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Research Article

Strong Convergence Theorems by Hybrid Methods for Strict Pseudocontractions and Equilibrium Problems

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Let $\{S_i\}_{i=1}^{N}$ be N strict pseudocontractions defined on a closed convex subset C of a real Hilbert space H. Consider the problem of finding a common element of the set of fixed point of these mappings and the set of solutions of an equilibrium problem with the parallel and cyclic algorithms. In this paper, we propose new iterative schemes for solving this problem and prove these schemes converge strongly by hybrid methods.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of such solutions is denoted by $\mathrm{EP}(f)$.

A mapping S of C is said to be a κ-strict pseudocontraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2 \tag{1.2}$$

for all $x, y \in C$; see [1]. We denote the set of fixed points of S by $F(S)$ (i.e., $F(S) = \{x \in C : Sx = x\}$).

Note that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings which are mapping S on C such that

$$\|Sx - Sy\| \leq \|x - y\| \tag{1.3}$$

for all $x, y \in C$. That is, S is nonexpansive if and only if S is a 0-strict pseudocontraction.

Numerous problems in physics, optimization, and economics reduce to finding a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem (1.1); see for instance [2–5]. In particular, Combettes and Hirstoaga [6] proposed several methods for solving the equilibrium problem. On the other hand, Mann [7], Nakajo and Takahashi [8] considered iterative schemes for finding a fixed point of a nonexpansive mapping.

Recently, Acedo and Xu [9] considered the problem of finding a common fixed point of a finite family of strict pseudocontractive mappings by the parallel and cyclic algorithms. Very recently, Liu [3] considered a general iterative method for equilibrium problems and strict pseudocontractions. In this paper, motivated by [3, 5, 9–12], applying parallel and cyclic algorithms, we obtain strong convergence theorems for finding a common element of the set of fixed points of a finite family of strict pseudocontractions and the set of solutions of the equilibrium problem (1.1) by the hybrid methods.

We will use the notation

(1) $\rightharpoonup$ for weak convergence and $\rightarrow$ for strong convergence,

(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

We need some facts and tools in a real Hilbert space H which are listed as below.

Lemma 2.1. *Let H be a real Hilbert space. There hold the following identities.*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$, *for all* $t \in [0,1]$, *for all* $x, y \in H$.

Lemma 2.2 (see [4]). *Let H be a real Hilbert space. Given a nonempty closed convex subset $C \subset H$ and points $x, y, z \in H$ and given also a real number $a \in \mathbb{R}$, the set*

$$\left\{v \in C : \left\|y - v\right\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\right\} \tag{2.1}$$

is convex (and closed).

Recall that given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \tag{2.2}$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C.

Lemma 2.3 (see [4]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in C$, then $y = P_C x$ if and only if there holds the relation*

$$\langle x - y, y - z \rangle \geq 0 \quad \forall z \in C. \tag{2.3}$$

Lemma 2.4 (see [13]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ is a sequence in H and $u \in H$. Let $q = P_C u$. Suppose $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\| \quad \forall n. \tag{2.4}$$

Then $x_n \to q$.

Lemma 2.5 (see [9]). *Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ is a sequence in H and $u \in H$. Assume*

(i) *the weak ω-limit set $\omega_w(x_n) \subset C$,*

(ii) *for each $z \in C$, $\lim_{n\to\infty}\|x_n - z\|$ exists.*

Then $\{x_n\}$ is weakly convergent to a point in C.

Proposition 2.6 (see [9]). *Assume C be a nonempty closed convex subset of a real Hilbert space H.*

(i) *If $T : C \to C$ is a κ-strict pseudocontraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leq \frac{1+\kappa}{1-\kappa}\|x - y\|, \quad \forall x, y \in C. \tag{2.5}$$

(ii) *If $T : C \to C$ is a κ-strict pseudocontraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$.*

(iii) *If $T : C \to C$ is a κ-strict pseudocontraction, then the fixed point set of $F(T)$ of T is closed and convex so that the projection $P_{F(T)}$ is well defined.*

(iv) *Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : C \to C$ be a κ_i-strict pseudocontraction for some $0 \leq \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then $\sum_{i=1}^N \lambda_i T_i$ is a κ-strict pseudocontraction, with $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$.*

(v) *Let $\{T_i\}_{i=1}^N$ and $\{\lambda_i\}_{i=1}^N$ be given as in (iv) above. Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F\left(\sum_{i=1}^N \lambda_i T_i\right) = \bigcap_{i=1}^N F(T_i). \tag{2.6}$$

Lemma 2.7 (see [1]). *Let $S : C \to H$ be a κ-strict pseudocontraction. Define $T : C \to H$ by $Tx = \lambda x + (1-\lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F(T) = F(S)$.*

For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$,

(A2) f is monotone, that is, $f(x,y) + f(y,x) \leq 0$ for any $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim\sup_{t \to 0} f(tz + (1-t)x, y) \leq f(x,y)$,

(A4) $f(x,\cdot)$ is convex and lower semicontionuous for each $x \in C$.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.8 (see [14]). *Let C be a nonempty closed convex subset of H, let f be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4) and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.7}$$

Lemma 2.9 (see [6]). *For $r > 0$, $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C \mid f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{2.8}$$

for all $x \in H$. Then, the following statements hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\|T_r x - T_r y\right\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{2.9}$$

(iii) *$F(T_r) = EP(f)$;*

(iv) *$EP(f)$ is closed and convex.*

3. Parallel Algorithm

In this section, we apply the hybrid methods to the parallel algorithm for finding a common element of the set of fixed points of strict pseudocontractions and the set of solutions of the equilibrium problem (1.1) in Hilbert spaces.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a κ_i-strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the set $F = \bigcap_{i=1}^{N} F(S_i) \cap EP(f) \neq \emptyset$. Assume also $\{\eta_i^{(n)}\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \eta_i^{(n)} = 1$ for all $n \in \mathbb{N}$ and $\inf_{n \geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping A_n be defined by*

$$A_n = \sum_{i=1}^{N} \eta_i^{(n)} S_i. \tag{3.1}$$

Given $x_1 \in C$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by the following algorithm:

$$
\begin{aligned}
u_n &= T_{r_n} x_n, \\
A_n^{\lambda_n} &= \lambda_n I + (1-\lambda_n) A_n, \\
y_n &= \alpha_n x_n + (1-\alpha_n) A_n^{\lambda_n} u_n, \\
C_n &= \{ z \in C : \|y_n - z\| \le \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_1 - x_n \rangle \ge 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_1
\end{aligned}
\tag{3.2}
$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,a]$ for some $a \in [0,1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converge strongly to $P_F x_1$.

Proof. The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is well defined.

It is obvious that C_n is closed and Q_n is closed convex for every $n \in \mathbb{N}$. From Lemma 2.2, we also get C_n is convex.

Step 2. Show $F \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$.

Indeed, take $p \in F$, from $u_n = T_{r_n} x_n$, we have

$$
\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\| \tag{3.3}
$$

for all $n \in \mathbb{N}$. From Proposition 2.6, Lemma 2.7, and (3.3), we get

$$
\begin{aligned}
\|y_n - p\| &= \left\| \alpha_n x_n + (1-\alpha_n) A_n^{\lambda_n} u_n - p \right\| \\
&\le \alpha_n \|x_n - p\| + (1-\alpha_n) \left\| A_n^{\lambda_n} u_n - p \right\| \\
&\le \|x_n - p\|.
\end{aligned}
\tag{3.4}
$$

So $p \in C_n$ for all n. Next we show that $F \subset Q_n$ for all $n \in \mathbb{N}$ by induction. For $n = 1$, we have $F \subset C = Q_1$. Assume that $F \subset Q_n$ for some $n \ge 1$. Since $x_{n+1} = P_{C_n \cap Q_n} x_1$, we obtain

$$
\langle x_{n+1} - z, x_1 - x_{n+1} \rangle \ge 0, \quad \forall z \in C_n \cap Q_n. \tag{3.5}
$$

As $F \subset C_n \cap Q_n$ by induction assumption, the inequality holds, in particular, for all $z \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset Q_n$ holds for all $n \ge 1$.

Step 3. Show that

$$
\|x_n - x_1\| \le \|x_1 - q\|, \quad \text{where } q = P_F x_1. \tag{3.6}
$$

Notice that the definition of Q_n actually $x_n = P_{Q_n} x_1$. This together with the fact $F \subset Q_n$ further implies

$$\|x_n - x_1\| \le \|x_1 - p\| \quad \forall p \in F. \tag{3.7}$$

Then $\{x_n\}$ is bounded and (3.6) holds. From (3.3), (3.4), and Proposition 2.6(i), we also obtain $\{u_n\}$, $\{y_n\}$, and $\{S_i x_n\}$ are bounded.

Step 4. Show that

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{3.8}$$

From $x_n = P_{Q_n} x_1$ and $x_{n+1} \in Q_n$, we get $\langle x_{n+1} - x_n, x_n - x_1 \rangle \ge 0$. This together with Lemma 2.1(i) implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x_1 - (x_n - x_1)\|^2 \\ &= \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2 - 2\langle x_{n+1} - x_n, x_n - x_1 \rangle \\ &\le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2. \end{aligned} \tag{3.9}$$

Then $\|x_n - x_1\| \le \|x_{n+1} - x_1\|$, that is, the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. Since $\{\|x_n - x_1\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_1\|$ exists. Then (3.8) holds.

Step 5. Show that

$$\|A_n x_n - x_n\| \longrightarrow 0. \tag{3.10}$$

From $x_{n+1} \in C_n$, we have

$$\|y_n - x_n\| \le \|x_{n+1} - x_n\| + \|y_n - x_{n+1}\| \le 2\|x_{n+1} - x_n\|. \tag{3.11}$$

By (3.8), we obtain

$$\|y_n - x_n\| \longrightarrow 0. \tag{3.12}$$

For $p \in F$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \le \langle T_{r_n} x_n - T_{r_n} p, x_n - p \rangle \\ &= \langle u_n - p, x_n - p \rangle = \frac{1}{2}\left(\|u_n - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2\right), \end{aligned} \tag{3.13}$$

hence,

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.14}$$

Therefore, by the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned} \|y_n - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left\|A_n^{\lambda_n} u_n - p\right\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left(\|x_n - p\|^2 - \|x_n - u_n\|^2\right) \\ &= \|x_n - p\|^2 - (1-\alpha_n)\|x_n - u_n\|^2. \end{aligned} \tag{3.15}$$

Since $\{\alpha_n\} \subset [0, a]$, we get

$$\begin{aligned} (1-\alpha_n)\|x_n - u_n\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|). \end{aligned} \tag{3.16}$$

It follows that

$$\|x_n - u_n\| \longrightarrow 0 \tag{3.17}$$

from (3.12). Observe that $\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\|$, we also have $\|y_n - u_n\| \to 0$. On the other hand, from $y_n = \alpha_n x_n + (1-\alpha_n)A_n^{\lambda_n} u_n$, we compute

$$\begin{aligned} (1-\alpha_n)\left\|A_n^{\lambda_n} u_n - u_n\right\| &= \left\|(1-\alpha_n)\left(A_n^{\lambda_n} u_n - u_n\right)\right\| \\ &= \|y_n - u_n - \alpha_n(x_n - u_n)\| \\ &\leq \|y_n - u_n\| + \alpha_n\|x_n - u_n\|. \end{aligned} \tag{3.18}$$

From $\{\alpha_n\} \subset [0, a]$, (3.17), and $\|y_n - u_n\| \to 0$, we obtain $\|A_n^{\lambda_n} u_n - u_n\| \to 0$. It is easy to get

$$\begin{aligned} \left\|A_n^{\lambda_n} x_n - x_n\right\| &\leq \left\|A_n^{\lambda_n} x_n - A_n^{\lambda_n} u_n\right\| + \left\|A_n^{\lambda_n} u_n - u_n\right\| + \|u_n - x_n\| \\ &\leq 2\|u_n - x_n\| + \left\|A_n^{\lambda_n} u_n - u_n\right\|. \end{aligned} \tag{3.19}$$

Combining the above results, we obtain $\|A_n^{\lambda_n} x_n - x_n\| \to 0$. From (3.2), we have

$$\begin{aligned} \left\|A_n^{\lambda_n} x_n - x_n\right\| &= \|\lambda_n x_n + (1-\lambda_n)A_n x_n - x_n\| \\ &= (1-\lambda_n)\|A_n x_n - x_n\| \longrightarrow 0. \end{aligned} \tag{3.20}$$

It follows from $\{\lambda_n\} \subset [\kappa, b]$ that $\|A_n x_n - x_n\| \to 0$.

Step 6. Show that

$$\omega_w(x_n) \subset F. \tag{3.21}$$

We first show $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i)$. To see this, we take $\omega \in \omega_w(x_n)$ and assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$ for some subsequence $\{x_{n_j}\}$ of x_n.

Without loss of generality, we may assume that

$$\eta_i^{(n_j)} \longrightarrow \eta_i \quad (\text{as } j \longrightarrow \infty),\ 1 \le i \le N. \tag{3.22}$$

It is easily seen that each $\eta_i > 0$ and $\sum_{i=1}^N \eta_i^{(n)} = 1$. We also have

$$A_{n_j} x \longrightarrow Ax \quad (\text{as } j \longrightarrow \infty)\ \forall x \in C, \tag{3.23}$$

where $A = \sum_{i=1}^N \eta_i S_i$. Note that by Proposition 2.6, A is κ-strict pseudocontraction and $F(A) = \bigcap_{i=1}^N F(S_i)$. Since

$$\begin{aligned} \left\| Ax_{n_j} - x_{n_j} \right\| &\le \left\| A_{n_j} x_{n_j} - Ax_{n_j} \right\| + \left\| A_{n_j} x_{n_j} - x_{n_j} \right\| \\ &\le \sum_{i=1}^N \left| \eta_i^{(n_j)} - \eta_i \right| \left\| S_i x_{n_j} \right\| + \left\| A_{n_j} x_{n_j} - x_{n_j} \right\|, \end{aligned} \tag{3.24}$$

we obtain by virtue of (3.10) and (3.22)

$$\left\| Ax_{n_j} - x_{n_j} \right\| \longrightarrow 0. \tag{3.25}$$

So by the demiclosedness principle (Proposition 2.6(ii)), it follows that $\omega \in F(A) = \bigcap_{i=1}^N F(S_i)$ and hence $\omega_w(x_n) \subset \bigcap_{i=1}^N F(S_i)$ holds.

Next we show $\omega_w(x_n) \subset \mathrm{EP}(f)$, take $\omega \in \omega_w(x_n)$, and assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$ for some subsequence $\{x_{n_j}\}$ of x_n. From (3.17), we obtain $u_{n_j} \rightharpoonup \omega$. Since $\{u_{n_j}\} \subset C$ and C is closed convex, we get $\omega \in C$.

By $u_n = T_{r_n} x_n$, we have

$$f(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C. \tag{3.26}$$

From the monotonicity of f, we get

$$\frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge f(y, u_n), \quad \forall y \in C, \tag{3.27}$$

hence

$$\left\langle y - u_{n_j}, \frac{u_{n_j} - x_{n_j}}{r_{n_j}} \right\rangle \geq f\left(y, u_{n_j}\right), \quad \forall y \in C. \tag{3.28}$$

From (3.17) and condition (A4), we have

$$0 \geq f(y, \omega), \quad \forall y \in C. \tag{3.29}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)\omega$. Since $y \in C$ and $\omega \in C$, we obtain $y_t \in C$ and hence $f(y_t, \omega) \leq 0$. So, we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \omega) \leq tf(y_t, y). \tag{3.30}$$

Dividing by t, we get

$$f(y_t, y) \geq 0, \quad \forall y \in C. \tag{3.31}$$

Letting $t \to 0$ and from (A3), we get

$$f(\omega, y) \geq 0 \tag{3.32}$$

for all $y \in C$ and $\omega \in \mathrm{EP}(f)$. Hence (3.21) holds.

Step 7. From (3.6) and Lemma 2.4, we conclude that $x_n \to q$, where $q = P_F x_1$.

□

A very similar result obtained in a way completely different is Theorem 3 of [11].

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $S_i : C \to C$ be a κ_i-strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 1 \leq i \leq N\}$. Assume the set $F = \bigcap_{i=1}^{N} F(S_i) \cap EP(f) \neq \emptyset$. Assume also $\{\eta_i^{(n)}\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \eta_i^{(n)} = 1$ for all n and $\inf_{n \geq 1} \eta_i^{(n)} > 0$ for all $1 \leq i \leq N$. Let the mapping A_n be defined by*

$$A_n = \sum_{i=1}^{N} \eta_i^{(n)} S_i. \tag{3.33}$$

Given $x_1 \in C = C_1$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by the following algorithm:

$$
\begin{aligned}
cu_n &= T_{r_n} x_n, \\
A_n^{\lambda_n} &= \lambda_n I + (1 - \lambda_n) A_n, \\
y_n &= \alpha_n x_n + (1 - \alpha_n) A_n^{\lambda_n} u_n, \\
C_{n+1} &= \{ z \in C_n : \| y_n - z \| \le \| x_n - z \| \}, \\
x_{n+1} &= P_{C_{n+1}} x_1
\end{aligned}
\tag{3.34}
$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$. Then, $\{x_n\}$ converge strongly to $P_F x_1$.

Proof. The proof of this theorem is similar to that of Theorem 3.1.

Step 1. $\{x_n\}$ is well defined for all $n \ge 1$.

We show C_n is closed convex for all n by induction. For $n = 1$, we have $C = C_1$ is closed convex. Assume that C_n for some $n \ge 1$ is closed convex, from Lemma 2.2, we have C_{n+1} is also closed convex. The assumption holds.

Step 2. $F \subset C_n$.

Step 3. $\|x_n - x_1\| \le \|q - x_1\|$ for all n, where $q = P_F x_1$.

Step 4. $\|x_{n+1} - x_n\| \to 0$.

Step 5. $\|A_n x_n - x_n\| \to 0$.

Step 6. $\omega_w(x_n) \subset F$.

Step 7. $x_n \to q$.

The proof of Steps 2–7 is similar to that of Theorem 3.1. □

A very similar result obtained in a way completely different is Theorem 3.1 of [10].

4. Cyclic Algorithm

Let C be a closed convex subset of a Hilbert space H and let $\{S_i\}_{i=0}^{N-1}$ be $N\kappa_i$-strict pseudocontractions on C such that the common fixed point set

$$
\bigcap_{i=0}^{N-1} F(S_i) \ne \emptyset. \tag{4.1}
$$

Let $x_0 \in C$ and let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence in $(0,1)$. The cyclic algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ in the following way:

$$
\begin{aligned}
x_1 &= \alpha_0 x_0 + (1-\alpha_0) S_0 x_0, \\
x_2 &= \alpha_1 x_1 + (1-\alpha_1) S_1 x_1, \\
&\ \vdots \\
x_N &= \alpha_{N-1} x_{N-1} + (1-\alpha_{N-1}) S_{N-1} x_{N-1}, \\
x_{N+1} &= \alpha_N x_N + (1-\alpha_N) S_0 x_N, \\
&\ \vdots
\end{aligned}
\tag{4.2}
$$

In general, x_{n+1} is defined by

$$
x_{n+1} = \alpha_n x_n + (1-\alpha_n) S_{[n]} x_n, \tag{4.3}
$$

where $S_{[n]} = S_i$, with $i = n(\mathrm{mod})N$, $0 \le i \le N-1$.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $S_i : C \to C$ be a κ_i-strict pseudocontraction for some $0 \le \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 0 \le i \le N-1\}$. Assume the set $F = \bigcap_{i=0}^{N-1} F(S_i) \cap EP(f) \ne \emptyset$. Given $x_0 \in C$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by the following algorithm:*

$$
\begin{aligned}
cu_n &= T_{r_n} x_n, \\
S_{[n]}^{\lambda_n} &= \lambda_n I + (1-\lambda_n) S_{[n]}, \\
y_n &= \alpha_n x_n + (1-\alpha_n) S_{[n]}^{\lambda_n} u_n, \\
C_n &= \{ z \in C : \|y_n - z\| \le \|x_n - z\| \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x_0
\end{aligned}
\tag{4.4}
$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,a]$ for some $a \in [0,1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_n\} \subset (0,\infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converge strongly to $P_F x_0$.

Proof. The proof of this theorem is similar to that of Theorem 3.1. The main points include the following.

Step 1. $\{x_n\}$ is well defined for all $n \ge 1$.

Step 2. $F \subset C_n \cap Q_n$.

Step 3. $\|x_n - x_0\| \leq \|q - x_0\|$ for all n, where $q = P_F x_0$.

Step 4. $\|x_{n+1} - x_n\| \to 0$.

Step 5. $\|S_{[n]}x_n - x_n\| \to 0$.

To prove the above steps, one simply replaces A_n with $S_{[n]}$ in the proof of Theorem 3.1.

Step 6. Show that $\omega_w(x_n) \subset F$.

Indeed, assume $\omega \in \omega_w(x_n)$ and $x_{n_i} \rightharpoonup \omega$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We may further assume $l = n_i (\operatorname{mod} N)$ for all i. Since by $\|x_{n+1} - x_n\| \to 0$, we also have $x_{n_i+j} \rightharpoonup \omega$ for all $j \geq 0$, we deduce that

$$\left\|x_{n_i+j} - S_{[l+j]}x_{n_i+j}\right\| = \left\|x_{n_i+j} - S_{[n_i+j]}x_{n_i+j}\right\| \longrightarrow 0. \tag{4.5}$$

Then the demiclosedness principle (Proposition 2.6(ii)) implies that $\omega \in F(S_{[l+j]})$ for all j. This ensures that $\omega \in \bigcap_{i=0}^{N-1} F(S_i)$.

The proof of $\omega \in \mathrm{EP}(f)$ is similar to that of Theorem 3.1.

Step 7. Show that $x_n \to q$.

The strong convergence to q of $\{x_n\}$ is the consequence of Step 3, Step 5, and Lemma 2.4. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H and f a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $N \geq 1$ be an integer. Let, for each $0 \leq i \leq N-1$, $S_i : C \to C$ be a κ_i-strict pseudocontraction for some $0 \leq \kappa_i < 1$. Let $\kappa = \max\{\kappa_i : 0 \leq i \leq N-1\}$. Assume the set $F = \bigcap_{i=0}^{N-1} F(S_i) \cap EP(f) \neq \emptyset$. Given $x_0 \in C = C_0$, let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by the following algorithm:*

$$\begin{gathered}
cu_n = T_{r_n}x_n, \\
S_{[n]}^{\lambda_n} = \lambda_n I + (1 - \lambda_n)S_{[n]}, \\
y_n = \alpha_n x_n + (1 - \alpha_n)S_{[n]}^{\lambda_n}u_n, \\
C_{n+1} = \left\{z \in C_n : \left\|y_n - z\right\| \leq \|x_n - z\|\right\}, \\
x_{n+1} = P_{C_{n+1}}x_0
\end{gathered} \tag{4.6}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$, $\{\lambda_n\} \subset [\kappa, b]$ for some $b \in [\kappa, 1)$, and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$. Then, $\{x_n\}$ converge strongly to $P_F x_0$.

Proof. The proof of this theorem can consult Step 1 of Theorem 3.2 and Steps 2–7 of Theorem 4.1. □

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