Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 614867, 13 pages
doi:10.1155/2010/614867

Research Article

Approximate Endpoints for Set-Valued Contractions in Metric Spaces

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Received 18 March 2010; Accepted 26 April 2010

Academic Editor: Mohamed Amine Khamsi

The existence of approximate fixed points and approximate endpoints of the multivalued almost I-contractions is established. We also develop quantitative estimates of the sets of approximate fixed points and approximate endpoints for multivalued almost I-contractions. The proved results unify and improve recent results of Amini-Harandi (2010), M. Berinde and V. Berinde (2007), Ćirić (2009), M. Păcurar and R. V. Păcurar (2007) and many others.

1. Introduction and Preliminaries

In fixed point theory, one of the main directions of investigation concerns the study of the fixed point property in topological spaces. Recall that a topological space X is said to have the fixed point property if every continuous mapping $f : X \to X$ has a fixed point. The major contribution to this subject has been provided by Tychonoff. Every compact convex subset of a locally convex space has the fixed point property. Another important branch of fixed point theory is the study of the approximate fixed point property. Recently, the interest in approximate fixed point results arise in the study of some problems in economics and game theory, including, for example, the Nash equilibrium approximation in games; see [1–3] and references therein.

We establish some existence results concerning approximate fixed points, endpoints, and approximate endpoints of multivalued contractions. We also develop quantitative estimates of the sets of approximate fixed points and approximate endpoints for set-valued almost I-contractions. The results presented in this paper extend and improve the recent results of [4–10] and many others.

Now, we give some notions and definitions.

Let (X, d) be a metric space and let $\mathcal{P}(X)$ and $\mathrm{Cl}(X)$ denote the families of all nonempty subsets and nonempty closed subsets of X, respectively. Let X and Y be two Hausdorff topological spaces and $T : X \to \mathcal{P}(Y)$ a multivalued mapping with nonempty values. Then T is said to be

(1) *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is closed in X;

(2) *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, $T^{-1}(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$ is open in X;

(3) *continuous* if it is both u.s.c. and l.s.c.;

(4) *closed* if its graph $\mathrm{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed;

(5) *compact* if $\mathrm{cl}\, T(X)$ is a compact subset of Y.

For any subsets A, B, of a metric space X, we consider the following notions:

$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$: the distance between the sets A and B;

$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$: the diameter of the sets A and B;

$\delta(A) = \sup\{d(x, y) : x, y \in A\}$: the diameter of the set A;

$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$: the Hausdorff metric on $\mathrm{Cl}(X)$ induced by the metric d.

Let $T : X \to \mathcal{P}(X)$ be a multivalued mapping. An element $x \in X$ such that $x \in T(x)$ is called a fixed point of T. We denote by $F(T)$ the set of all fixed points of T, that is, $F(T) = \{x \in X : x \in T(x)\}$.

A mapping $T : X \to \mathcal{P}(X)$ is called

(mc) a *multivalued contraction* (or *multivalued k-contraction*) if there exists a number $0 < k < 1$ such that

$$H(Tx, Ty) \leq kd(x, y), \quad \forall x, y \in X, \tag{1.1}$$

(mac) a *multivalued almost contraction* [6] or a *multivalued (θ, L)-almost contraction* if there exist two constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(y, Tx), \quad \forall x, y \in X, \tag{1.2}$$

(gmac) a *generalized multivalued almost contraction* [6] if there exists a function $\alpha : [0, \infty) \to [0, 1)$ satisfying $\lim\sup_{r\to t^+} \alpha(r) < 1$ for every $t \in [0, \infty)$ such that

$$H(Tx, Ty) \leq \alpha(d(x, y))d(x, y) + Ld(y, Tx), \quad \forall x, y \in X. \tag{1.3}$$

It is important to note that any mapping satisfying Banach, Kannan, Chatterjea, Zamfirescu, or Ćirić (with the constant k in $]0, 1/2[$) type conditions is a single-valued almost contraction; see [5, 6, 8, 11].

2. Approximate Fixed Points of Multivalued Contractions

Definition 2.1. A multivalued mapping $T : X \to \mathcal{P}(X)$ is said to have the *approximate fixed point property* [2] provided

$$\inf_{x\in X} d(x, Tx) = 0 \tag{2.1}$$

or, equivalently, for any $\epsilon > 0$, there exists $z \in X$ such that

$$d(z, Tz) \leq \epsilon \tag{2.2}$$

or, equivalently, for any $\epsilon > 0$, there exists $x_\epsilon \in X$ such that

$$T(x_\epsilon) \cap B(x_\epsilon, \epsilon) \neq \emptyset, \tag{2.3}$$

where $B(x, r)$ denotes a closed ball of radius r centered at x.

We first prove that every generalized multivalued almost contraction has the approximate fixed point property.

Lemma 2.2. *Every generalized multivalued almost contraction has the approximate fixed point property.*

Proof. Let (X, d) be an arbitrary metric space and $T : X \to \mathrm{Cl}(X)$ a generalized multivalued almost contraction. Let $x_n \in X$ and $y_n \in T(x_n)$ be such that

$$\lim_{n\to\infty} d(x_n, y_n) = \inf_{x\in X} d(x, Tx). \tag{2.4}$$

By passing to the subsequences, if necessary, we may assume that the sequence $\{\alpha(d(x_n, y_n))\}$ is convergent. Then we have

$$\begin{aligned}
\inf_{x\in X} d(x, Tx) &\leq \inf\left\{ d(y, Ty) : y \in \bigcup_{x\in X} Tx \right\} = \inf_{x\in X} \inf_{y\in Tx} d(y, Ty) \leq \inf_{x\in X} \inf_{y\in Tx} H(Tx, Ty) \\
&\leq \inf_{x\in X} \inf_{y\in Tx} [\alpha(d(x, y))d(x, y) + L \cdot d(y, Tx)] = \inf_{x\in X} \inf_{y\in Tx} [\alpha(d(x, y))d(x, y)] \\
&\leq \inf_{n\in\mathbb{N}} [\alpha(d(x_n, y_n))d(x_n, y_n)] \leq \lim_{n\to\infty} \alpha(d(x_n, y_n)) \lim_{n\to\infty} d(x_n, y_n) \\
&\leq \limsup_{r\to(\inf_{x\in X} d(x,Tx))^+} \alpha(r) \inf_{x\in X} d(x, Tx).
\end{aligned} \tag{2.5}$$

Since $\limsup_{r\to(\inf_{x\in X} d(x,Tx))^+} \alpha(r) < 1$, we get $\inf_{x\in X} d(x, Tx) = 0$. This completes the proof. □

Corollary 2.3 (see [5, Theorem 2.5], [10, Theorem 2.1]). *Let (X, d) be a metric space and $T : X \to X$ a single-valued almost contraction. Then T has the approximate fixed point property.*

The authors in [5, 10] obtained the following quantitative estimate of the diameter of the set, $F_\epsilon(T) = \{x \in X : d(x, Tx) \leq \epsilon\}$, of approximate fixed points of single-valued almost contraction T.

Theorem 2.4 (see [5, Theorem 3.5], [10, Theorem 2.2]). *Let (X, d) be a metric space. If $T : X \to X$ is a single-valued almost contraction with $\theta + L < 1$, then*

$$\delta(F_\epsilon(T)) \leq \frac{(2 + L)\epsilon}{1 - (\theta + L)}, \quad \forall \epsilon > 0. \tag{2.6}$$

The following simple example shows that the conclusion of Theorem 2.4 is not valid for set-valued almost contractions.

Example 2.5. Let $T : [0,1] \to \mathrm{Cl}([0,1])$ be defined by $T(x) = [0,1]$ for all $x \in [0,1]$. Then $0 = H(T(x), T(y)) \leq (1/2)d(x, y)$ and so T is multivalued almost contraction with $\theta + L = 1/2 < 1$. Further, $F_\epsilon(T) = [0,1]$ and so $\delta(F_\epsilon(T)) = 1$. This shows that conclusion of Theorem 2.4 is not true whenever T is multivalued almost contraction.

Theorem 2.6. *Let (X, d) be a metric space. If $T : X \to \mathrm{Cl}(X)$ is a generalized multivalued almost contraction, then T has a fixed point provided either (X, d) is compact and the function $f(x) = d(x, Tx)$ is lower semicontinuous or T is closed and compact.*

Proof. By Lemma 2.2, we have $\inf_{x\in X} f(x) = \inf_{x\in X} d(x, Tx) = 0$. The lower semicontinuity of the function $f(x) = d(x, Tx)$ and the compactness of X imply that the infimum is attained. Thus there exists an $x_0 \in X$ such that $f(x_0) = d(x_0, Tx_0) = 0$ and so $x_0 \in Tx_0$.

Suppose that T is closed and compact. According to Lemma 2.2, T has the approximate fixed point property. Therefore, for any $\epsilon > 0$, there exist $x_\epsilon \in X$ and $y_\epsilon \in X$ such that

$$y_\epsilon \in T(x_\epsilon) \cap B(x_\epsilon, \epsilon). \tag{2.7}$$

Now, since $Y := \mathrm{cl}(T(X))$ is compact, we may assume that $\{y_\epsilon\}$ converges to a point $z \in Y$ as $\epsilon \to 0$. Consequently, $\{x_\epsilon\}$ also converges to z as $\epsilon \to 0$. Since T is closed, then $z \in T(z)$. This completes the proof. □

Let $I : X \to X$ be a single-valued mapping and $T : X \to \mathrm{Cl}(X)$ a multivalued mapping. Then T is called a *multivalued almost I-contraction* [6, 8] if there exist constants $\theta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(Ix, Iy) + L d(Iy, Tx), \quad \forall x, y \in X. \tag{2.8}$$

We say that T is a *generalized multivalued almost I-contraction* if there exists a function $\alpha : [0,\infty) \to [0,1)$ satisfying $\limsup_{r \to t^+} \alpha(r) < 1$ for every $t \in [0,\infty)$ such that

$$H(Tx, Ty) \leq \alpha(d(Ix, Iy)) d(Ix, Iy) + L d(Iy, Tx), \quad \forall x, y \in X. \tag{2.9}$$

The mappings I and T are said to have an *approximate coincidence point property* provided

$$\inf_{x\in X} d(Ix, Tx) = 0 \tag{2.10}$$

or, equivalently, for any $\epsilon > 0$, there exists $z \in X$ such that

$$d(Iz, Tz) \le \epsilon. \tag{2.11}$$

A point $x \in X$ is called a *coincidence (common fixed) point* of I and T if $Ix \in Tx$ ($x = Ix \in Tx$).

Theorem 2.7. *Every generalized multivalued almost I-contraction in a metric space (X, d) has the approximate coincidence point property provided each Tx is I-invariant. Further, if (X, d) is compact and the function $f(x) = d(Ix, Tx)$ is lower semicontinuous, then I and T have a coincidence point.*

Proof. Let $T : X \to \mathrm{Cl}(X)$ be a generalized multivalued almost I-contraction and let $x_n \in X$ and $y_n \in T(x_n)$ be such that

$$\lim_{n\to\infty} d(Ix_n, y_n) = \inf_{x\in X} d(Ix, Tx). \tag{2.12}$$

By passing to the subsequences, if necessary, we may assume that the sequence $\{\alpha(d(Ix_n, y_n))\}$ is convergent. Then we have

$$\begin{aligned}
\inf_{x\in X} d(Ix, Tx) &\le \inf\left\{ d(Iy, Ty) : y \in \bigcup_{x\in X} Tx \right\} = \inf_{x\in X} \inf_{y\in Tx} d(Iy, Ty) \le \inf_{x\in X} \inf_{y\in Tx} H(Tx, Ty) \\
&\le \inf_{x\in X} \inf_{y\in Tx} \left[\alpha(d(Ix, Iy))d(Ix, Iy) + Ld(Iy, Tx)\right] \\
&= \inf_{x\in X} \inf_{y\in Tx} \left[\alpha(d(Ix, Iy))d(Ix, Iy)\right] \le \inf_{n\in\mathbb{N}} \left[\alpha(d(Ix_n, y_n))d(Ix_n, y_n)\right] \\
&\le \lim_{n\to\infty} \alpha(d(Ix_n, y_n)) \lim_{n\to\infty} d(Ix_n, y_n) \le \limsup_{r\to(\inf_{x\in X} d(Ix, Tx))^+} \alpha(r) \inf_{x\in X} d(Ix, Tx)
\end{aligned} \tag{2.13}$$

since each Tx is I-invariant, that is, for each $y \in Tx$, we have $Iy \in Tx$. Since

$$\limsup_{r\to(\inf_{x\in X} d(Ix, Tx))^+} \alpha(r) < 1, \tag{2.14}$$

we get $\inf_{x\in X} d(Ix, Tx) = 0$.

Further, the lower semi-continuity of the function $f(x) = d(Ix, Tx)$ and the compactness of X imply that the infimum is attained. Thus there exists $z \in X$ such that $f(z) = d(Iz, Tz) = 0$ and so $Iz \in Tz$ as required. This completes the proof. □

Corollary 2.8. *Every multivalued almost I-contraction in a metric space (X, d) has the approximate coincidence point property provided each Tx is I-invariant. Further, if (X, d) is compact and the function $f(x) = d(Ix, Tx)$ is lower semicontinuous, then I and T have a coincidence point.*

Recently, Ćirić [7] has introduced multivalued contractions and obtained some interesting results which are proper generalizations of the recent results of Klim and Wardowski [9], Feng and Liu [12], and many others. In the results to follow, we obtain approximate fixed point property for these multivalued contractions.

Theorem 2.9. *Let (X, d) be a metric space and T a multivalued mapping from X into $\mathrm{Cl}(X)$. Suppose that there exist a function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t} \varphi(r) < 1, \quad \forall t \in [0, \infty), \tag{2.15}$$

and $x_n \in X$ and $y_n \in Tx_n$ satisfying the following two conditions:

$$\begin{gathered} \lim_{n \to \infty} d(x_n, y_n) = \inf_{x \in X} d(x, Tx), \\ f(y_n) \le \varphi(f(x_n)) d(x_n, y_n), \end{gathered} \tag{2.16}$$

where $f(x) = d(x, Tx)$. Then T has the approximate fixed point property. Further, T has a fixed point provided either (X, d) is compact and the function $f(x)$ is lower semicontinuous or T is closed and compact.

Proof. Let $x_n \in X$ and $y_n \in Tx_n$ be the sequences that satisfy (2.16). By passing to subsequences, if necessary, we may assume that both of the sequences $f(x_n)$ and $\varphi(f(x_n))$ are convergent (note that $f(x_n)$ is bounded since $f(x_n) \le d(x_n, y_n)$). Then we have

$$\begin{aligned} \inf_{x \in X} f(x) = \inf_{x \in X} d(x, Tx) &\le \inf_{x \in X} \inf_{y \in Tx} d(y, Ty) \le \inf_{n \in \mathbb{N}} \inf_{y \in Tx_n} d(y, Ty) \\ &\le \inf_{n \in \mathbb{N}} d(y_n, Ty_n) \le \inf_{n \in \mathbb{N}} \varphi(f(x_n)) d(x_n, y_n) \le \lim_{n \to \infty} \varphi(f(x_n)) \lim_{n \to \infty} d(x_n, y_n) \\ &\le \limsup_{r \to (\lim_{n \to \infty} f(x_n))} \varphi(r) \inf_{x \in X} f(x). \end{aligned} \tag{2.17}$$

Since $\limsup_{r \to (\lim_{n \to \infty} f(x_n))} \varphi(r) < 1$, we get $\inf_{x \in X} f(x) = \inf_{x \in X} d(x, Tx) = 0$.

Further, the lower semi-continuity of the function $f(x) = d(x, Tx)$ and the compactness of X imply that the infimum is attained. Thus there exists $z_0 \in X$ such that $f(z_0) = d(z_0, Tz_0) = 0$ and so $z_0 \in Tz_0$.

The second assertion follows as in the proof of Theorem 2.6. This completes the proof. □

Theorem 2.10. *Let (X, d) be a metric space and T a multivalued mapping from X into $\mathrm{Cl}(X)$. Suppose that there exist a function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t+} \varphi(r) < 1, \quad \forall t \in [0, \infty), \tag{2.18}$$

and $x_n \in X$ *and* $y_n \in Tx_n$ *satisfying the following two conditions:*

$$\lim_{n\to\infty} d(x_n, y_n) = \inf_{x\in X} d(x, Tx),$$
$$f(y_n) \leq \varphi(d(x_n, y_n))d(x_n, y_n), \tag{2.19}$$

where $f(x) = d(x, Tx)$*. Then* T *has the approximate fixed point property. Further,* T *has a fixed point provided either* (X, d) *is compact and the function* $f(x)$ *is lower semicontinuous or* T *is closed and compact.*

Proof. Let $x_n \in X$ and $y_n \in Tx_n$ satisfy (2.19). By passing to subsequences, if necessary, we may assume that the sequence $\varphi(d(x_n, y_n)$ is convergent. Then we have

$$\begin{aligned} \inf_{x\in X} f(x) &= \inf_{x\in X} d(x, Tx) \leq \inf_{x\in X} \inf_{y\in Tx} d(y, Ty) \leq \inf_{n\in\mathbb{N}} \inf_{y\in Tx_n} d(y, Ty) \\ &\leq \inf_{n\in\mathbb{N}} d(y_n, Ty_n) \leq \inf_{n\in\mathbb{N}} \varphi(d(x_n, y_n))d(x_n, y_n) \\ &\leq \lim_{n\to\infty} \varphi(d(x_n, y_n)) \lim_{n\to\infty} d(x_n, y_n) \leq \limsup_{r\to(\inf_{x\in X} f(x))^+} \varphi(r) \inf_{x\in X} f(x). \end{aligned} \tag{2.20}$$

Since $\limsup_{r\to(\inf_{x\in X} f(x))^+} \varphi(r) < 1$, we get $\inf_{x\in X} f(x) = 0$.

Further, the lower semi-continuity of the function $f(x) = d(x, Tx)$ and the compactness of X imply that the infimum is attained. Thus there exists $z_0 \in X$ such that $f(z_0) = d(z_0, Tz_0) = 0$ and so $z_0 \in Tz_0$.

The second assertion follows as in the proof of Theorem 2.6. This completes the proof. □

3. Endpoints of Multivalued Nonlinear Contractions

Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a *endpoint (or stationary point)* [13] of T if $Tx = \{x\}$. We say that a multivalued mapping $T : X \to 2^X$ has the *approximate endpoint property* [4] if

$$\inf_{x\in X} \sup_{y\in Tx} d(x, y) = 0. \tag{3.1}$$

Let $I : X \to X$ be a single-valued mapping and $T : X \to \mathrm{Cl}(X)$ a multivalued contraction. We say that the mappings I and T have an *approximate endpoint property* provided

$$\inf_{x\in X} \sup_{y\in Tx} d(Ix, y) = 0. \tag{3.2}$$

A point $x \in X$ is called an *endpoint* of I and T if $Tx = \{Ix\}$.

For each $\epsilon > 0$, set

$$E_\epsilon(I,T) = \left\{ x \in X : \sup_{y \in Tx} d(Ix, y) \le \epsilon \right\}. \tag{3.3}$$

Lemma 3.1. *Let (X, d) be a metric space. Let $I : X \to X$ be a single-valued mapping such that $rd(x,y) \le d(Ix, Iy)$ for all $x, y \in X$, where $r > 0$ is a constant. If $T : X \to \mathrm{Cl}(X)$ is a multivalued almost I-contraction with $\theta + L < 1$, then*

$$\delta(E_\epsilon(I,T)) \le \frac{(2+L)\epsilon}{r(1-(\theta+L))}, \quad \forall \epsilon > 0. \tag{3.4}$$

Proof. For any $x, y \in E_\epsilon(I,T)$, we have

$$\begin{aligned} d(Ix, Iy) &= H(\{Ix\}, \{Iy\}) \\ &\le H(\{Ix\}, Tx) + H(Tx, Ty) + H(\{Iy\}, Ty) \\ &\le 2\epsilon + \theta d(Ix, Iy) + Ld(Iy, Tx) \\ &\le 2\epsilon + \theta d(Ix, Iy) + Ld(Iy, Ix) + Ld(Ix, Tx) \\ &\le \epsilon(2+L) + (\theta+L)d(Ix, Iy) \end{aligned} \tag{3.5}$$

and so

$$d(Ix, Iy) \le \frac{(2+L)\epsilon}{1-(\theta+L)}. \tag{3.6}$$

Since $rd(x,y) \le d(Ix, Iy)$, from (3.6), we have

$$\delta(E_\epsilon(I,T)) \le \frac{(2+L)\epsilon}{r(1-(\theta+L))}, \quad \forall \epsilon > 0. \tag{3.7}$$

□

The following simple example shows that under the assumptions of Lemma 3.1, $E_\epsilon(I,T)$ may be empty.

Example 3.2. Let $T : [0,1] \to \mathrm{Cl}([0,1])$ be a multivalued mapping defined by $T(x) = [0,1]$ for each $x \in [0,1]$ and I the identity mapping. Then $0 = H(T(x), T(y)) \le (1/2)d(x,y)$ and so T is a multivalued almost I-contraction with $\theta + L = 1/2 < 1$. However, $E_\epsilon(I,T) = \emptyset$ for each $0 < \epsilon < 1/2$.

Lemma 3.3. *Let (X, d) be a metric space. Let $I : X \to X$ be a continuous single-valued mapping and $T : X \to \mathrm{Cl}(X)$ a lower semicontinuous multivalued mapping. Then, for each $\epsilon > 0$, $E_\epsilon(I,T)$ is closed.*

Proof. Let $x_n \in E_\epsilon(I,T)$ be such that with $x_n \to x$ as $n \to \infty$. Let $z \in Tx$. Since T is lower semicontinuous, then there exists $z_n \in Tx_n$ such that $z_n \to z$. Since $x_n \in E_\epsilon(I,T)$, then

$\sup_{y\in Tx_n} d(Ix_n, y) \le \epsilon$ and so $d(Ix_n, z_n) \le \epsilon$. Since I is continuous, $d(x, z) \le \epsilon$. Therefore, $\sup_{y\in Tx} d(Ix, y) \le \epsilon$, that is, $x \in E_\epsilon(I, T)$. This completes the proof. □

Theorem 3.4. *Let (X, d) be a complete metric space. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \le d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to \mathrm{Cl}(X)$ be a lower semicontinuous multivalued almost I-contraction. Then I and T have a unique endpoint if and only if I and T have the approximate endpoint property.*

Proof. It is clear that, if I and T have an endpoint, then I and T have the approximate endpoint property. Conversely, suppose that I and T have the approximate endpoint property. Then

$$C_n = \left\{ x \in X : \sup_{y\in Tx} d(Ix, y) \le \frac{1}{n} \right\} \ne \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.8}$$

Also it is clear that, for each $n \in \mathbb{N}$, $C_n \supseteq C_{n+1}$. By Lemma 3.3, C_n is closed for each $n \in \mathbb{N}$. Since I and T have the approximate endpoint property, then $C_n \ne \emptyset$ for each $n \in \mathbb{N}$. Now, we show that $\lim_{n\to\infty} \delta(C_n) = 0$. To show this, let $x, y \in C_n$. Then, from Lemma 3.1,

$$\delta(C_n) = \delta(E_{1/n}(I, T)) \le \frac{(2 + L)(1/n)}{r(1 - (\theta + L))} \tag{3.9}$$

and so $\lim_{n\to\infty} \delta(C_n) = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in\mathbb{N}} C_n = \{x_0\}. \tag{3.10}$$

Thus x_0 is the unique endpoint of I and T. □

If I is the identity mapping on X, then the above result reduces to the following.

Corollary 3.5. *Let (X, d) be a metric space. If $T : X \to \mathrm{Cl}(X)$ is a multivalued almost contraction with $\theta + L < 1$, then*

$$\delta(E_\epsilon(T)) \le \frac{(2 + L)\epsilon}{1 - (\theta + L)}, \quad \forall \epsilon > 0, \tag{3.11}$$

where $E_\epsilon(T) = \{x \in X : \sup_{y\in Tx} d(x, y) \le \epsilon\}$.

Corollary 3.6. *Let (X, d) be a complete metric space. Let $T : X \to \mathrm{Cl}(X)$ be a lower semicontinuous multivalued almost contraction with $\theta + L < 1$. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Corollary 3.7 (see [4, Corollary 2.2]). *Let (X, d) be a complete metric space. Let $T : X \to \mathrm{Cl}(X)$ be a multivalued k-contraction. Then T has a unique endpoint if and only if T has the approximate endpoint property.*

Theorem 3.8. *Let (X, d) be a complete metric space and T a multivalued mapping from X into* $\mathrm{Cl}(X)$. *Suppose that there exist a function $\varphi : [0, \infty) \to [0, 1)$ such that*

$$\limsup_{r \to t^+} \varphi(r) < 1, \quad \forall t \in [0, \infty), \tag{3.12}$$

and $x_n \in X$ and $y_n \in Tx_n$ satisfying the two following conditions:

$$\begin{gathered} \lim_{n \to \infty} d(x_n, y_n) = \inf_{x \in X} F(x), \\ F(y_n) \le \varphi(d(x_n, y_n)) d(x_n, y_n), \end{gathered} \tag{3.13}$$

where $F(x) = \sup_{y \in T(x)} d(x, y)$. Then T has the approximate endpoint property. Further, T has an endpoint provided (X, d) is compact and the function $F(x)$ is lower semicontinuous.

Proof. We first prove that T has the approximate endpoint property. Let $x_n \in X$ and $y_n \in Tx_n$ that satisfy (3.13). By passing to subsequences, if necessary, we may assume that the sequence $\{\varphi(d(x_n, y_n))\}$ is convergent. Then we have

$$\begin{aligned} \inf_{x \in X} F(x) &\le \inf_{n \in \mathbb{N}} \inf_{y \in Tx_n} F(y) \le \inf_{n \in \mathbb{N}} F(y_n) \le \inf_{n \in \mathbb{N}} \varphi(d(x_n, y_n)) d(x_n, y_n) \\ &\le \lim_{n \to \infty} \varphi(d(x_n, y_n)) \lim_{n \to \infty} d(x_n, y_n) \le \limsup_{r \to (\inf_{x \in X} F(x))^+} \varphi(r) \inf_{x \in X} F(x). \end{aligned} \tag{3.14}$$

Since $\limsup_{r \to (\inf_{x \in X} F(x))^+} \varphi(r) < 1$, we get

$$\inf_{x \in X} F(x) = 0. \tag{3.15}$$

Thus T has the approximate endpoint property. The lower semi-continuity of the function $F(x)$ and the compactness of X imply that the infimum is attained. Thus there exists $z_0 \in X$ such that $F(z_0) = 0$. Therefore, $T(z_0) = \{z_0\}$. This completes the proof. □

The following theorem extends and improves Theorem 2.1 in [4].

Theorem 3.9. *Let (X, d) be a complete metric space. Let $I : X \to X$ be a continuous single-valued mapping such that $rd(x, y) \le d(Ix, Iy)$, where $r > 0$ is a constant. Let $T : X \to \mathrm{Cl}(X)$ be a multivalued mapping satisfying*

$$H(Tx, Ty) \le \psi(d(Ix, Iy)), \quad \forall x, y \in X, \tag{3.16}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function such that $\limsup_{r \to t} \psi(r) \le \psi(t)$ and $\psi(t) < t$ for each $t > 0$. Then I and T have a unique endpoint if and only if I and T have the approximate endpoint property.

Proof. It is clear that, if I and T have an endpoint, then I and T have the approximate endpoint property. Conversely, suppose that I and T have the approximate endpoint property. Then

$$C_n = \left\{ x \in X : H(\{Ix\}, Tx) = \sup_{y \in Tx} d(Ix, y) \leq \frac{1}{n} \right\} \neq \emptyset, \quad \forall n \in \mathbb{N}. \tag{3.17}$$

Also it is clear that, for each $n \in \mathbb{N}$, $C_n \supseteq C_{n+1}$. Since the mapping $x \to \sup_{y \in Tx} d(Ix, y)$ is continuous (note that I and T are continuous), we have that C_n is closed. Now we show that $\lim_{n\to\infty} \delta(C_n) = 0$. On the contrary, assume that $\lim_{n\to\infty} \delta(C_n) > 0$. Since $\delta(I(C_n)) \geq r\delta(C_n)$, then $\lim_{n\to\infty} \delta(I(C_n)) = r_0 > 0$ (note that the sequences $\{\delta(I(C_n))\}$ and $\{\delta(C_n)\}$ are nonincreasing and bounded below and then they have the limits). Let $x_{k,n}$ $y_{k,n} \in C_n$ be such that $\lim_{k\to\infty} d(Ix_{k,n}, Iy_{k,n}) = \delta(I(C_n))$. Given $x, y \in C_n$, from (3.16) and triangle inequality, we have

$$d(Ix, Iy) = H(\{Ix\}, \{Iy\}) \leq H(\{Ix\}, Tx) + H(Tx, Ty) + H(\{Iy\}, Ty) \leq \frac{2}{n} + \psi(d(Ix, Iy)). \tag{3.18}$$

Therefore, we have

$$d(Ix, Iy) - \psi(d(Ix, Iy)) \leq \frac{2}{n}, \quad \forall x, y \in C_n. \tag{3.19}$$

From (3.19), we have $0 \leq d(Ix_{k,n}, Iy_{k,n}) - \psi(d(Ix_{k,n}, Iy_{k,n})) \leq 2/n$ for each $k \in \mathbb{N}$ and so we get

$$\begin{aligned} \frac{2}{n} &\geq \liminf_{k\to\infty} \left[d(Ix_{k,n}, Iy_{k,n}) - \psi(d(Ix_{k,n}, Iy_{k,n})) \right] \\ &\geq \liminf_{k\to\infty} d(Ix_{k,n}, Iy_{k,n}) + \liminf_{k\to\infty} \left[-\psi(d(Ix_{k,n}, Iy_{k,n})) \right] \\ &= \delta(I(C_n)) - \limsup_{k\to\infty} \psi(d(Ix_{k,n}, Iy_{k,n})) \geq \delta(I(C_n)) - \psi(\delta(I(C_n))). \end{aligned} \tag{3.20}$$

Hence we have

$$0 \leq \delta(I(C_n)) - \psi(\delta(I(C_n))) \leq \frac{2}{n}, \quad \forall n \in \mathbb{N}. \tag{3.21}$$

From (3.21), we obtain

$$\lim_{n\to\infty} \left(\delta(I(C_n)) - \psi(\delta(I(C_n))) \right) = 0. \tag{3.22}$$

Since $\lim_{n\to\infty} \delta(I(C_n)) = r_0$, from (3.22), we get $\lim_{n\to\infty} \psi(\delta(I(C_n))) = r_0$. Thus

$$r_0 = \lim_{n\to\infty} \psi(\delta(I(C_n))) \leq \limsup_{r\to r_0} \psi(r) \leq \psi(r_0) < r_0, \tag{3.23}$$

which is a contradiction and so $r_0 = 0$. It follows from the Cantor intersection theorem that

$$\bigcap_{n\in\mathbb{N}} C_n = \{x_0\}. \tag{3.24}$$

Thus $H(\{Ix_0\}, T(x_0)) = \sup_{y\in Tx_0} d(Ix_0, y) = 0$ and hence $T(x_0) = \{Ix_0\}$. To prove the uniqueness of the endpoints of I and T, let x be an arbitrary endpoint of I and T. Then $H(\{Ix\}, Tx)$=0 and so $x \in \bigcap_{n\in\mathbb{N}} C_n = \{x_0\}$. Thus $x = x_0$. This completes the proof. □

From Theorem 3.9, we obtain the following improved version of the main result of [4].

Corollary 3.10. *Let (X, d) be a complete metric space. Let $T : X \to \mathrm{Cl}(X)$ be a multivalued mapping satisfying*

$$H(Tx, Ty) \le \psi(d(x, y)), \quad \forall x, y \in X, \tag{3.25}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a function such that $\limsup_{r\to t} \psi(r) \le \psi(t)$ and $\psi(t) < t$ for each $t > 0$. Then T has a unique endpoint if and only if T has the approximate endpoint property.

Example 3.11. Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Let $T : X \to \mathrm{Cl}(X)$ be a multivalued mapping defined by $Tx = [x/2, 1]$ and $\psi : [0, \infty) \to [0, \infty)$ be a function defined by

$$\psi(t) = \begin{cases} \dfrac{t}{2} & \text{if } 0 \le t \le 1, \\[2mm] \dfrac{t^2}{1+t} & \text{if } t \ge 1. \end{cases} \tag{3.26}$$

Then

$$H(Tx, Ty) = \frac{1}{2}|x - y| = \psi(|x - y|) = \psi(d(x, y)). \tag{3.27}$$

Then T and ψ satisfy the conditions of Corollary 3.10, but the conditions of Theorem 2.1 in [4] are not satisfied (note that $\lim t_{t\to\infty}(t - \psi(t)) = 1$).

Acknowledgments

The authors would like to thank the referees for their valuable suggestions to improve the paper. This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

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