the sake of brevity and appropriate reference, the sequence $\{Tx_n\}$ will be called Jungck sequence of iterates or simply Jungck iterates. Notice that the sequence $\{Tx_n\}$ is the usual Picard sequence of iterates when T is the identity map on $Y = X$. In the case of Example 5.1, take $x_0 = 11/10$. Then $\{Tx_n\} = \{1/2, 0, 0, \ldots\}$ which converges to 0. However, in general, under the assumptions of Theorems 2.2 or 3.2, there may not exist a sequence $\{Tx_n\}$ which converges. The following examples illustrate this fact.

Example 5.2. Let $X = [-11, -10] \cup \{0\} \cup [10, 11]$,

$$Sx = \begin{cases} \dfrac{11x + 100}{x + 9}, & \text{if } -11 \le x < -10, \\ 0, & \text{if } x = -10, 0, 10, \\ -\dfrac{11x - 100}{x - 9}, & \text{if } 10 < x \le 11, \end{cases} \tag{5.2}$$

and $Tx = x$, $x \in X$.

Suzuki [7] has shown that S satisfies the assumption of Theorem 2.2 with $Y = X$ and T the identity map on X. It is also shown in [7] that the Picard sequence of iterates of the map S does not converge when the initial choice x_0 falls in $X - \{-10, 0, 10\}$, although S satisfies all the hypotheses of Theorem 2.1. Thus, under the hypotheses of Theorem 2.2, Jungck iterates for (S, T) need not converge.

Example 5.3. Let $X = [3, 7]$, and let $S,\ T : X \to X$ be such that

$$Sx = \begin{cases} 3 & \text{if } x \neq 6, \\ 5 & \text{if } x = 6, \end{cases}$$
$$Tx = \begin{cases} 3 & \text{if } x = 3, \\ 5 & \text{if } x \neq 3,\ x \neq 6, \\ 7 & \text{if } x = 6. \end{cases} \tag{5.3}$$

Evidently, S is not nonexpansive. Further, S is also not Suzuki nonexpansive. Indeed, for $x = 6,\ y = 5$,

$$\frac{1}{2}|x - Sx| = \frac{1}{2}|6 - 5| = \frac{1}{2} \leq 1 = |x - y|, \tag{5.4}$$

while

$$|Sx - Sy| = |5 - 3| = 2 \nleq 1 = |x - y|. \tag{5.5}$$

Notice that $S(X) \subseteq T(X)$, and the assumption (3.2), namely,

$$\frac{1}{2}|Tx - Sx| \leq |Tx - Ty| \Longrightarrow |Sx - Sy| \leq |Tx - Ty|, \tag{5.6}$$

is satisfied for all $x, y \in X$. Also $Sx = Sy$ whenever $Tx = Ty$ for any $x, y \in X$.

The sequence $\{Tx_n\}$ constructed before Example 5.2 may be used to approximate the coincidence values of the maps S and T under the hypotheses of Theorem 3.2. Note that if z is such that $Sz = Tz = w$, then w is the coincidence value of S and T at their coincidence point. For example, in the case of Example 5.3, for any $x_0 \in X$, the sequence $\{Tx_n\}$ converges to 3. The following example reveals some strange pattern regarding the convergence of Jungck iterates $\{Tx_n\}$ under the assumption (3.2).

Example 5.4. Let $X = [3,7]$ and $S,\ T : X \to X$ be defined by

$$Sx = \begin{cases} 3, & \text{if } x \neq 6 \\ 5, & \text{if } x = 6, \end{cases}$$
$$Tx = \begin{cases} 7, & \text{if } x = 3 \\ 3, & \text{if } x \neq 3, \neq 6 \\ 5 & \text{if } x = 6. \end{cases} \tag{5.7}$$

Notice the following.

(1) S is not nonexpansive.

(2) S is not Suzuki nonexpansive (take $x = 5, y = 6$).

(3) $S(X) \subseteq T(X)$.

(4) S and T satisfy assumption (3.2) with $Y = E = X$.

(5) $Sx = Sy$ whenever $Tx = Ty$ for any $x, y \in X$.

(6) For any $z \neq 3, \neq 6, Sz = Tz = 3$. Note that coincidence point z is different from the coincidence value $w = 3$.

(7) As regards the Jungck sequence of iterates $\{Tx_n\}$, we examine some cases below.

(i) For $x_0 = 3$, consider $x_n = 3 + n/(n+1),\ n = 1, 2, \ldots$. Evidently, $Tx_n \to 3$.

(ii) For $x_0 = 6$ and $x_n = 6,\ n = 1,\ 2, \ldots, Tx_n \to 5$ and $S6 = T6 = 5$.

Here it is very interesting to note that S and T are commuting at $x = 6$, which is not a common fixed point of S and T.

The following example illustrates the validity and superiority of the Kikkawa-Suzuki generalized contraction for a pair of maps.

Example 5.5. Let $X = [0, \infty)$ and for every $x \in X$, define $Px = [0, 3x]$ and $Tx = 5x$. Then P does not satisfy the assumption (KSMC) of Theorem 4.1. Indeed, for any $r \in [0,1)$ and $x = 3$ and $y = 1, \eta(r)d(3, P3) = 0 \leq d(3,1)$ and $H(P3, P1) = 6 > d(3,1)$. Further, as $d(1, P1) = d(2, P2) = 0$, the map P does not satisfy either of the conditions studied by Moţ and Petruşel [3] and Dhompongsa and Yingtaweesittikul [4] (see Remarks 4.2(ii)–(iii)). However, for every $x, y \in X, H(Px, Py) \leq ad(Tx, Ty)$, where $a \in [3/5, 1), b = c = 0$. So, P and T satisfy the assumption (KSG) of Theorem 4.3 with $Y = X$.

The following example shows the usefulness of domain Y different from X in Theorem 4.3.

Example 5.6. Let R be the set of real numbers, $Y = C$ (the set of complex numbers) and $X = [0, \infty)$. For $x, y \in R$ and $z = (x, y) \in Y$, define $Pz = [0, x^2 + y^2]$ and $Tz = 2(x^2 + y^2)$. Then $P(Y) \subseteq T(Y)$ and P satisfies the assumption (KSG) with $a = (1/2), b = c = 0$. Evidently Theorem 4.3 applies and $Tz \in Pz$ for $z = (0,0)$.

In view of the foregoing discussion regarding the convergence of Jungck iterates of Suzuki class of nonexpansive pair of maps, we present the following.

Conjecture 5.7. *Let C be a nonempty subset of a Banach space E and $S,T : C \to C$ satisfying assumption* (3.2). *Let $S(C) \subseteq T(C)$ and let $T(C)$ be a compact convex subset of E. For $x_0 \in C$, define a sequence $\{Tx_n\}$ such that*

$$Tx_{n+1} = \lambda Sx_n + (1-\lambda)Tx_n, \quad n = 0,1,2,\ldots, \tag{5.8}$$

where $\lambda \in [1/2,1)$. Then the sequence $\{Tx_n\}$ converges to a coincidence point of S and T.

We remark that its particular case with $Y = X$ and T the identity map is Theorem 2 of Suzuki [6].

6. Applications

Throughout this section, we assume that U and V are Banach spaces, $W \subseteq U$ and $D \subseteq V$. Let $\mathbb{R}$ denote the field of reals, $\tau : W \times D \to W$, $f, g : W \times D \to \mathbb{R}$ and $G, F : W \times D \times \mathbb{R} \to \mathbb{R}$. Considering W and D as the state and decision spaces respectively, the problem of dynamic programming reduces to the problem of solving the functional equations:

$$p := \sup_{y \in D}\{f(x,y) + G(x,y,p(\tau(x,y)))\}, \quad x \in W, \tag{6.1}$$

$$q := \sup_{y \in D}\{g(x,y) + F(x,y,q(\tau(x,y)))\}, \quad x \in W. \tag{6.2}$$

In the multistage process, some functional equations arise in a natural way (cf., Bellman [19] and Bellman and Lee [20]) (see also [37–39]). In this section, we study the existence of a common solution of the functional equations (6.1) and (6.2) arising in dynamic programming.

Let $B(W)$ denote the set of all bounded real-valued functions on W. For an arbitrary $h \in B(W)$, define $\|h\| = \sup_{x \in W}|h(x)|$. Then $(B(W), \|\cdot\|)$ is a Banach space. Suppose that the following conditions hold.

(DP-1) G, F, f and g are bounded.

(DP-2a) Assume that for every $(x,y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,

$$\frac{1}{2}|Kh(t) - Jh(t)| \le |Jh(t) - Jk(t)| \tag{6.3}$$

implies

$$\bigl|G(x,y,h(t)) - G(x,y,k(t))\bigr| \le |Jh(t) - Jk(t)|, \tag{6.4}$$

where K and J are defined as follows:

$$Kh(x) = \sup_{y\in D}\{f(x,y) + G(x,y,h(\tau(x,y)))\}, \quad x \in W,\ h \in B(W), \tag{$*$}$$

$$Jh(x) = \sup_{y\in D}\{g(x,y) + F(x,y,h(\tau(x,y)))\}, \quad x \in W,\ h \in B(W). \tag{6.5}$$

(DP-2b) Let β be defined as in Section 4. Assume that there exist $a,\ b,\ c \in [0,1)$ such that for every $(x,y) \in W \times D, h, k \in B(W)$ and $t \in W$,

$$\beta|Kh(t) - Jh(t)| \le |Jh(t) - Jk(t)| \tag{6.6}$$

implies

$$\begin{aligned}\left|G(x,y,h(t)) - G(x,y,k(t))\right| \le a|Jh(t) - Jk(t)| + b[|Jh(t) - Kh(t)| + |Jk(t) - Kk(t)|]\\ + c[|Jh(t) - Kk(t)| + |Jk(t) - Kh(t)|],\end{aligned} \tag{6.7}$$

where $a + 2b + 2c < 1$.

(DP-2c) $Jh_1 = Jh_2$ implies $Kh_1 = Kh_2$.

(DP-3) For any $h \in B(W)$, there exists $k \in B(W)$ such that

$$Kh(x) = Jk(x), \quad x \in W. \tag{6.8}$$

(DP-4) There exists $h \in B(W)$ such that

$$Jh(x) = Kh(x) \quad \text{implies } JKh(x) = KJh(x). \tag{6.9}$$

Theorem 6.1. *Assume that conditions (DP-1), (DP-2a), (DP-2c) and (DP-3) are satisfied. If $J(B(W))$ is a compact convex subspace of $B(W)$, then the functional equations* (6.1) *and* (6.2) *have a conicidence bounded solution.*

Proof. Let d be the metric induced by the supremum norm on $B(W)$. Then $B(W)$ is a complete metric space. By (DP-1), J and K are self-maps of $B(W)$. Condition (DP-3) implies that $K(B(W)) \subseteq J(B(W))$.

Let λ be an arbitrary positive number and $h_1, h_2 \in B(W)$. Let $x \in W$ be arbitrary and choose $y_1, y_2 \in D$ such that

$$Kh_j < f(x,y_j) + G(x,y_j,h_j(x_j)) + \lambda, \tag{6.10}$$

where $x_j = \tau(x,y_j),\ j = 1,2$.

Further,

$$Kh_1(x) \geq f(x, y_2) + G(x, y_2, h_1(x_2)), \tag{6.11}$$

$$Kh_2(x) \geq f(x, y_1) + G(x, y_1, h_2(x_1)). \tag{6.12}$$

Therefore, the first inequality in (DP-2a) becomes

$$\frac{1}{2}|Kh_1(x) - Jh_1(x)| \leq |Jh_1(x) - Jh_2(x)|, \tag{6.13}$$

and this together with (6.10) and (6.12) implies

$$\begin{aligned} Kh_1(x) - Kh_2(x) &< G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1)) + \lambda \\ &\leq \left|G(x, y_1, h_1(x_1)) - G(x, y_1, h_2(x_1))\right| + \lambda, \end{aligned} \tag{6.14}$$

that is

$$Kh_1(x) - Kh_2(x) \leq |Jh_1(x) - Jh_2(x)| + \lambda. \tag{6.15}$$

Similarly, (6.10), (6.11) and (6.13) imply

$$Kh_2(x) - Kh_1(x) \leq |Jh_1(x) - Jh_2(x)| + \lambda. \tag{6.16}$$

So, from (6.15) and (6.16), we have

$$|Kh_1(x) - Kh_2(x)| \leq |Jh_1(x) - Jh_2(x)| + \lambda. \tag{6.17}$$

Since $x \in W$ and $\lambda > 0$ is arbitrary, we find from (6.13) that

$$\frac{1}{2}d(Kh_1, Jh_1) \leq d(Jh_1, Jh_2) \tag{6.18}$$

implies

$$d(Kh_1, Kh_2) \leq d(Jh_1, Jh_2). \tag{6.19}$$

Hence taking also the notice of (DP-2c), we see that Theorem 3.2(i) applies, wherein K and J correspond, respectively, to the maps S and T So, K and J have a coincidence point h^*, that is, $h^*(x)$ is a bounded coincidence solution of the functional equations (6.1) and (6.2). □

Corollary 6.2. *Suppose that the following conditions hold:*

(i) *G and f are bounded.*

(ii) *for every $(x, y) \in W \times D$, $h, k \in B(W)$ and $t \in W$,*

$$\frac{1}{2}|h(t) - Kh(t)| \leq |h(t) - k(t)| \tag{6.20}$$

implies

$$\left|G(x, y, h(t)) - G(x, y, k(t))\right| \leq |h(t) - k(t)|, \tag{6.21}$$

where K is defined by $(*)$*. Then the functional equation* (6.1) *has a bounded solution in W provided that $B(W)$ is compact.*

Proof. It comes from Theorem 6.1 when $g = 0, \tau(x, y) = x$ and $F(x, y, t) = t$ as the assumptions (DP-2c) and (DP-3) become redundant in this context. □

We remark that Theorem 6.1 does not guarantee the existence of a common solution even if we add to it the commutativity requirement (DP-4). Further, a solution guaranteed by Corollary 6.2 need not be unique. These observations add importance to the following formulation regarding the existence of a unique common bounded solution.

Theorem 6.3. *Assume that conditions (DP-1), (DP-2b), (DP-3), and (DP-4) are satisfied. If $K(B(W))$ or $J(B(W))$ is a closed convex subspace of $B(W)$, then the functional equations* (6.1) *and* (6.2) *have a unique common bounded solution.*

Proof. Recall that $(B(W), d)$ is a complete metric space. The self-maps J and K of $B(W)$ are commuting at their coincidence points by (DP-4). Proceeding as in the proof of Theorem 6.1, we see that K and J correspond, respectively, to the maps g and T of Corollary 4.5. Hence K and J have a unique bounded common solution $h^*(x)$ of the functional equations (6.1) and (6.2). □

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