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Research Article

A Common End Point Theorem for Set-Valued Generalized (ψ, φ)-Weak Contraction

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We introduce the class of generalized (ψ, φ)-weak contractive set-valued mappings on a metric space. We establish that such mappings have a unique common end point under certain weak conditions. The theorem obtained generalizes several recent results on single-valued as well as certain set-valued mappings.

1. Introduction and Preliminaries

Alber and Guerre-Delabriere [1] defined weakly contractive maps on a Hilbert space and established a fixed point theorem for such a map. Afterwards, Rhoades [2], using the notion of weakly contractive maps, obtained a fixed point theorem in a complete metric space. Dutta and Choudhury [3] generalized the weak contractive condition and proved a fixed point theorem for a selfmap, which in turn generalizes theorem 1 in [2] and the corresponding result in [1]. The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. Beg and Abbas [4] obtained a common fixed point theorem extending weak contractive condition for two maps. In this direction, Zhang and Song [5] introduced the concept of a generalized φ-weak contraction condition and obtained a common fixed point for two maps, and Đorić [6] proved a common fixed point theorem for generalized (ψ, φ)-weak contractions. On the other hand, there are many theorems in the existing literature which deal with fixed point of multivalued mappings. In some cases, multivalued mapping T defined on a nonempty set X assumes a compact value Tx for each x in X. There are the situations when, for each x in X, Tx is assumed to be closed and bounded subset of X. To prove existence of fixed point of such mappings, it is essential for mappings to satisfy certain contractive conditions which involve Hausdorff metric.

The aim of this paper is to obtain the common end point, a special case of fixed point, of two multivlaued mappings without appeal to continuity of any map involved therein. It is also noted that our results do not require any commutativity condition to prove an existence of common end point of two mappings. These results extend, unify, and improve the earlier comparable results of a number of authors.

Let (X, d) be a metric space, and let $B(X)$ be the class of all nonempty bounded subsets of X. We define the functions $\delta : B(X) \times B(X) \to R^+$ and $D : B(X) \times B(X) \to R^+$ as follows:

$$\begin{aligned} \delta(A,B) &= \sup\{d(a,b) \ : a \in A,\ b \in B\}, \\ D(A,B) &= \inf\{d(a,b) \ : a \in A,\ b \in B\}, \end{aligned} \tag{1.1}$$

where R^+ denotes the set of all positive real numbers. For $\delta(\{a\}, B)$ and $\delta(\{a\}, \{b\})$, we write $\delta(a, B)$ and $d(a, b)$, respectively. Clearly, $\delta(A, B) = \delta(B, A)$. We appeal to the fact that $\delta(A, B) = 0$ if and only if $A = B = \{x\}$ for $A, B \in B(X)$ and

$$0 \leq \delta(A,B) \leq \delta(A,B) + \delta(A,B), \tag{1.2}$$

for $A, B, C \in B(X)$. A point $x \in X$ is called a fixed point of T if $x \in Tx$. If there exists a point $x \in X$ such that $Tx = \{x\}$, then x is termed as an end point of the mapping T.

2. Main Results

In this section, we established an end point theorem which is a generalization of fixed point theorem for generalized (ψ, φ)-weak contractions. The idea is in line with Theorem 2.1 in [6] and theorem 1 in [5].

Definition 2.1. Two set-valued mappings $T, S : X \to B(X)$ are said to satisfy the property of *generalized (ψ, ϕ)-weak contraction* if the inequality

$$\psi(\delta(Sx, Ty)) \leq \psi(M(x,y)) - \varphi(M(x,y)), \tag{2.1}$$

where

$$M(x,y) = \max\left\{ d(x,y), \delta(x,Sx), \delta(y,Ty), \frac{1}{2}[D(x,Ty) + D(y,Sx)] \right\} \tag{2.2}$$

holds for all $x, y \in X$ and for given functions $\psi, \varphi : R^+ \to R^+$.

Theorem 2.2. *Let (X, d) be a complete metric space, and let $T, S : X \to B(X)$ be two set-valued mappings that satisfy the property of generalized (ψ, ϕ)-weak contraction, where*

(a) ψ *is a continuous monotone nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$,

(b) φ *is a lower semicontinuous function with* $\varphi(t) = 0$ *if and only if* $t = 0$

then there exists the unique point $u \in X$ *such that* $\{u\} = Tu = Su$.

Proof. We construct the convergent sequence $\{x_n\}$ in X and prove that the limit point of that sequence is a unique common fixed point for T and S. For a given $x_0 \in X$ and nonnegative integer n let

$$x_{2n+1} \in Sx_{2n} = A_{2n}, \qquad x_{2n+2} \in Tx_{2n+1} = A_{2n+1}, \tag{2.3}$$

and let

$$a_n = \delta(A_n, A_{n+1}), \qquad c_n = d(x_n, x_{n+1}). \tag{2.4}$$

The sequences a_n *and* c_n *are convergent*. Suppose that n is an odd number. Substituting $x = x_{n+1}$ and $y = x_n$ in (2.1) and using properties of functions ψ and φ, we obtain

$$\begin{aligned} \psi(\delta(A_{n+1}, A_n)) &= \psi\delta(Sx_{n+1}, Tx_n) \\ &\leq \psi(M(x_{n+1}, x_n)) - \varphi(M(x_{n+1}, x_n)) \\ &\leq \psi(M(x_{n+1}, x_n)), \end{aligned} \tag{2.5}$$

which implies that

$$\delta(A_{n+1}, A_n) \leq M(x_{n+1}, x_n). \tag{2.6}$$

Now from (2.2) and from triangle inequality for δ, we have

$$\begin{aligned} &M(x_{n+1}, x_n) \\ &= \max\left\{ d(x_{n+1}, x_n), \delta(x_{n+1}, S_{n+1}), \delta(x_n, T_n), \frac{1}{2}[D(x_{n+1}, T_n) + D(x_n, S_{n+1})] \right\} \\ &\leq \max\left\{ \delta(A_n, A_{n-1}), \delta(A_n, A_{n+1}), \delta(A_{n-1}, A_n), \frac{1}{2}[D(x_{n+1}, A_n) + \delta(A_{n-1}, A_{n+1})] \right\} \\ &= \max\left\{ \delta(A_n, A_{n-1}), \delta(A_n, A_{n+1}), \frac{1}{2}\delta(A_{n-1}, A_{n+1}) \right\} \\ &\leq \max\left\{ \delta(A_n, A_{n-1}), \delta(A_n, A_{n+1}), \frac{1}{2}[\delta(A_{n-1}, A_n) + \delta(A_n, A_{n+1})] \right\} \\ &= \max\{\delta(A_{n-1}, A_n), \delta(A_n, A_{n+1})\}. \end{aligned} \tag{2.7}$$

If $\delta(A_n, A_{n+1}) > \delta(A_{n-1}, A_n)$, then

$$M(x_n, x_{n+1}) \leq \delta(A_{n+1}, A_n). \tag{2.8}$$

From (2.6) and (2.8) it follows that

$$M(x_n, x_{n+1}) = \delta(A_{n+1}, A_n) > \delta(A_{n-1}, A_n) \geq 0. \tag{2.9}$$

It furthermore implies that

$$\begin{aligned}\psi(\delta(A_n, A_{n+1})) &\leq \psi(M(x_n, x_{n+1})) - \varphi(M(x_n, x_{n+1})) \\ &< \psi(M(x_{n+1}, x_n)) \\ &= \psi(\delta(A_n, A_{n+1}))\end{aligned} \tag{2.10}$$

which is a contradiction. So, we have

$$\delta(A_n, A_{n+1}) \leq M(x_n, x_{n+1}) \leq \delta(A_{n-1}, A_n). \tag{2.11}$$

Similarly, we can obtain inequalities (2.11) also in the case when n is an even number. Therefore, the sequence $\{a_n\}$ defined in (2.4) is monotone nonincreasing and bounded. Let $a_n \to a$ when $n \to \infty$. From (2.11), we have

$$\lim_{n\to\infty} \delta(A_n, A_{n+1}) = \lim_{n\to\infty} M(x_n, x_{n+1}) = a \geq 0. \tag{2.12}$$

Letting $n \to \infty$ in inequality

$$\psi(\delta(A_{2n}, A_{2n+1})) \leq \psi(M(x_{2n}, x_{2n+1})) - \varphi(M(x_{2n}, x_{2n+1})), \tag{2.13}$$

we obtain

$$\psi(a) \leq \psi(a) - \varphi(a), \tag{2.14}$$

which is a contradiction unless $a = 0$. Hence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \delta(A_n, A_{n+1}) = 0. \tag{2.15}$$

From (2.15) and (2.3), it follows that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.16}$$

The sequence $\{x_n\}$ *is a Cauchy sequence*. First, we prove that for each $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that

$$m, n \geq n_0 \Rightarrow \delta(A_{2m}, A_{2n}) < \varepsilon. \tag{2.17}$$

Suppose opposite that (2.17) does not hold then there exists $\varepsilon > 0$ for which we can find nonnegative integer sequences $\{m(k)\}$ and $\{n(k)\}$, such that $n(k)$ is the smallest element of the sequence $\{n(k)\}$ for which

$$n(k) > m(k) > k, \quad \delta\left(A_{2m(k)}, A_{2n(k)}\right) \geq \varepsilon. \tag{2.18}$$

This means that

$$\delta\left(A_{2m(k)}, A_{2n(k)-2}\right) < \varepsilon. \tag{2.19}$$

From (2.19) and triangle inequality for δ, we have

$$\begin{aligned} \varepsilon &\leq \delta\left(A_{2m(k)}, A_{2n(k)}\right) \\ &\leq \delta\left(A_{2m(k)}, A_{2n(k)-2}\right) + \delta\left(A_{2n(k)-2}, A_{2n(k)-1}\right) + \delta\left(A_{2n(k)-1}, A_{2n(k)}\right) \\ &< \varepsilon + \delta\left(A_{2n(k)-2}, A_{2n(k)-1}\right) + \delta\left(A_{2n(k)-1}, A_{2n(k)}\right). \end{aligned} \tag{2.20}$$

Letting $k \to \infty$ and using (2.15), we can conclude that

$$\lim_{k \to \infty} \delta\left(A_{2m(k)}, A_{2n(k)}\right) = \varepsilon. \tag{2.21}$$

Moreover, from

$$\begin{aligned} \left|\delta\left(A_{2m(k)}, A_{2n(k)+1}\right) - \delta\left(A_{2m(k)}, A_{2n(k)}\right)\right| &\leq \delta\left(A_{2n(k)}, A_{2n(k)+1}\right), \\ \left|\delta\left(A_{2m(k)-1}, A_{2n(k)}\right) - \delta\left(A_{2m(k)}, A_{2n(k)}\right)\right| &\leq \delta\left(A_{2m(k)}, A_{2m(k)-1}\right), \end{aligned} \tag{2.22}$$

using (2.15) and (2.21), we get

$$\lim_{k \to \infty} \delta\left(A_{2m(k)-1}, A_{2n(k)}\right) = \lim_{k \to \infty} \delta\left(A_{2m(k)}, A_{2n(k)+1}\right) = \varepsilon, \tag{2.23}$$

and from

$$\left|\delta\left(A_{2m(k)-1}, A_{2n(k)+1}\right) - \delta\left(A_{2m(k)-1}, A_{2n(k)}\right)\right| \leq \delta\left(A_{2n(k)}, A_{2n(k)+1}\right), \tag{2.24}$$

using (2.15) and (2.23), we get

$$\lim_{k \to \infty} \delta\left(A_{2m(k)-1}, A_{2n(k)+1}\right) = \varepsilon. \tag{2.25}$$

Also, from the definition of M (2.2) and from (2.15), (2.23), and (2.25), we have

$$\lim_{k\to\infty} M(x_{2m(k)}, x_{2n(k)+1}) = \varepsilon. \tag{2.26}$$

Putting $x = x_{2m(k)}$, $y = x_{2n(k)+1}$ in (2.1), we have

$$\begin{aligned}\psi(\delta(A_{2m(k)}, A_{2n(k)+1})) &= \psi(\delta(Sx_{2m(k)}, Tx_{2n(k)+1})) \\ &\leq \psi(M(x_{2m(k)}, x_{2n(k)+1})) - \varphi(M(x_{2m(k)}, x_{2n(k)+1})).\end{aligned} \tag{2.27}$$

Letting $k \to \infty$ and using (2.23), (2.26), we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon), \tag{2.28}$$

which is a contradiction with $\varepsilon > 0$.

Therefore, conclusion (2.17) is true. From the construction of the sequence $\{x_n\}$, it follows that the same conclusion holds for $\{x_n\}$. Thus, for each $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that

$$m, n \geq n_0 \Rightarrow d(x_{2m}, x_{2n}) < \varepsilon. \tag{2.29}$$

From (2.4) and (2.29), we conclude that $\{x_n\}$ is a Cauchy sequence.

In complete metric space X, there exists u such that $x_n \to u$ as $n \to \infty$.

The point u is end point of S. As the limit point u is independent of the choice of $x_n \in A_n$, we also get

$$\lim_{n\to\infty} \delta(Sx_{2n}, u) = \lim_{n\to\infty} \delta(Tx_{2n+1}, u) = 0. \tag{2.30}$$

From

$$\begin{aligned}M(u, x_{2n+1}) = \max\Big\{ & d(u, x_{2n+1}), \delta(u, Su), \delta(x_{2n+1}, Tx_{2n+1}), \\ & \frac{1}{2}[D(u, Tx_{2n+1}) + D(x_{2n+1}, Su)]\Big\},\end{aligned} \tag{2.31}$$

we have $M(u, x_{2n+1}) \to \delta(u, Su)$ as $n \to \infty$. Since

$$\psi(\delta(Su, Tx_{2n+1})) \leq \psi(M(u, x_{2n+1})) - \varphi(M(u, x_{2n+1})), \tag{2.32}$$

letting $n \to \infty$ and using (2.30), we obtain

$$\psi(\delta(Su,u)) \leq \psi(\delta(u,Su)) - \varphi(\delta(u,Su)), \tag{2.33}$$

which implies $\varphi(\delta(u,Su)) = 0$. Hence, $\delta(u,Su) = 0$ or $Su = \{u\}$.

The point u is also end point for T. It is easy to see that $M(u,u) = \delta(u,Tu)$. Using that u is fixed point for S, we have

$$\begin{aligned}\psi(\delta(u,Tu)) &= \psi(\delta(Su,Tu)) \\ &\leq \psi(M(u,u)) - \varphi(M(u,u)) \\ &= \psi(\delta(u,Tu)) - \varphi(\delta(u,Tu)),\end{aligned} \tag{2.34}$$

and using an argument similar to the above, we conclude that $\delta(u,Tu) = 0$ or $\{u\} = Tu$.

The point u is a unique end point for S and T. If there exists another fixed point $v \in X$, then $M(u,v) = d(u,v)$ and from

$$\begin{aligned}\psi(d(u,v)) &= \psi(\delta(Su,Tv)) \\ &\leq \psi(M(u,v)) - \varphi(M(u,v)) \\ &= \psi(d(u,v)) - \varphi(d(u,v)),\end{aligned} \tag{2.35}$$

we conclude that $u = v$.

The proof is completed. □

The Theorem 2.2 established that set-valued mappings S and T under weak condition (2.1) have the unique common end point u. Now, we give an example to support our result.

Example 2.3. Consider $X = \{1,2,3,4,5\}$ as a subspace of real line with usual metric, $d(x,y) = |y-x|$. Let $S,T : X \to B(X)$ be defined as

$$S(x) = \begin{cases} \{4,5\} & \text{for } x \in \{1,2\} \\ \{4\} & \text{for } x \in \{3,4\}, \\ \{3,4\} & \text{for } x = 5 \end{cases} \qquad T(x) = \begin{cases} \{3,4\} & \text{for } x \in \{1,2\} \\ \{4\} & \text{for } x \in \{3,4\}. \\ \{3\} & \text{for } x = 5 \end{cases} \tag{2.36}$$

and take $\psi, \varphi : [0,\infty) \to [0,\infty)$ as $\psi(t) = 2t$ and $\varphi(t) = t/2$.

From Tables 1 and 2 , it is easy to verify that mappings S and T satisfy condition (2.1).

Therefore, S and T satisfy the property of generalized (ψ,ϕ) – weak contraction. Note that S and T have unique common end point. $S4 = T4 = \{4\}$. Also, note that for $\psi(t) = t$ condition (2.1), which became analog to condition (2.1) in [5], does not hold. For example, $\delta(S2,T1) = 2$ while $M(2,1) - \phi(M(2,1)) = 3/2$.

Table 1

$\delta(Sx, Ty)$	1	2	3	4	5
1	2	2	1	1	2
2	2	2	1	1	2
3	1	1	0	0	1
4	1	1	0	0	1
5	1	1	1	1	1

Table 2

$M(x, y)$	1	2	3	4	5
1	4	4	4	4	4
2	3	3	3	3	3
3	3	2	1	1	2
4	3	2	1	0	2
5	4	3	2	2	2

Remark 2.4. The Theorem 2.2 generalizes recent results on single-valued weak contractions given in [3, 5, 6]. The example above shows that function ψ in (2.1) gives an improvement over condition (2.1) in [5].

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