

## Research Article

# Hierarchical Convergence of a Double-Net Algorithm for Equilibrium Problems and Variational Inequality Problems

Yonghong Yao,<sup>1</sup> Yeong-Cheng Liou,<sup>2</sup> and Chia-Ping Chen<sup>3</sup>

<sup>1</sup> Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China

<sup>2</sup> Department of Information Management, Cheng Shiu University, Kaohsiung 833, Taiwan

<sup>3</sup> Department of Computer Science and Engineering, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

Correspondence should be addressed to Chia-Ping Chen, cpchen@cse.nsysu.edu.tw

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We consider the following hierarchical equilibrium problem and variational inequality problem (abbreviated as HEVP): find a point  $x^* \in EP(F, B)$  such that  $\langle Ax^*, x - x^* \rangle \geq 0$ , for all  $x \in EP(F, B)$ , where  $A, B$  are two monotone operators and  $EP(F, B)$  is the solution of the equilibrium problem of finding  $z \in C$  such that  $F(z, y) + \langle Bz, y - z \rangle \geq 0$ , for all  $y \in C$ . We note that the problem (HEVP) includes some problems, for example, mathematical program and hierarchical minimization problems as special cases. For solving (HEVP), we propose a double-net algorithm which generates a net  $\{x_{s,t}\}$ . We prove that the net  $\{x_{s,t}\}$  hierarchically converges to the solution of (HEVP); that is, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution  $x_t \in EP(F, B)$  of the equilibrium problem, and as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^*$  of (HEVP).

## 1. Introduction

Let  $H$  be a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\| \cdot \|$ , respectively, and let  $C$  be a nonempty closed convex subset of  $H$ . Recall that a mapping  $A$  of  $C$  into  $H$  is called monotone if

$$\langle Au - Av, u - v \rangle \geq 0, \quad (1.1)$$

for all  $u, v \in C$  and  $A : C \rightarrow H$  is called  $\alpha$ -inverse strongly monotone mapping if there exists a positive real number  $\alpha$  such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad (1.2)$$

for all  $u, v \in C$ . It is obvious that any  $\alpha$ -inverse strongly monotone mapping  $A$  is monotone and  $1/\alpha$ -Lipschitz continuous.

Recently, the following problem has attracted much attention: find hierarchically a fixed point of a nonexpansive mapping  $T$  with respect to a nonexpansive mapping  $P$ , namely,

$$\text{Find } \tilde{x} \in \text{Fix}(T) \text{ such that } \langle \tilde{x} - P\tilde{x}, \tilde{x} - x \rangle \leq 0, \quad \forall x \in \text{Fix}(T). \quad (1.3)$$

Some algorithms for solving the hierarchical fixed point problem (1.3) have been introduced by many authors. For related works, please see, for instance, [1–9] and the references therein.

*Remark 1.1.* It is not hard to check that solving (1.3) is equivalent to the fixed point problem

$$\text{Find } \tilde{x} \in C \text{ such that } \tilde{x} = \text{proj}_{\text{Fix}(T)} \cdot P\tilde{x}, \quad (1.4)$$

where  $\text{proj}_{\text{Fix}(T)}$  stands for the metric projection on the closed convex set  $\text{Fix}(T)$ . By using the definition of the normal cone to  $\text{Fix}(T)$ , that is,

$$N_{\text{Fix}(T)} : x \mapsto \begin{cases} \{u \in H \mid \langle u, y - x \rangle \leq 0, \forall y \in \text{Fix}(T)\} & \text{if } x \in \text{Fix}(T), \\ \emptyset, & \text{otherwise,} \end{cases} \quad (1.5)$$

we easily prove that (1.3) is equivalent to the variational inequality

$$0 \in (I - P)\tilde{x} + N_{\text{Fix}(T)}\tilde{x}. \quad (1.6)$$

At this point, we wish to point out the link with some monotone variational inequalities and convex programming problems as follows.

*Example 1.2.* Setting  $P = I - \gamma A$ , where  $A$  is  $\eta$ -Lipschitzian and  $k$ -strongly monotone with  $\gamma \in (0, 2k/\eta^2]$ , then (1.3) reduces to

$$\text{Find } \tilde{x} \in \text{Fix}(T) \text{ such that } \langle A\tilde{x}, x - \tilde{x} \rangle \geq 0, \quad \forall x \in \text{Fix}(T), \quad (1.7)$$

a variational inequality studied by Yamada and Ogura [10].

*Example 1.3.* Let  $A$  be a maximal monotone operator. Taking  $T = J_\lambda^A := (I + \lambda A)^{-1}$  and  $P = I - \gamma \nabla \psi$ , where  $\psi$  is a convex function such that  $\nabla \psi$  is  $\eta$ -Lipschitzian (which is equivalent to the fact that  $\nabla \psi$  is  $\eta^{-1}$  cocoercive) with  $\gamma \in (0, 2/\eta]$ , and  $\text{Fix}(J_\lambda^A) = A^{-1}(0)$ . Then (1.3) reduces

to the following mathematical program with generalized equation constraint:

$$\min_{0 \in A(x)} \psi(x), \quad (1.8)$$

a problem considered by Luo et al. [11].

*Example 1.4.* Taking  $A = \partial\psi$ , where  $\partial\psi$  is the subdifferential of a lower semicontinuous convex function, then (1.8) reduces to the following hierarchical minimization problem considered in Cabot [12] and Solodov [13]:

$$\min_{x \in \arg \min \varphi} \psi(x). \quad (1.9)$$

Let  $B : C \rightarrow H$  be a nonlinear mapping, and let  $F$  be a bifunction of  $C \times C$  into  $\mathbf{R}$ . Consider the following equilibrium problem of finding  $z \in C$  such that

$$F(z, y) + \langle Bz, y - z \rangle \geq 0, \quad \forall y \in C. \quad (1.10)$$

If  $B = 0$ , then (1.10) reduces to

$$F(z, y) \geq 0, \quad \forall y \in C. \quad (1.11)$$

The solution set of equilibrium problems (1.10) and (1.11) are denoted by  $EP(F, B)$  and  $EP(F)$ , respectively. The equilibrium problem (1.10) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, fixed point problems, minimax problems, Nash equilibrium problem in noncooperative games, and others. We remind the readers to refer to [14–30] and the references therein.

Motivated and inspired by the above works, in this paper, we consider the following hierarchical equilibrium problem and variational inequality problem: find a point  $x^* \in EP(F, B)$  such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in EP(F, B), \quad (1.12)$$

where  $A, B$  are two monotone operators. The solution set of (1.12) is denoted by  $\Omega$ .

*Remark 1.5.* It is clear that the hierarchical variational inequality problem and equilibrium problem (1.12) includes the variational inequality problem studied by Yamada and Ogura [10], mathematical program studied by Luo et al. [11], hierarchical minimization problem considered by Cabot [12] and Solodov [13], as special cases.

For solving (1.12), we propose a double-net algorithm which generates a net  $\{x_{s,t}\}$ . We prove that the net  $\{x_{s,t}\}$  hierarchically converges to the solution of (1.12); that is, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution  $x_t \in EP(F, B)$  of the equilibrium problem, and as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^* \in \Omega$  of (1.12).

## 2. Preliminaries

Let  $H$  be a real Hilbert space. Throughout this paper, let us assume that a bifunction  $F : H \times H \rightarrow \mathbf{R}$  satisfies the following conditions:

- (F1)  $F(x, x) = 0$  for all  $x \in H$ ;
- (F2)  $F$  is monotone, that is,  $F(x, y) + F(y, x) \leq 0$  for all  $x, y \in H$ ;
- (F3) for each  $x, y, z \in H$ ,  $\limsup_{t \searrow 0} F(tz + (1-t)x, y) \leq F(x, y)$ ;
- (F4) for each  $x \in H$ ,  $y \mapsto F(x, y)$  is convex and lower semicontinuous.

On the equilibrium problems, we have the following important lemma. You can find it in [31].

**Lemma 2.1.** *Let  $H$  be a real Hilbert space, and let  $F$  be a bifunction of  $H \times H$  into  $\mathbf{R}$  satisfying conditions (F1)–(F4). Let  $r > 0$ , and  $x \in H$ . Then, there exists  $z \in H$  such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \forall y \in H. \quad (2.1)$$

Further, if  $T_r(x) = \{z \in H \mid F(z, y) + (1/r) \langle y - z, z - x \rangle \geq 0, \text{ for all } y \in H\}$ , then the following hold:

- (1)  $T_r$  is single-valued;
- (2)  $T_r$  is firmly nonexpansive; that is, for any  $x, y \in H$ ,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad (2.2)$$

- (3)  $\text{Fix}(T_r) = \text{EP}(F)$ ;
- (4)  $\text{EP}(F)$  is closed and convex.

Below we gather some basic facts that are needed in the argument of the subsequent sections.

**Lemma 2.2** (see [32]). *Let  $H$  be a real Hilbert space. Let the mapping  $A : H \rightarrow H$  be  $\alpha$ -inverse strongly monotone, and let  $\lambda > 0$  be a constant. Then, one has*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2, \quad \forall x, y \in H. \quad (2.3)$$

In particular, if  $0 \leq \lambda \leq 2\alpha$ , then  $I - \lambda A$  is nonexpansive.

**Lemma 2.3** (demiclosedness principle for nonexpansive mappings, see [33]). *Let  $C$  be a nonempty closed convex subset of a real Hilbert space  $H$  and let  $T : C \rightarrow C$  be a nonexpansive mapping with  $\text{Fix}(T) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in  $C$  weakly converging to  $x$ , and if  $\{(I - T)x_n\}$  converges strongly to  $y$ , then  $(I - T)x = y$ ; in particular, if  $y = 0$ , then  $x \in \text{Fix}(T)$ .*

**Lemma 2.4.** *Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let the mapping  $A : H \rightarrow H$  be  $\alpha$ -inverse strongly monotone. Let  $\lambda \in (0, 2\alpha)$ , and  $t \in (0, 1)$ . Then the variational inequality*

$$x^* \in \text{EP}(F, B), \quad \langle tf(z) + (1-t)(I - \lambda A)z - z, x^* - z \rangle \geq 0, \quad \forall z \in \text{EP}(F, B) \quad (2.4)$$

*is equivalent to the dual variational inequality*

$$x^* \in \text{EP}(F, B), \quad \langle tf(x^*) + (1-t)(I - \lambda A)x^* - x^*, x^* - z \rangle \geq 0, \quad \forall z \in \text{EP}(F, B). \quad (2.5)$$

*Proof.* Assume that  $x^* \in \text{EP}(F, B)$  solves (2.4). For all  $z \in \text{EP}(F, B)$ , set

$$x = x^* + s(z - x^*) \in \text{EP}(F, B), \quad 0 < s < 1. \quad (2.6)$$

We note that

$$\langle tf(x) + (1-t)(I - \lambda A)x - x, x^* - x \rangle \geq 0. \quad (2.7)$$

Hence, we have

$$\langle tf(x^* + s(z - x^*)) + (1-t)(I - \lambda A)(x^* + s(z - x^*)) - x^* - s(z - x^*), s(x^* - z) \rangle \geq 0, \quad (2.8)$$

which implies that

$$\langle tf(x^* + s(z - x^*)) + (1-t)(I - \lambda A)(x^* + s(z - x^*)) - x^* - s(z - x^*), x^* - z \rangle \geq 0. \quad (2.9)$$

Letting  $s \rightarrow 0$ , we have

$$\langle tf(x^*) + (1-t)(I - \lambda A)x^* - x^*, x^* - z \rangle \geq 0, \quad (2.10)$$

which is exactly (2.5).

Assume that  $x^*$  solves (2.5). Hence,

$$\langle tf(x^*) + (1-t)(I - \lambda A)x^* - x^*, x^* - z \rangle \geq 0. \quad (2.11)$$

Noting that  $I - f$  and  $A$  are monotone, we have

$$\begin{aligned} \langle (I - f)z - (I - f)x^*, z - x^* \rangle &\geq 0, \\ \langle Az - Ax^*, z - x^* \rangle &\geq 0. \end{aligned} \quad (2.12)$$

It follows that

$$t\langle (I - f)z - (I - f)x^*, z - x^* \rangle + (1-t)\lambda\langle Az - Ax^*, z - x^* \rangle \geq 0, \quad (2.13)$$

which implies that

$$\langle tf(z) + (1-t)(I - \lambda A)z - z, x^* - z \rangle \geq \langle tf(x^*) + (1-t)(I - \lambda A)x^* - x^*, x^* - z \rangle \geq 0. \quad (2.14)$$

This implies that  $x^*$  solves (2.4). The proof is completed.  $\square$

### 3. Main Results

In this section, we first introduce our double-net algorithm.

Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let the mappings  $A, B : H \rightarrow H$  be  $\alpha$ -inverse strongly monotone and  $\beta$ -inverse strongly monotone, respectively. Let  $F$  be a bifunction from  $H \times H \rightarrow \mathbf{R}$ , and let  $\lambda \in (0, 2\alpha)$  and  $r \in (0, 2\beta)$  be two constants. For  $s, t \in (0, 1)$ , we define the following mapping:

$$x \mapsto W_{s,t}x := s[tf(x) + (1-t)(x - \lambda Ax)] + (1-s)T_r(x - rBx), \quad (3.1)$$

where  $T_r(x)$  is defined by Lemma 2.1. We note that the mapping  $W_{s,t}$  is a contraction. As a matter of fact, we have

$$\begin{aligned} \|W_{s,t}x - W_{s,t}y\| &= \|s[tf(x) + (1-t)(x - \lambda Ax)] + (1-s)T_r(x - rBx) \\ &\quad - s[tf(y) + (1-t)(y - \lambda Ay)] - (1-s)T_r(y - rBy)\| \\ &\leq st\|f(x) - f(y)\| + s(1-t)\|(x - \lambda Ax) - (y - \lambda Ay)\| \\ &\quad + (1-s)\|T_r(x - rBx) - T_r(y - rBy)\| \\ &\leq st\rho\|x - y\| + s(1-t)\|x - y\| + (1-s)\|x - y\| \\ &= [1 - (1 - \rho)st]\|x - y\|, \end{aligned} \quad (3.2)$$

which implies that the mapping  $W_{s,t}$  is contractive. Hence, by Banach's contraction principle,  $W_{s,t}$  has a unique fixed point which is denoted  $x_{s,t} \in H$ ; that is,  $x_{s,t}$  is the unique solution in  $H$  of the fixed point equation

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)(x_{s,t} - \lambda Ax_{s,t})] + (1-s)T_r(x_{s,t} - rBx_{s,t}), \quad s, t \in (0, 1). \quad (3.3)$$

Below is our main result of this paper which displays the behavior of the net  $\{x_{s,t}\}$  as  $s \rightarrow 0$  and  $t \rightarrow 0$  successively.

**Theorem 3.1.** *Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let the mappings  $A, B : H \rightarrow H$  be  $\alpha$ -inverse strongly monotone and  $\beta$ -inverse strongly monotone, respectively. Let  $\lambda \in (0, 2\alpha)$  and  $r \in (0, 2\beta)$  be two constants. Let  $F$  be a bifunction from  $H \times H \rightarrow \mathbf{R}$  satisfying (F1)–(F4). Suppose the solution set  $\Omega$  of (1.12) is nonempty. Let, for each  $(s, t) \in (0, 1)^2$ ,  $x_{s,t}$  be defined implicitly by (3.3). Then, the net  $\{x_{s,t}\}$  hierarchically converges to the unique solution  $x^*$  of the hierarchical equilibrium problem and variational inequality problem (1.12). That is to say, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution*

$x_t \in \text{EP}(F, B)$  of the equilibrium problem (1.10). Moreover, as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^* \in \Omega$ . Furthermore,  $x^*$  also solves the following variational inequality:

$$x^* \in \Omega, \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \quad (3.4)$$

We divide our detailed proofs into several conclusions as follows. Throughout, we assume all assumptions of Theorem 3.1 are satisfied.

*Conclusion 1.* For each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  is bounded.

*Proof.* Take any  $z \in \text{EP}(F, B)$ . It is clear that  $z = T_r(z - rBz)$ . Set  $u_{s,t} = T_r(x_{s,t} - rBx_{s,t})$  for all  $s, t \in (0, 1)$ . Since  $T_r$ ,  $I - \lambda A$  and  $I - rB$  are nonexpansive (by Lemmas 2.1 and 2.2), we have from (3.3) that

$$\begin{aligned} \|x_{s,t} - z\| &= \|s[t f(x_{s,t}) + (1-t)(I - \lambda A)x_{s,t}] + (1-s)T_r(x_{s,t} - rBx_{s,t}) - z\| \\ &\leq s\|t f(x_{s,t}) + (1-t)(I - \lambda A)x_{s,t} - z\| + (1-s)\|T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz)\| \\ &\leq s[t\|f(x_{s,t}) - f(z)\| + t\|f(z) - z\| \\ &\quad + (1-t)\|(I - \lambda A)x_{s,t} - (I - \lambda A)z\| + (1-t)\|(I - \lambda A)z - z\|] + (1-s)\|x_{s,t} - z\| \\ &\leq s[t\rho\|x_{s,t} - z\| + t\|f(z) - z\| + (1-t)\|x_{s,t} - z\| + (1-t)\lambda\|Az\|] + (1-s)\|x_{s,t} - z\| \\ &= [1 - (1 - \rho)st]\|x_{s,t} - z\| + st\|f(z) - z\| + s(1-t)\lambda\|Az\|. \end{aligned} \quad (3.5)$$

This implies that

$$\begin{aligned} \|x_{s,t} - z\| &\leq \frac{1}{(1-\rho)t} (t\|f(z) - z\| + (1-t)\lambda\|Az\|) \\ &\leq \frac{1}{(1-\rho)t} \max\{\|f(z) - z\|, \lambda\|Az\|\}. \end{aligned} \quad (3.6)$$

It follows that for each fixed  $t \in (0, 1)$ ,  $\{x_{s,t}\}$  is bounded, so are the nets  $\{f(x_{s,t})\}$ ,  $\{(I - \lambda A)x_{s,t}\}$  and  $\{u_{s,t}\}$ . Note that we use  $M_t$  as a positive constant which bounds all bounded terms appearing in the following.  $\square$

*Conclusion 2.*  $x_{s,t} \rightarrow x_t \in \text{EP}(F, B)$  as  $s \rightarrow 0$ .

*Proof.* From Lemma 2.2, we have

$$\begin{aligned} \|x_{s,t} - \lambda Ax_{s,t} - (z - \lambda Az)\|^2 &\leq \|x_{s,t} - z\|^2 + \lambda(\lambda - 2\alpha)\|Ax_{s,t} - Az\|^2, \\ \|u_{s,t} - z\|^2 &= \|T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz)\|^2 \\ &\leq \|x_{s,t} - rBx_{s,t} - (z - rBz)\|^2 \\ &\leq \|x_{s,t} - z\|^2 + r(r - 2\beta)\|Bx_{s,t} - Bz\|^2. \end{aligned} \quad (3.7)$$

By (3.3), we have

$$\begin{aligned}
\|x_{s,t} - z\|^2 &= st\langle f(x_{s,t}) - f(z), x_{s,t} - z \rangle + st\langle f(z) - z, x_{s,t} - z \rangle \\
&\quad + s(1-t)\langle (I - \lambda A)x_{s,t} - (I - \lambda A)z, x_{s,t} - z \rangle \\
&\quad + s(1-t)\langle (I - \lambda A)z - z, x_{s,t} - z \rangle \\
&\quad + (1-s)\langle T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz), x_{s,t} - z \rangle \\
&\leq st\|f(x_{s,t}) - f(z)\|\|x_{s,t} - z\| + st\langle f(z) - z, x_{s,t} - z \rangle \\
&\quad + s(1-t)\|(I - \lambda A)x_{s,t} - (I - \lambda A)z\|\|x_{s,t} - z\| - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle \\
&\quad + (1-s)\|T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz)\|\|x_{s,t} - z\| \\
&\leq st\rho\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle \\
&\quad + s(1-t)\|(I - \lambda A)x_{s,t} - (I - \lambda A)z\|\|x_{s,t} - z\| \\
&\quad + (1-s)\|(I - rB)x_{s,t} - (I - rB)z\|\|x_{s,t} - z\| \\
&\leq st\rho\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle \\
&\quad + \frac{s(1-t)}{2}\left(\|(I - \lambda A)x_{s,t} - (I - \lambda A)z\|^2 + \|x_{s,t} - z\|^2\right) \\
&\quad + \frac{1-s}{2}\left(\|(I - rB)x_{s,t} - (I - rB)z\|^2 + \|x_{s,t} - z\|^2\right).
\end{aligned} \tag{3.8}$$

This together with (3.7) implies that

$$\begin{aligned}
\|x_{s,t} - z\|^2 &\leq st\rho\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle \\
&\quad + \frac{s(1-t)}{2}\left(\|x_{s,t} - z\|^2 + \lambda(\lambda - 2\alpha)\|Ax_{s,t} - Az\|^2 + \|x_{s,t} - z\|^2\right) \\
&\quad + \frac{1-s}{2}\left(\|x_{s,t} - z\|^2 + r(r - 2\beta)\|Bx_{s,t} - Bz\|^2 + \|x_{s,t} - z\|^2\right) \\
&= [1 - (1 - \rho)st]\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle \\
&\quad + \frac{s(1-t)}{2}\lambda(\lambda - 2\alpha)\|Ax_{s,t} - Az\|^2 + \frac{1-s}{2}r(r - 2\beta)\|Bx_{s,t} - Bz\|^2.
\end{aligned} \tag{3.9}$$

It follows that

$$\begin{aligned}
&(1-s)r(2\beta - r)\|Bx_{s,t} - Bz\|^2 \\
&\quad \leq -2(1-\rho)st\|x_{s,t} - z\|^2 + 2st\|f(z) - z\|\|x_{s,t} - z\| \\
&\quad \quad - 2s(1-t)\lambda\|Az\|\|x_{s,t} - z\| + s(1-t)\lambda(\lambda - 2\alpha)\|Ax_{s,t} - Az\|^2 \\
&\quad \longrightarrow 0 \quad \text{as } s \longrightarrow 0 \text{ for each fixed } t \in (0, 1).
\end{aligned} \tag{3.10}$$



Therefore

$$\lim_{s \rightarrow 0} \|Bx_{s,t} - Bz\| = 0. \quad (3.11)$$

Using Lemma 2.1, we obtain

$$\begin{aligned} \|u_{s,t} - z\|^2 &= \|T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz)\|^2 \\ &\leq \langle (x_{s,t} - rBx_{s,t}) - (z - rBz), u_{s,t} - z \rangle \\ &= \frac{1}{2} \left( \|(x_{s,t} - rBx_{s,t}) - (z - rBz)\|^2 + \|u_{s,t} - z\|^2 \right. \\ &\quad \left. - \|(x_{s,t} - z) - r(Bx_{s,t} - Bz) - (u_{s,t} - z)\|^2 \right) \\ &\leq \frac{1}{2} \left( \|x_{s,t} - z\|^2 + \|u_{s,t} - z\|^2 - \|(x_{s,t} - u_{s,t}) - r(Bx_{s,t} - Bz)\|^2 \right) \\ &= \frac{1}{2} \left( \|x_{s,t} - z\|^2 + \|u_{s,t} - z\|^2 - \|x_{s,t} - u_{s,t}\|^2 \right. \\ &\quad \left. + 2r \langle x_{s,t} - u_{s,t}, Bx_{s,t} - Bz \rangle - r^2 \|Bx_{s,t} - Bz\|^2 \right), \end{aligned} \quad (3.12)$$

which implies that

$$\begin{aligned} \|u_{s,t} - z\|^2 &\leq \|x_{s,t} - z\|^2 - \|x_{s,t} - u_{s,t}\|^2 + 2r \langle x_{s,t} - u_{s,t}, Bx_{s,t} - Bz \rangle - r^2 \|Bx_{s,t} - Bz\|^2 \\ &\leq \|x_{s,t} - z\|^2 - \|x_{s,t} - u_{s,t}\|^2 + 2r \|x_{s,t} - u_{s,t}\| \|Bx_{s,t} - Bz\|. \end{aligned} \quad (3.13)$$

From (3.3), we have

$$\begin{aligned} \|x_{s,t} - z\| &= \|(1-s)(u_{s,t} - z) + s[tf(x_{s,t}) + (1-t)(x_{s,t} - \lambda Ax_{s,t}) - z]\| \\ &\leq \|u_{s,t} - z\| + sM_t. \end{aligned} \quad (3.14)$$

Hence,

$$\begin{aligned} \|x_{s,t} - z\|^2 &\leq \|u_{s,t} - z\|^2 + sM_t \\ &\leq \|x_{s,t} - z\|^2 - \|x_{s,t} - u_{s,t}\|^2 + M_t \|Bx_{s,t} - Bz\| + sM_t. \end{aligned} \quad (3.15)$$

It follows that

$$\|x_{s,t} - u_{s,t}\|^2 \leq M_t \|Bx_{s,t} - Bz\| + sM_t \longrightarrow 0 \quad \text{as } s \longrightarrow 0 \text{ for each fixed } t \in (0, 1). \quad (3.16)$$

Next, we show that, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  is relatively norm-compact as  $s \rightarrow 0$ . It follows from (3.8) that

$$\begin{aligned}
\|x_{s,t} - z\|^2 &= st\langle f(x_{s,t}) - f(z), x_{s,t} - z \rangle + st\langle f(z) - z, x_{s,t} - z \rangle \\
&\quad + s(1-t)\langle (I - \lambda A)x_{s,t} - (I - \lambda A)z, x_{s,t} - z \rangle \\
&\quad + s(1-t)\langle (I - \lambda A)z - z, x_{s,t} - z \rangle \\
&\quad + (1-s)\langle T_r(x_{s,t} - rBx_{s,t}) - T_r(z - rBz), x_{s,t} - z \rangle \tag{3.17} \\
&\leq st\rho\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle + s(1-t)\|x_{s,t} - z\|^2 \\
&\quad + s(1-t)\langle (I - \lambda A)z - z, x_{s,t} - z \rangle + (1-s)\|x_{s,t} - z\|^2 \\
&= [1 - (1-\rho)st]\|x_{s,t} - z\|^2 + st\langle f(z) - z, x_{s,t} - z \rangle - s(1-t)\lambda\langle Az, x_{s,t} - z \rangle.
\end{aligned}$$

It turns out that

$$\|x_{s,t} - z\|^2 \leq \frac{1}{(1-\rho)t} \langle tf(z) + (1-t)(I - \lambda A)z - z, x_{s,t} - z \rangle, \quad z \in \text{EP}(F, B). \tag{3.18}$$

Assume that  $\{s_n\} \subset (0, 1)$  is such that  $s_n \rightarrow 0$  as  $n \rightarrow \infty$ . By (3.18), we conclude immediately that

$$\|x_{s_n,t} - z\|^2 \leq \frac{1}{(1-\rho)t} \langle tf(z) + (1-t)(I - \lambda A)z - z, x_{s_n,t} - z \rangle, \quad z \in \text{EP}(F, B). \tag{3.19}$$

Since  $\{x_{s_n,t}\}$  is bounded, without loss of generality, we may assume that as  $s_n \rightarrow 0$ ,  $\{x_{s_n,t}\}$  converges weakly to a point  $x_t$ . Note that  $\{u_{s_n,t}\}$  also converges weakly to a point  $x_t$ .

Now we show that  $x_t \in \text{EP}$ . Since  $u_{s_n,t} = T_r(x_{s_n,t} - rBx_{s_n,t})$ , for any  $y \in H$ , we have

$$F(u_{s_n,t}, y) + \frac{1}{r} \langle y - u_{s_n,t}, u_{s_n,t} - (x_{s_n,t} - rBx_{s_n,t}) \rangle \geq 0. \tag{3.20}$$

From the monotonicity of  $F$ , we have

$$\frac{1}{r} \langle y - u_{s_n,t}, u_{s_n,t} - (x_{s_n,t} - rBx_{s_n,t}) \rangle \geq F(y, u_{s_n,t}), \quad \forall y \in H. \tag{3.21}$$

Hence,

$$\left\langle y - u_{s_n,t}, \frac{u_{s_n,t} - x_{s_n,t}}{r} + Bx_{s_n,t} \right\rangle \geq F(y, u_{s_n,t}), \quad \forall y \in H. \tag{3.22}$$

Put  $z_k = ky + (1 - k)x_t$  for all  $k \in (0, 1]$  and  $y \in H$ . From (3.22), we have

$$\begin{aligned} \langle z_k - u_{s_{n_i}, t}, Bz_k \rangle &\geq \langle z_k - u_{s_{n_i}, t}, Bz_k \rangle - \left\langle z_k - u_{s_{n_i}, t}, \frac{u_{s_{n_i}, t} - x_{s_{n_i}, t}}{r} + Bx_{s_{n_i}, t} \right\rangle + F(z_k, u_{s_{n_i}, t}) \\ &= \langle z_k - u_{s_{n_i}, t}, Bz_k - Bu_{s_{n_i}, t} \rangle + \langle z_k - u_{s_{n_i}, t}, Bu_{s_{n_i}, t} - Bx_{s_{n_i}, t} \rangle \\ &\quad - \left\langle z_k - u_{s_{n_i}, t}, \frac{u_{s_{n_i}, t} - x_{s_{n_i}, t}}{r} \right\rangle + F(z_k, u_{s_{n_i}, t}). \end{aligned} \quad (3.23)$$

Note that  $\|Bu_{s_{n_i}, t} - Bx_{s_{n_i}, t}\| \leq (1/\beta)\|u_{s_{n_i}, t} - x_{s_{n_i}, t}\| \rightarrow 0$ . Further, from monotonicity of  $B$ , we have  $\langle z_k - u_{s_{n_i}, t}, Bz_k - Bu_{s_{n_i}, t} \rangle \geq 0$ . Letting  $i \rightarrow \infty$  in (3.23), we have

$$\langle z_k - x_t, Bz_k \rangle \geq F(z_k, x_t). \quad (3.24)$$

From (F1), (F4), and (3.24), we also have

$$\begin{aligned} 0 = F(z_k, z_k) &\leq kF(z_k, y) + (1 - k)F(z_k, x_t) \\ &\leq kF(z_k, y) + (1 - k)\langle z_k - x_t, Bz_k \rangle \\ &= kF(z_k, y) + (1 - k)k\langle y - x_t, Bz_k \rangle, \end{aligned} \quad (3.25)$$

and hence

$$0 \leq F(z_k, y) + (1 - k)\langle Bz_k, y - x_t \rangle. \quad (3.26)$$

Letting  $k \rightarrow 0$  in (3.26), we have, for each  $y \in H$ ,

$$0 \leq F(x_t, y) + \langle y - x_t, Bx_t \rangle. \quad (3.27)$$

This implies that  $x_t \in \text{EP}(F, B)$ .

We can then substitute  $x_t$  for  $z$  in (3.19) to get

$$\|x_{s_{n,t}} - x_t\|^2 \leq \frac{1}{(1 - \rho)t} \langle tf(x_t) + (1 - t)(I - \lambda A)x_t - x_t, x_{s_{n,t}} - x_t \rangle. \quad (3.28)$$

Consequently, the weak convergence of  $\{x_{s_{n,t}}\}$  to  $x_t$  actually implies that  $x_{s_{n,t}} \rightarrow x_t$  strongly. This has proved the relative norm-compactness of the net  $\{x_{s,t}\}$  as  $s \rightarrow 0$ .

Now we return to (3.19) and take the limit, as  $n \rightarrow \infty$ , to get

$$\|x_t - z\|^2 \leq \frac{1}{(1 - \rho)t} \langle tf(z) + (1 - t)(I - \lambda A)z - z, x_t - z \rangle, \quad \forall z \in \text{EP}(F, B). \quad (3.29)$$

In particular,  $x_t$  solves the following variational inequality:

$$x_t \in \text{EP}(F, B), \quad \langle tf(z) + (1-t)(I - \lambda A)z - z, x_t - z \rangle \geq 0, \quad \forall z \in \text{EP}(F, B), \quad (3.30)$$

or the equivalent dual variational inequality (see Lemma 2.4)

$$x_t \in \text{EP}(F, B), \quad \langle tf(x_t) + (1-t)(I - \lambda A)x_t - x_t, x_t - z \rangle \geq 0, \quad \forall z \in \text{EP}(F, B). \quad (3.31)$$

Notice that (3.31) is equivalent to the fact that  $x_t = P_{\text{EP}(F, B)}(tf + (1-t)(I - \lambda A))x_t$ . That is,  $x_t$  is the unique element in  $\text{EP}(F, B)$  of the contraction  $P_{\text{EP}(F, B)}(tf + (1-t)(I - \lambda A))$ . Clearly, this is sufficient to conclude that the entire net  $\{x_{s,t}\}$  converges in norm to  $x_t \in \text{EP}(F, B)$  as  $s \rightarrow 0$ .  $\square$

*Conclusion 3.* The net  $\{x_t\}$  is bounded.

*Proof.* In (3.31), we take any  $y \in \Omega$  to deduce

$$\langle tf(x_t) + (1-t)(I - \lambda A)x_t - x_t, x_t - y \rangle \geq 0. \quad (3.32)$$

By virtue of the monotonicity of  $A$  and the fact that  $y \in \Omega$ , we have

$$\langle (I - \lambda A)x_t - x_t, x_t - y \rangle \leq \langle (I - \lambda A)y - y, x_t - y \rangle \leq 0. \quad (3.33)$$

It follows from (3.32) and (3.33) that

$$\langle f(x_t) - x_t, x_t - y \rangle \geq 0, \quad \forall y \in \Omega. \quad (3.34)$$

Hence,

$$\begin{aligned} \|x_t - y\|^2 &\leq \langle x_t - y, x_t - y \rangle + \langle f(x_t) - x_t, x_t - y \rangle \\ &= \langle f(x_t) - f(y), x_t - y \rangle + \langle f(y) - y, x_t - y \rangle \\ &\leq \rho \|x_t - y\|^2 + \langle f(y) - y, x_t - y \rangle. \end{aligned} \quad (3.35)$$

Therefore,

$$\|x_t - y\|^2 \leq \frac{1}{1-\rho} \langle f(y) - y, x_t - y \rangle, \quad \forall y \in \Omega. \quad (3.36)$$

In particular,

$$\|x_t - y\| \leq \frac{1}{1-\rho} \|f(y) - y\|, \quad \forall t \in (0, 1), \quad (3.37)$$

which implies that  $(x_t)$  is bounded.  $\square$

*Conclusion 4.* The net  $x_t \rightarrow x^* \in \Omega$  which solves the variational inequality VI (3.4).

*Proof.* First, we note that the solution of the variational inequality VI (3.4) is unique.

We next prove that  $\omega_w(x_t) \subset \Omega$ ; namely, if  $(t_n)$  is a null sequence in  $(0, 1)$  such that  $x_{t_n} \rightarrow x'$  weakly as  $n \rightarrow \infty$ , then  $x' \in \Omega$ . To see this, we use (3.31) to get

$$\langle \lambda Ax_t, z - x_t \rangle \geq \frac{t}{1-t} \langle (I-f)x_t, x_t - z \rangle, \quad z \in \text{EP}(F, B). \quad (3.38)$$

However, since  $A$  is monotone,

$$\langle Az, z - x_t \rangle \geq \langle Ax_t, z - x_t \rangle. \quad (3.39)$$

Combining the last two relations yields

$$\langle \lambda Az, z - x_t \rangle \geq \frac{t}{1-t} \langle (I-f)x_t, x_t - z \rangle, \quad z \in \text{EP}(F, B). \quad (3.40)$$

Letting  $t = t_n \rightarrow 0$  as  $n \rightarrow \infty$  in (3.40), we get

$$\langle Az, z - x' \rangle \geq 0, \quad z \in \text{EP}(F, B), \quad (3.41)$$

which is equivalent to its dual variational inequality

$$\langle Ax', z - x' \rangle \geq 0, \quad z \in \text{EP}(F, B). \quad (3.42)$$

Namely,  $x'$  is a solution of VI (1.12); hence,  $x' \in \Omega$ .

We further prove that  $x' = x^*$ , the unique solution of VI (3.4). As a matter of fact, we have by (3.36)

$$\|x_{t_n} - x'\|^2 \leq \frac{1}{1-\rho} \langle f(x') - x', x_{t_n} - x' \rangle, \quad x' \in \Omega. \quad (3.43)$$

Therefore, the weak convergence to  $x'$  of  $\{x_{t_n}\}$  implies that  $x_{t_n} \rightarrow x'$  in norm. Now we can let  $t = t_n \rightarrow 0$  in (3.36) to get

$$\langle f(x') - x', y - x' \rangle \leq 0, \quad \forall y \in \Omega. \quad (3.44)$$

It turns out that  $x' \in \Omega$  solves VI (3.4). By uniqueness, we have  $x' = x^*$ . This is sufficient to guarantee that  $x_t \rightarrow x^*$  in norm, as  $t \rightarrow 0$ . The proof is complete.  $\square$

*Proof.* By Conclusions 1–4, the proof of Theorem 3.1 is completed.  $\square$

Take  $B = 0$ . Then (1.12) reduces to the following: find a point  $x^* \in \text{EP}(F)$  such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{EP}(F). \quad (3.45)$$

The solution of (3.45) is denoted by  $\Omega_1$ .

**Corollary 3.2.** *Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let the mapping  $A : H \rightarrow H$  be  $\alpha$ -inverse strongly monotone. Let  $\lambda \in (0, 2\alpha)$  be a constant. Let  $F$  be a bifunction from  $H \times H \rightarrow \mathbf{R}$  satisfying (F1)–(F4). Suppose the solution set  $\Omega_1$  is nonempty. Let, for each  $(s, t) \in (0, 1)^2$ ,  $x_{s,t}$  be defined implicitly by*

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)(x_{s,t} - \lambda Ax_{s,t})] + (1-s)T_r(x_{s,t}), \quad s, t \in (0, 1). \quad (3.46)$$

*Then, the net  $\{x_{s,t}\}$  hierarchically converges to the unique solution  $x^*$  of the hierarchical equilibrium problem and variational inequality problem (3.45). That is to say, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution  $x_t \in \text{EP}(F)$  of the equilibrium problem (1.11). Moreover, as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^* \in \Omega_1$ . Furthermore,  $x^*$  solves the following variational inequality:*

$$x^* \in \Omega_1, \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega_1. \quad (3.47)$$

Taking  $A = 0$  in Theorem 3.1, we have the following corollary.

**Corollary 3.3.** *Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let the mapping  $B : H \rightarrow H$  be  $\beta$ -inverse strongly monotone. Let  $r \in (0, 2\beta)$  be a constant. Let  $F$  be a bifunction from  $H \times H \rightarrow \mathbf{R}$  satisfying (F1)–(F4). Suppose that the solution set  $\text{EP}(F, B)$  of (1.10) is nonempty. Let, for each  $(s, t) \in (0, 1)^2$ ,  $x_{s,t}$  be defined implicitly by*

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)x_{s,t}] + (1-s)T_r(x_{s,t} - rBx_{s,t}), \quad s, t \in (0, 1). \quad (3.48)$$

*Then, the net  $\{x_{s,t}\}$  hierarchically converges to the unique solution  $x^*$  of the equilibrium problem (1.10). That is to say, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution  $x_t \in \text{EP}(F, B)$  of the equilibrium problem (1.10). Moreover, as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^* \in \text{EP}(F, B)$ . Furthermore,  $x^*$  solves the following variational inequality:*

$$x^* \in \text{EP}(F, B), \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{EP}(F, B). \quad (3.49)$$

Taking  $A = B = 0$  in Theorem 3.1, we have the following corollary.

**Corollary 3.4.** *Let  $H$  be a real Hilbert space. Let  $f : H \rightarrow H$  be a  $\rho$ -contraction with coefficient  $\rho \in [0, 1)$ . Let  $F$  be a bifunction from  $H \times H \rightarrow \mathbf{R}$  satisfying (F1)–(F4). Suppose the solution set  $\text{EP}(F)$  of (1.11) is nonempty. Let, for each  $(s, t) \in (0, 1)^2$ ,  $x_{s,t}$  be defined implicitly by*

$$x_{s,t} = s[tf(x_{s,t}) + (1-t)x_{s,t}] + (1-s)T_r(x_{s,t}), \quad s, t \in (0, 1). \quad (3.50)$$

Then, the net  $\{x_{s,t}\}$  hierarchically converges to the unique solution  $x^*$  of the equilibrium problem (1.11). That is to say, for each fixed  $t \in (0, 1)$ , the net  $\{x_{s,t}\}$  converges in norm, as  $s \rightarrow 0$ , to a solution  $x_t \in \text{EP}(F)$  of the equilibrium problem (1.11). Moreover, as  $t \rightarrow 0$ , the net  $\{x_t\}$  converges in norm to the unique solution  $x^* \in \text{EP}(F)$ . Furthermore,  $x^*$  solves the following variational inequality:

$$x^* \in \text{EP}(F), \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{EP}(F). \quad (3.51)$$

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