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*Research Article*

# Ishikawa Iterative Process for a Pair of Single-valued and Multivalued Nonexpansive Mappings in Banach Spaces

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Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ and $T : E \to KC(E)$ be a single-valued nonexpansive mapping and a multivalued nonexpansive mapping, respectively. Assume in addition that $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ and $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We prove that the sequence of the modified Ishikawa iteration method generated from an arbitrary $x_0 \in E$ by $y_n = (1-\beta_n)x_n + \beta_n z_n$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n t y_n$, where $z_n \in Tx_n$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$, converges strongly to a common fixed point of $t$ and $T$; that is, there exists $x \in E$ such that $x = tx \in Tx$.

## 1. Introduction

Let $X$ be a Banach space, and let $E$ be a nonempty subset of $X$. We will denote by $FB(E)$ the family of nonempty bounded closed subsets of $E$ and by $KC(E)$ the family of nonempty compact convex subsets of $E$. Let $H(\cdot,\cdot)$ be the Hausdorff distance on $FB(X)$, that is,

$$H(A,B) = \max\left\{\sup_{a\in A} \mathrm{dist}(a,B),\ \sup_{b\in B} \mathrm{dist}(b,A)\right\}, \quad A,B \in FB(X), \tag{1.1}$$

where $\mathrm{dist}(a,B) = \inf\{\|a-b\| : b \in B\}$ is the distance from the point $a$ to the subset $B$.

A mapping $t : E \to E$ is said to be *nonexpansive* if

$$\|tx - ty\| \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.2}$$

A point $x$ is called a fixed point of $t$ if $tx = x$.

A multivalued mapping $T : E \to FB(X)$ is said to be *nonexpansive* if

$$H(Tx, Ty) \leq \|x - y\|, \quad \forall x, y \in E. \tag{1.3}$$

A point $x$ is called a fixed point for a multivalued mapping $T$ if $x \in Tx$.

We use the notation $\mathrm{Fix}(T)$ standing for the set of fixed points of a mapping $T$ and $\mathrm{Fix}(t) \cap \mathrm{Fix}(T)$ standing for the set of common fixed points of $t$ and $T$. Precisely, a point $x$ is called a common fixed point of $t$ and $T$ if $x = tx \in Tx$.

In 2006, S. Dhompongsa et al. [1] proved a common fixed point theorem for two nonexpansive commuting mappings.

**Theorem 1.1** (see [1, Theorem 4.2]). *Let $E$ be a nonempty bounded closed convex subset of a uniformly Banach space $X$, and let $t : E \to E$, and $T : E \to KC(E)$ be a nonexpansive mapping and a multivalued nonexpansive mapping, respectively. Assume that $t$ and $T$ are commuting; that is, if for every $x, y \in E$ such that $x \in Ty$ and $ty \in E$, there holds $tx \in Tty$. Then, $t$ and $T$ have a common fixed point.*

In this paper, we introduce an iterative process in a new sense, called the modified Ishikawa iteration method with respect to a pair of single-valued and multivalued nonexpansive mappings. We also establish the strong convergence theorem of a sequence from such process in a nonempty compact convex subset of a uniformly convex Banach space.

## 2. Preliminaries

The important property of the uniformly convex Banach space we use is the following lemma proved by Schu [2] in 1991.

**Lemma 2.1** (see [2]). *Let $X$ be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of $X$ such that $\limsup_{n\to\infty}\|x_n\| \leq a$, $\limsup_{n\to\infty}\|y_n\| \leq a$, and $\lim_{n\to\infty}\|u_n x_n + (1 - u_n)y_n\| = a$ for some $a \geq 0$. Then, $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

The following observation will be used in proving our results, and the proof is straightforward.

**Lemma 2.2.** *Let $X$ be a Banach space, and let $E$ be a nonempty closed convex subset of $X$. Then,*

$$\mathrm{dist}(y, Ty) \leq \|y - x\| + \mathrm{dist}(x, Tx) + H(Tx, Ty), \tag{2.1}$$

*where $x, y \in E$ and $T$ is a multivalued nonexpansive mapping from $E$ into $FB(E)$.*

A fundamental principle which plays a key role in ergodic theory is the demiclosedness principle. A mapping $t$ defined on a subset $E$ of a Banach space $X$ is said to be demiclosed if any sequence $\{x_n\}$ in $E$ the following implication holds: $x_n \rightharpoonup x$ and $tx_n \to y$ implies $tx = y$.

**Theorem 2.3** (see [3]). *Let $E$ be a nonempty closed convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ be a nonexpansive mapping. If a sequence $\{x_n\}$ in $E$ converges weakly to $p$ and $\{x_n - tx_n\}$ converges to 0 as $n \to \infty$, then $p \in \text{Fix}(t)$.*

In 1974, Ishikawa introduced the following well-known iteration.

*Definition 2.4* (see [4]). Let $X$ be a Banach space, let $E$ be a closed convex subset of $X$, and let $t$ be a selfmap on $E$. For $x_0 \in E$, the sequence $\{x_n\}$ of Ishikawa iterates of $t$ is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n t x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n, \quad n \geq 0, \end{aligned} \tag{2.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences.

A nonempty subset $K$ of $E$ is said to be proximinal if, for any $x \in E$, there exists an element $y \in K$ such that $\|x-y\| = \text{dist}(x, K)$. We will denote $P(K)$ by the family of nonempty proximinal bounded subsets of $K$.

In 2005, Sastry and Babu [5] defined the Ishikawa iterative scheme for multivalued mappings as follows.

Let $E$ be a compact convex subset of a Hilbert space $X$, and let $T : E \to P(E)$ be a multivalued mapping, and fix $p \in \text{Fix}(T)$.

$$\begin{aligned} x_0 &\in E, \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{aligned} \tag{2.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ with $z_n \in Tx_n$ such that $\|z_n - p\| = \text{dist}(p, Tx_n)$ and $\|z'_n - p\| = \text{dist}(p, Ty_n)$.

They also proved the strong convergence of the above Ishikawa iterative scheme for a multivalued nonexpansive mapping $T$ with a fixed point $p$ under some certain conditions in a Hilbert space.

Recently, Panyanak [6] extended the results of Sastry and Babu [5] to a uniformly convex Banach space and also modified the above Ishikawa iterative scheme as follows.

Let $E$ be a nonempty convex subset of a uniformly convex Banach space $X$, and let $T : E \to P(E)$ be a multivalued mapping

$$\begin{aligned} x_0 &\in E, \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{aligned} \tag{2.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$ with $z_n \in Tx_n$ and $u_n \in \text{Fix}(T)$ such that $\|z_n - u_n\| = \text{dist}(u_n, Tx_n)$ and $\|x_n - u_n\| = \text{dist}(x_n, \text{Fix}(T))$, respectively. Moreover, $z'_n \in Tx_n$ and $v_n \in \text{Fix}(T)$ such that $\|z'_n - v_n\| = \text{dist}(v_n, Tx_n)$ and $\|y_n - v_n\| = \text{dist}(y_n, \text{Fix}(T))$, respectively.

Very recently, Song and Wang [7, 8] improved the results of [5, 6] by means of the following Ishikawa iterative scheme.

Let $T : E \to FB(E)$ be a multivalued mapping, where $\alpha_n, \beta_n \in [0,1)$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{aligned} x_0 &\in E, \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{aligned} \tag{2.5}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$ such that $\|z_n - z'_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - z'_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$, respectively. Moreover, $\gamma_n \in (0, +\infty)$ such that $\lim_{n\to\infty}\gamma_n = 0$.

At the same period, Shahzad and Zegeye [9] modified the Ishikawa iterative scheme $\{x_n\}$ and extended the result of [7, Theorem 2] to a multivalued quasinonexpansive mapping as follows.

Let $K$ be a nonempty convex subset of a Banach space $X$, and let $T : E \to FB(E)$ be a multivalued mapping, where $\alpha_n, \beta_n \in [0,1]$. The Ishikawa iterative scheme $\{x_n\}$ is defined by

$$\begin{aligned} x_0 &\in E, \\ y_n &= (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n z'_n, \quad \forall n \geq 0, \end{aligned} \tag{2.6}$$

where $z_n \in Tx_n$ and $z'_n \in Ty_n$.

In this paper, we introduce a new iteration method modifying the above ones and call it the modified Ishikawa iteration method.

*Definition 2.5.* Let $E$ be a nonempty closed bounded convex subset of a Banach space $X$, let $t : E \to E$ be a single-valued nonexpansive mapping, and let $T : E \to FB(E)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n t y_n, \end{aligned} \tag{2.7}$$

where $x_0 \in E$, $z_n \in Tx_n$, and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

## 3. Main Results

We first prove the following lemmas, which play very important roles in this section.

**Lemma 3.1.** *Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ and $T : E \to FB(E)$ be a single-valued and a multivalued nonexpansive mapping,* *respectively, and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying* $Tw = \{w\}$ *for all* $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$*. Let* $\{x_n\}$ *be the sequence of the modified Ishikawa iteration defined by* (2.7)*. Then,* $\lim_{n\to\infty}\|x_n - w\|$ *exists for all* $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$*.*

*Proof.* Letting $x_0 \in E$ and $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, we have

$$\begin{aligned}
\|x_{n+1} - w\| &= \left\|(1-\alpha_n)x_n + \alpha_n t\big((1-\beta_n)x_n + \beta_n z_n\big) - w\right\| \\
&= \left\|(1-\alpha_n)x_n + \alpha_n t\big((1-\beta_n)x_n + \beta_n z_n\big) - (1-\alpha_n)w - \alpha_n w\right\| \\
&\le (1-\alpha_n)\|x_n - w\| + \alpha_n\left\|t\big((1-\beta_n)x_n + \beta_n z_n\big) - w\right\| \\
&\le (1-\alpha_n)\|x_n - w\| + \alpha_n\left\|(1-\beta_n)x_n + \beta_n z_n - w\right\| \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n\left\|(1-\beta_n)x_n + \beta_n z_n - (1-\beta_n)w - \beta_n w\right\| \\
&\le (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n\|z_n - w\| \\
&= (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n \operatorname{dist}(z_n, Tw) \\
&\le (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n H(Tx_n, Tw) \\
&\le (1-\alpha_n)\|x_n - w\| + \alpha_n(1-\beta_n)\|x_n - w\| + \alpha_n\beta_n\|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned} \tag{3.1}$$

Since $\{\|x_n - w\|\}$ is a decreasing and bounded sequence, we can conclude that the limit of $\{\|x_n - w\|\}$ exists. □

We can see how Lemma 2.1 is useful via the following lemma.

**Lemma 3.2.** *Let* $E$ *be a nonempty compact convex subset of a uniformly convex Banach space* $X$*, and let* $t : E \to E$ *and* $T : E \to FB(E)$ *be a single-valued and a multivalued nonexpansive mapping, respectively, and* $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ *satisfying* $Tw = \{w\}$ *for all* $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$*. Let* $\{x_n\}$ *be the sequence of the modified Ishikawa iteration defined by* (2.7)*. If* $0 < a \le \alpha_n \le b < 1$ *for some* $a,\ b \in \mathbb{R}$*, then,* $\lim_{n\to\infty}\|ty_n - x_n\| = 0$*.*

*Proof.* Let $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. By Lemma 3.1, we put $\lim_{n\to\infty}\|x_n - w\| = c$ and consider

$$\begin{aligned}
\left\|ty_n - w\right\| &\le \left\|y_n - w\right\| \\
&= \left\|(1-\beta_n)x_n + \beta_n z_n - w\right\| \\
&\le (1-\beta_n)\|x_n - w\| + \beta_n\|z_n - w\| \\
&= (1-\beta_n)\|x_n - w\| + \beta_n \operatorname{dist}(z_n, Tw) \\
&\le (1-\beta_n)\|x_n - w\| + \beta_n H(Tx_n, Tw) \\
&\le (1-\beta_n)\|x_n - w\| + \beta_n\|x_n - w\| \\
&= \|x_n - w\|.
\end{aligned} \tag{3.2}$$

Then, we have

$$\limsup_{n\to\infty}\|ty_n - w\| \leq \limsup_{n\to\infty}\|y_n - w\| \leq \limsup_{n\to\infty}\|x_n - w\| = c. \tag{3.3}$$

Further, we have

$$\begin{aligned} c &= \lim_{n\to\infty}\|x_{n+1} - w\| \\ &= \lim_{n\to\infty}\|(1-\alpha_n)x_n + \alpha_n ty_n - w\| \\ &= \lim_{n\to\infty}\|\alpha_n ty_n - \alpha_n w + x_n - \alpha_n x_n + \alpha_n w - w\| \\ &= \lim_{n\to\infty}\|\alpha_n(ty_n - w) + (1-\alpha_n)(x_n - w)\|. \end{aligned} \tag{3.4}$$

By Lemma 2.1, we can conclude that $\lim_{n\to\infty}\|(ty_n - w) - (x_n - w)\| = \lim_{n\to\infty}\|ty_n - x_n\| = 0$. □

**Lemma 3.3.** *Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ and $T : E \to FB(E)$ be a single-valued and a multivalued nonexpansive mapping, respectively, and $\mathrm{Fix}(t) \cap \mathrm{Fix}(T) \neq \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.7). If $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some $a,\ b \in \mathbb{R}$, then $\lim_{n\to\infty}\|x_n - z_n\| = 0$.*

*Proof.* Let $w \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$. We put, as in Lemma 3.2, $\lim_{n\to\infty}\|x_n - w\| = c$. For $n \geq 0$, we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|(1-\alpha_n)x_n + \alpha_n ty_n - w\| \\ &= \|(1-\alpha_n)x_n + \alpha_n ty_n - (1-\alpha_n)w - \alpha_n w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|ty_n - w\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|y_n - w\|, \end{aligned} \tag{3.5}$$

and hence

$$\begin{aligned} \|x_{n+1} - w\| - \|x_n - w\| &\leq -\alpha_n\|x_n - w\| + \alpha_n\|y_n - w\|, \\ \|x_{n+1} - w\| - \|x_n - w\| &\leq \alpha_n\left(\|y_n - w\| - \|x_n - w\|\right), \\ \frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n} &\leq \|y_n - w\| - \|x_n - w\|. \end{aligned} \tag{3.6}$$

Therefore, since $0 < a \leq \alpha_n \leq b < 1$,

$$\left(\frac{\|x_{n+1} - w\| - \|x_n - w\|}{\alpha_n}\right) + \|x_n - w\| \leq \|y_n - w\|. \tag{3.7}$$

Thus,

$$\liminf_{n\to\infty}\left\{\left(\frac{\|x_{n+1}-w\|-\|x_n-w\|}{\alpha_n}\right)+\|x_n-w\|\right\}\leq\liminf_{n\to\infty}\|y_n-w\|. \tag{3.8}$$

It follows that

$$c\leq\liminf_{n\to\infty}\|y_n-w\|. \tag{3.9}$$

Since, from (3.3), $\limsup_{n\to\infty}\|y_n-w\|\leq c$, we have

$$\begin{aligned}c&=\lim_{n\to\infty}\|y_n-w\|\\&=\lim_{n\to\infty}\|(1-\beta_n)x_n+\beta_n z_n-w\|\\&=\lim_{n\to\infty}\|(1-\beta_n)(x_n-w)+\beta_n(z_n-w)\|.\end{aligned} \tag{3.10}$$

Recall that

$$\begin{aligned}\|z_n-w\|&=\operatorname{dist}(z_n,Tw)\\&\leq H(Tx_n,Tw)\\&\leq\|x_n-w\|.\end{aligned} \tag{3.11}$$

Hence, we have

$$\limsup_{n\to\infty}\|z_n-w\|\leq\limsup_{n\to\infty}\|x_n-w\|=c. \tag{3.12}$$

Using the fact that $0<a\leq\beta_n\leq b<1$ and by (3.10), we can conclude that $\lim_{n\to\infty}\|x_n-z_n\|=0$. □

The following lemma allows us to go on.

**Lemma 3.4.** *Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t:E\to E$ and $T:E\to FB(E)$ be a single-valued and a multivalued nonexpansive mapping, respectively, and $\operatorname{Fix}(t)\cap\operatorname{Fix}(T)\neq\emptyset$ satisfying $Tw=\{w\}$ for all $w\in\operatorname{Fix}(t)\cap\operatorname{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.7). If $0<a\leq\alpha_n,\beta_n\leq b<1$, then $\lim_{n\to\infty}\|tx_n-x_n\|=0$.*

*Proof.* Consider

$$\begin{aligned} \|tx_n - x_n\| &= \left\|tx_n - ty_n + ty_n - x_n\right\| \\ &\le \left\|tx_n - ty_n\right\| + \left\|ty_n - x_n\right\| \\ &\le \left\|x_n - y_n\right\| + \left\|ty_n - x_n\right\| \\ &= \left\|x_n - (1-\beta_n)x_n - \beta_n z_n\right\| + \left\|ty_n - x_n\right\| \\ &= \left\|x_n - x_n + \beta_n x_n - \beta_n z_n\right\| + \left\|ty_n - x_n\right\| \\ &= \beta_n\|x_n - z_n\| + \left\|ty_n - x_n\right\|. \end{aligned} \tag{3.13}$$

Then, we have

$$\lim_{n\to\infty}\|tx_n - x_n\| \le \lim_{n\to\infty}\beta_n\|x_n - z_n\| + \lim_{n\to\infty}\left\|ty_n - x_n\right\|. \tag{3.14}$$

Hence, by Lemmas 3.2 and 3.3, $\lim_{n\to\infty}\|tx_n - x_n\| = 0$. □

We give the sufficient conditions which imply the existence of common fixed points for single-valued mappings and multivalued nonexpansive mappings, respectively, as follows

**Theorem 3.5.** *Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ and $T : E \to FB(E)$ be a single-valued and a multivalued nonexpansive mapping, respectively, and $\mathrm{Fix}(t)\cap\mathrm{Fix}(T) \ne \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$. Let $\{x_n\}$ be the sequence of the modified Ishikawa iteration defined by (2.7). If $0 < a \le \alpha_n, \beta_n \le b < 1$, then $x_{n_i} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$.*

*Proof.* Assume that $\lim_{n\to\infty}\|x_{n_i} - y\| = 0$. From Lemma 3.4, we have

$$0 = \lim_{n\to\infty}\|tx_{n_i} - x_{n_i}\| = \lim_{n\to\infty}\|(I-t)(x_{n_i})\|. \tag{3.15}$$

Since $I - t$ is demiclosed at 0, we have $(I-t)(y) = 0$, and hence $y = ty$, that is, $y \in \mathrm{Fix}(t)$. By Lemma 2.2 and by Lemma 3.4, we have

$$\begin{aligned} \mathrm{dist}(y, Ty) &\le \left\|y - x_{n_i}\right\| + \mathrm{dist}(x_{n_i}, Tx_{n_i}) + H(Tx_{n_i}, Ty) \\ &\le \left\|y - x_{n_i}\right\| + \|x_{n_i} - z_{n_i}\| + \left\|x_{n_i} - y\right\| \longrightarrow 0, \quad \text{as } i \to \infty. \end{aligned} \tag{3.16}$$

It follows that $y \in \mathrm{Fix}(T)$. Therefore $y \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$ as desired. □

Hereafter, we arrive at the convergence theorem of the sequence of the modified Ishikawa iteration. We conclude this paper with the following theorem.

**Theorem 3.6.** *Let $E$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and let $t : E \to E$ and $T : E \to FB(E)$ be a single-valued and a multivalued nonexpansive mapping, respectively, and $\mathrm{Fix}(t)\cap\mathrm{Fix}(T) \ne \emptyset$ satisfying $Tw = \{w\}$ for all $w \in \mathrm{Fix}(t)\cap\mathrm{Fix}(T)$. Let $\{x_n\}$ be* *the sequence of the modified Ishikawa iteration defined by* (2.7) *with* $0 < a \leq \alpha_n, \beta_n \leq b < 1$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* $t$ *and* $T$.

*Proof.* Since $\{x_n\}$ is contained in $E$ which is compact, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in E$, that is, $\lim_{i\to\infty}\|x_{n_i} - y\| = 0$. By Theorem 3.5, we have $y \in \mathrm{Fix}(t) \cap \mathrm{Fix}(T)$, and by Lemma 3.1, we have that $\lim_{n\to\infty}\|x_n - y\|$ exists. It must be the case in which $\lim_{n\to\infty}\|x_n - y\| = \lim_{i\to\infty}\|x_{n_i} - y\| = 0$. Therefore, $\{x_n\}$ converges strongly to a common fixed point $y$ of $t$ and $T$. □

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