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Research Article

Convergence of Inexact Iterative Schemes for Nonexpansive Set-Valued Mappings

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Taking into account possibly inexact data, we study iterative schemes for approximating fixed points and attractors of contractive and nonexpansive set-valued mappings, respectively. More precisely, we are concerned with the existence of convergent trajectories of nonstationary dynamical systems induced by approximations of a given set-valued mapping.

1. Introduction

The study of iterative schemes for various classes of nonexpansive mappings is a central topic in Nonlinear Functional Analysis. It began with the classical Banach theorem [1] on the existence of a unique fixed point for a strict contraction. This celebrated result also yields convergence of iterates to the unique fixed point. Since Banach's seminal result, many developments have taken place in this area. We mention, in particular, existence and approximation results regarding fixed points of those nonexpansive mappings which are not necessarily strictly contractive [2, 3]. Such results were obtained for general nonexpansive mappings in special Banach spaces, while for self-mappings of general complete metric spaces most of the results were established for several classes of contractive mappings [4]. More recently, interesting developments have occurred for nonexpansive set-valued mappings, where the situation is more difficult and less understood. See, for instance, [5–8] and the references cited therein. As we have already mentioned, one of the methods for proving the classical Banach result is to show the convergence of Picard iterations, which holds for any initial point. In the case of set-valued mappings, not all the trajectories of the dynamical system induced by the given mapping converge. Therefore, convergent trajectories have to be constructed in a special way. For example, in the setting of [9], if at the moment $t = 0, 1, \ldots$ we reach a point x_t, then the next iterate x_{t+1} is an element of $T(x_t)$, where

T is the given mapping, which approximates the best approximation of x_t in $T(x_t)$. Since T is assumed to act on a general complete metric space, we cannot, in general, choose x_{t+1} to be the best approximation of x_t by elements of $T(x_t)$. Instead, we choose x_{t+1} so that it provides an approximation up to a positive number ϵ_t, such that the sequence $\{\epsilon_t\}_{t=0}^{\infty}$ is summable. This method allowed Nadler [9] to obtain the existence of a fixed point of a strictly contractive set-valued mapping and the authors of [10] to obtain more general results.

In view of the above discussion, it is obviously important to study convergence properties of the iterates of (set-valued) nonexpansive mappings in the presence of errors and possibly inaccurate data. The present paper is a contribution in this direction. More precisely, we are concerned with the existence of convergent trajectories of nonstationary dynamical systems induced by approximations of a given set-valued mapping. In the second section of the paper, we consider an iterative scheme for approximating fixed points of closed-valued strict contractions in metric spaces and prove our first convergence theorem (see Theorem 2.1 below). Our second convergence theorem (Theorem 3.1) is established in the third section of our paper. We show there that if for any initial point, there exists a trajectory of the dynamical system induced by a nonexpansive set-valued mapping T, which converges to a given invariant set F, then a convergent trajectory also exists for a nonstationary dynamical system induced by approximations of T.

2. Convergence to a Fixed Point of a Contractive Mapping

In this section we consider iterative schemes for approximating fixed points of closed-valued strict contractions in metric spaces.

We begin with a few notations.

Throughout this paper, (X, ρ) is a complete metric space.

For $x \in X$ and a nonempty subset A of X, set

$$\rho(x, A) = \inf\{\rho(x, y) : y \in A\}. \tag{2.1}$$

For each pair of nonempty $A, B \subset X$, put

$$H(A, B) = \max\left\{\sup_{x\in A} \rho(x, B), \sup_{y\in B} \rho(y, A)\right\}. \tag{2.2}$$

Let $T : X \to 2^X \setminus \{\emptyset\}$ be such that $T(x)$ is a closed subset of X for each $x \in X$ and

$$H(T(x), T(y)) \le c\rho(x, y), \quad \forall x, y \in X, \tag{2.3}$$

where $c \in [0, 1)$ is a constant.

Theorem 2.1. *Let $\{\epsilon_i\}_{i=0}^{\infty} \subset (0, \infty)$ and $\{\delta_i\}_{i=0}^{\infty} \subset (0, \infty)$ satisfy*

$$\sum_{i=0}^{\infty} \epsilon_i < \infty, \qquad \sum_{i=0}^{\infty} \delta_i < \infty. \tag{2.4}$$

Let $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfy, for each integer $i \geq 0$,

$$H(T(x), T_i(x)) \leq \epsilon_i, \quad \forall x \in X. \tag{2.5}$$

Assume that $x_0 \in X$ and that for each integer $i \geq 0$,

$$x_{i+1} \in T_i(x_i), \quad \rho(x_i, x_{i+1}) \leq \rho(x_i, T_i(x_i)) + \delta_i. \tag{2.6}$$

Then $\{x_i\}_{i=0}^{\infty}$ converges to a fixed point of T.

Proof. We first show that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence. To this end, let $i \geq 0$ be an integer. Then by (2.6) and (2.5),

$$\begin{aligned}
\rho(x_{i+1}, x_{i+2}) &\leq \rho(x_{i+1}, T_{i+1}(x_{i+1})) + \delta_{i+1} \\
&\leq \rho(x_{i+1}, T(x_{i+1})) + \sup\{\rho(z, T_{i+1}(x_{i+1})) : z \in T(x_{i+1})\} + \delta_{i+1} \\
&\leq \rho(x_{i+1}, T(x_{i+1})) + \epsilon_{i+1} + \delta_{i+1} \\
&\leq H(T_i(x_i), T(x_{i+1})) + \epsilon_{i+1} + \delta_{i+1} \\
&\leq H(T(x_i), T(x_{i+1})) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1} \\
&\leq c\rho(x_i, x_{i+1}) + \epsilon_i + \epsilon_{i+1} + \delta_{i+1}.
\end{aligned} \tag{2.7}$$

By (2.7),

$$\rho(x_1, x_2) \leq c\rho(x_0, x_1) + \epsilon_1 + \delta_1 + \epsilon_0, \tag{2.8}$$

$$\rho(x_2, x_3) \leq c\rho(x_1, x_2) + \epsilon_1 + \epsilon_2 + \delta_2 \tag{2.9}$$

$$\leq c^2\rho(x_0, x_1) + c(\epsilon_1 + \epsilon_0 + \delta_1) + \epsilon_1 + \epsilon_2 + \delta_2. \tag{2.10}$$

Now we show by induction that for each integer $n \geq 1$,

$$\rho(x_n, x_{n+1}) \leq c^n\rho(x_0, x_1) + \sum_{i=0}^{n-1} c^i(\epsilon_{n-i} + \delta_{n-i} + \epsilon_{n-i-1}). \tag{2.11}$$

In view of (2.8) and (2.10), inequality (2.11) holds for $n = 1, 2$.

Assume that $k \geq 1$ is an integer and that (2.11) holds for $n = k$. When combined with (2.7), this implies that

$$\begin{aligned}
\rho(x_{k+1}, x_{k+2}) &\leq c\rho(x_k, x_{k+1}) + \epsilon_{k+1} + \delta_{k+1} + \epsilon_k \\
&\leq c^{k+1}\rho(x_0, x_1) + \sum_{i=0}^{k-1} c^{i+1}(\epsilon_{k-i} + \delta_{k-i} + \epsilon_{k-1-i}) + \epsilon_{k+1} + \delta_{k+1} + \epsilon_k \\
&= c^{k+1}\rho(x_0, x_1) + \sum_{i=0}^{k} c^{i}(\epsilon_{k+1-i} + \delta_{k+1-i} + \epsilon_{k-i}).
\end{aligned} \tag{2.12}$$

Thus (2.11) holds for $n = k + 1$. Therefore, we have shown by induction that (2.11) holds for all integers $n \geq 1$. By (2.11),

$$\begin{aligned}
\sum_{n=1}^{\infty}\rho(x_n, x_{n+1}) &\leq \sum_{n=1}^{\infty}\left(c^n\rho(x_0, x_1) + \sum_{i=1}^{n} c^{n-i}(\epsilon_i + \delta_i + \epsilon_{i-1})\right) \\
&\leq \rho(x_0, x_1)\sum_{n=1}^{\infty} c^n + \sum_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} c^j\right)(\epsilon_i + \delta_i + \epsilon_{i-1}) \\
&\leq \left(\sum_{n=0}^{\infty} c^n\right)\left[\rho(x_0, x_1) + \sum_{n=1}^{\infty}(\epsilon_n + \delta_n + \epsilon_{n-1})\right] < \infty.
\end{aligned} \tag{2.13}$$

Thus $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and there exists

$$x_* = \lim_{n\to\infty} x_n. \tag{2.14}$$

We claim that

$$x_* \in T(x_*). \tag{2.15}$$

Indeed, by (2.14), there is an integer $n_0 \geq 1$ such that for each integer $n \geq n_0$,

$$\rho(x_n, x_*) \leq \frac{\epsilon}{8}. \tag{2.16}$$

Let $n \geq n_0$ be an integer. By (2.3), (2.16) and (2.5),

$$\begin{aligned}
\rho(x_*, T(x_*)) &\leq \rho(x_*, x_{n+1}) + \rho(x_{n+1}, T(x_*)) \\
&\leq \rho(x_*, x_{n+1}) + \rho(x_{n+1}, T(x_n)) + H(T(x_n), T(x_*)) \\
&\leq \rho(x_*, x_{n+1}) + \rho(x_{n+1}, T(x_n)) + \rho(x_n, x_*) \\
&\leq \rho(x_{n+1}, T(x_n)) + \frac{\epsilon}{4} \\
&\leq \rho(x_{n+1}, T_n(x_n)) + H(T_n(x_n), T(x_n)) + \frac{\epsilon}{4} \\
&\leq \epsilon_n + \frac{\epsilon}{4} \longrightarrow \frac{\epsilon}{4}
\end{aligned} \tag{2.17}$$

as $n \to \infty$. Since ϵ is an arbitrary positive number, we conclude that

$$x_* \in T(x_*), \tag{2.18}$$

as claimed. Theorem 2.1 is proved. □

3. Convergence to an Attractor of a Nonexpansive Mapping

In this section we show that if for any initial point, there exists a trajectory of the dynamical system induced by a nonexpansive set-valued mapping T, which converges to an invariant set F, then a convergent trajectory also exists for a nonstationary dynamical system induced by approximations of T.

Let $T : X \to 2^X \setminus \{\emptyset\}$ be such that $T(x)$ is a closed set for each $x \in X$ and

$$H(T(x), T(y)) \leq \rho(x, y), \quad \forall x, y \in X. \tag{3.1}$$

Theorem 3.1. *Let $\{\epsilon_i\}_{i=0}^{\infty} \subset (0, \infty)$, $\sum_{i=0}^{\infty} \epsilon_i < \infty$, F a nonempty closed subset of X,*

$$T(F) \subset F, \tag{3.2}$$

and for each integer $i \geq 0$, let $T_i : X \to 2^X \setminus \{\emptyset\}$ satisfy

$$H(T(x), T_i(x)) \leq \epsilon_i, \quad \forall x \in X. \tag{3.3}$$

Assume that for each $x \in X$, there exists a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$\begin{gathered}
x_0 = x, \qquad x_{i+1} \in T(x_i), \quad i = 0, 1, \ldots, \\
\lim_{i \to \infty} \rho(x_i, F) = 0.
\end{gathered} \tag{3.4}$$

Then for each $x \in X$, *there is a sequence* $\{x_i\}_{i=0}^{\infty} \subset X$ *such that*

$$x_0 = x, \qquad x_{i+1} \in T_i(x_i), \quad i = 0, 1, \ldots,$$
$$\lim_{i \to \infty} \rho(x_i, F) = 0. \tag{3.5}$$

We begin the proof of Theorem 3.1 with two lemmata.

Lemma 3.2. *Let* $x \in X$, p *a natural number,* $\{x_i\}_{i=0}^{p} \subset X$,

$$x_0 = x, \qquad x_{i+1} \in T_i(x_i), \quad i = 0, \ldots, p-1, \tag{3.6}$$

and let $\delta > 0$. *Then there is a natural number* $q > p$ *and a sequence* $\{x_i\}_{i=p}^{q} \subset X$ *such that*

$$x_{i+1} \in T_i(x_i), \quad i = p, \ldots, q-1,$$
$$\rho(x_q, F) \le \delta. \tag{3.7}$$

Proof. Choose a natural number $p_1 > p$ such that

$$\sum_{i=p_1}^{\infty} \epsilon_i < \frac{\delta}{8} \tag{3.8}$$

and a sequence $\{x_i\}_{i=p}^{p_1} \subset X$ such that

$$x_{i+1} \in T_i(x_i), \quad i = p, \ldots, p_1 - 1. \tag{3.9}$$

There is a sequence $\{y_i\}_{i=p_1}^{\infty} \subset X$ such that

$$y_{p_1} = x_{p_1}, \qquad \lim_{i \to \infty} \rho(y_i, F) = 0, \tag{3.10}$$

$$y_{i+1} \in T(y_i), \quad \text{for all integers } i \ge p_1. \tag{3.11}$$

We are now going to define by induction a sequence $\{x_i\}_{i=p_1}^{\infty} \subset X$.

To this end, assume that $k \ge p_1$ is an integer and that we have already defined $x_i \in X$, $i = p_1, \ldots, k$, such that

$$x_{i+1} \in T_i(x_i), \quad i = p_1, \ldots, k-1, \tag{3.12}$$

$$\rho(x_k, y_k) \le 3\left(\sum_{i=p_1}^{k} \epsilon_i - \epsilon_k\right). \tag{3.13}$$

(Clearly, this assumption holds for $k = p_1$.)

By (3.11) and (3.1),

$$y_{k+1} \in T(y_k), \tag{3.14}$$

$$H(T(y_k), T(x_k)) \le \rho(x_k, y_k). \tag{3.15}$$

By (3.15), there is $\tilde{y}_{k+1} \in X$ such that

$$\tilde{y}_{k+1} \in T(x_k), \qquad \rho(y_{k+1}, \tilde{y}_{k+1}) \le \rho(x_k, y_k) + \epsilon_k. \tag{3.16}$$

Together with (3.3), this implies that

$$\rho(\tilde{y}_{k+1}, T_k(x_k)) \le \epsilon_k, \tag{3.17}$$

and there is

$$x_{k+1} \in T_k(x_k) \tag{3.18}$$

such that

$$\rho(\tilde{y}_{k+1}, x_{k+1}) \le 2\epsilon_k. \tag{3.19}$$

When combined with (3.16) and (3.13), this implies that

$$\rho(x_{k+1}, y_{k+1}) \le \rho(x_{k+1}, \tilde{y}_{k+1}) + \rho(\tilde{y}_{k+1}, y_{k+1}) \le \rho(x_k, y_k) + 3\epsilon_k \le 3\sum_{i=p_1}^{k} \epsilon_i. \tag{3.20}$$

Thus, by (3.18) and (3.20), the assumption we have made concerning k also holds for $k+1$. Therefore, we have indeed defined by induction a sequence $\{x_i\}_{i=p_1}^{\infty}$ such that

$$x_{i+1} \in T_i(x_i), \quad i = p_1, \ldots, \tag{3.21}$$

and (3.13) holds for all integers $k \ge p_1$. By (3.11), there is an integer $q > p_1 + 2$ such that

$$\rho(y_q, F) < \frac{\delta}{4}. \tag{3.22}$$

Together with (3.8) and (3.13), this inequality implies that

$$\rho(x_q, F) \le \rho(x_q, y_q) + \rho(y_q, F) \le \sum_{i=p_1}^{\infty} \epsilon_i + \frac{\delta}{4} < \frac{\delta}{2} + \frac{\delta}{4}. \tag{3.23}$$

Lemma 3.2 is proved. □

Lemma 3.3. *Let* $\{x_i\}_{i=0}^{\infty} \subset X$,

$$x_{i+1} \in T_i(x_i), \quad i = 0, 1, \ldots, \tag{3.24}$$

$\delta > 0$, *p a natural number,*

$$\rho(x_p, F) \le \delta, \tag{3.25}$$

$$\sum_{i=p}^{\infty} \epsilon_i < \delta. \tag{3.26}$$

Then $\rho(x_i, F) \le 3\delta$ *for all integers* $i \ge p$.

Proof. We intend to show by induction that for all integers $n \ge p$,

$$\rho(x_n, F) \le \delta + \sum_{i=p}^{n} (2\epsilon_i) - 2\epsilon_n. \tag{3.27}$$

Clearly, for $n = p$ inequality (3.27) does hold. Assume now that $n \ge p$ is an integer and (3.27) holds. Then there is

$$y_n \in F \tag{3.28}$$

such that

$$\rho(x_n, y_n) \le \delta + \sum_{i=p}^{n} 2\epsilon_i - \epsilon_n. \tag{3.29}$$

By (3.24) and (3.3), there is

$$\tilde{x}_{n+1} \in T(x_n) \tag{3.30}$$

such that

$$\rho(\tilde{x}_{n+1}, x_{n+1}) \le 2\epsilon_n. \tag{3.31}$$

By (3.29) and (3.1),

$$H(T(x_n), T(y_n)) \le \rho(x_n, y_n), \tag{3.32}$$

and, in view of (3.30), there is

$$y_{n+1} \in T(y_n) \tag{3.33}$$

such that

$$\rho(y_{n+1}, \tilde{x}_{n+1}) \le \rho(x_n, y_n) + \epsilon_n. \tag{3.34}$$

By (3.33), (3.28), and (3.2),

$$y_{n+1} \in F. \tag{3.35}$$

By (3.35), (3.31), (3.34), and (3.27),

$$\begin{aligned} \rho(x_{n+1}, F) &\le \rho(x_{n+1}, y_{n+1}) \le \rho(x_{n+1}, \tilde{x}_{n+1}) + \rho(\tilde{x}_{n+1}, y_{n+1}) \\ &\le 2\epsilon_n + \epsilon_n + \rho(x_n, y_n) \le \delta + 2\epsilon_n + \sum_{i=p}^{n} 2\epsilon_i. \end{aligned} \tag{3.36}$$

Thus, the assumption we have made concerning n also holds for $n+1$. Therefore, we may conclude that inequality (3.27) indeed holds for all integers $n \ge p$. Together with (3.26), this implies that for all integers $n \ge p$,

$$\rho(x_n, F) \le \delta + 2\delta = 3\delta. \tag{3.37}$$

Lemma 3.3 is proved. □

Completion of the Proof of Theorem 3.1

Let $x \in X$. Since $\sum_{i=0}^{\infty} \epsilon_i < \infty$, it follows from Lemma 3.2 that there exist a sequence $\{x_i\}_{i=0}^{\infty}$ and a strictly increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$, constructed by induction, such that

$$x_{i+1} \in T_i(x_i), \quad i = 0, 1, \ldots, \tag{3.38}$$

and for each integer $k \ge 1$,

$$\rho(x_{n_k}, F) \le 2^{-k}, \qquad \sum_{i=n_k}^{\infty} \epsilon_i < 2^{-k}. \tag{3.39}$$

It now follows from (3.39) and Lemma 3.3 that

$$\lim_{n \to \infty} \rho(x_n, F) = 0. \tag{3.40}$$

Theorem 3.1 is proved.

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