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Research Article

Random Periodic Point and Fixed Point Results for Random Monotone Mappings in Ordered Polish Spaces

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The measurability of order continuous random mappings in ordered Polish spaces is studied. Using order continuity, some random fixed point theorems and random periodic point theorems for increasing, decreasing, and mixed monotone random mappings are presented.

1. Introduction and Preliminaries

The study of random fixed points forms a central topic in probabilistic functional analysis. It was initiated by Špaček [1], Hanš [2], and Wang [3]. Some random fixed point theorems play an important role in the theory of random differential and random integral equations (see Bharucha-Reid [4, 5]). Since the recent 30 years, many interesting random fixed point theorems and applications have been developed, for example, see Beg and Shahzad [6, 7], Beg and Abbas [8], Chang [9], Ding [10], Fierro et al. [11], Itoh [12], Li and Duan [13], O'Regan et al. [14], Xiao and Tao [15], Xu [16], and Zhu and Xu [17].

In 1976, Caristi [18] introduced a partial ordering in metric spaces by a function and proved the famous Caristi fixed point theorem, which is one of the most important results in nonlinear analysis. From then on, there appeared many papers concerning fixed point theory and abstract monotone iterative technique in ordered metric spaces or ordered Banach spaces. In particular, some useful fixed point theorems for monotone mappings were proved by Zhang [19], Guo and Lakshmikantham [20], and Bhaskar and Lakshmikantham [21] under some weak assumptions.

In this paper, motivated by ideas in [18–21], we study random version of fixed point theorems for increasing, decreasing, and mixed monotone random mappings in ordered Polish spaces. In Section 2, we introduce order continuous random mapping and discuss its measurability. A well-known result is generalized (see Remark 2.4). In Sections 3–5, we present some existence results of random periodic point and fixed point for increasing, decreasing and, mixed monotone random mappings, respectively.

We begin with some definitions that are essential for this work. Let (X, d) be a metric space and $\mathcal{B}_X$ be a Borel algebra of X, where d is a metric function on X. If X is separable and complete, then (X, d) is called a Polish space. We denote by $(\Omega, \mathcal{A}, P)$ a complete probability measure space (briefly, a measure space), where $(\Omega, \mathcal{A})$ is a measurable space, $\mathcal{A}$ is a sigma algebra of subsets of Ω, and P is a probability measure. The notation "a.e." stands for "almost every."

Definition 1.1 (see [3, 5, 9, 12]). A mapping $y : \Omega \to X$ is said to be measurable if

$$y^{-1}(G) = \{\omega \in \Omega : y(\omega) \in G\} \in \mathcal{A} \tag{1.1}$$

for each open subset G of X. A measurable mapping is also called a random variable. A mapping $T : \Omega \times X \to X$ is called a random mapping, if for each fixed $x \in X$, the mapping $T(\cdot, x) : \Omega \to X$ is measurable. A random mapping is said to be continuous, if for $\omega \in \Omega$ a.e., the mapping $T(\omega, \cdot) : X \to X$ is continuous. A measurable mapping $y : \Omega \to X$ is said to be a random fixed point of the random mapping $T : \Omega \times X \to X$, if $T(\omega, y(\omega)) = y(\omega)$, for $\omega \in \Omega$ a.e. Let 2^X be the family of all nonempty subsets of X and $F : \Omega \to 2^X$ a set-valued mapping. F is said to be measurable, if

$$F^{-1}(G) = \{\omega \in \Omega : F(\omega) \cap G \neq \emptyset\} \in \mathcal{A} \tag{1.2}$$

for each open subset G of X. A mapping $y : \Omega \to X$ is said to be a measurable selection of a measurable mapping $F : \Omega \to 2^X$, if y is measurable and $y(\omega) \in F(\omega)$ a.e.

We denote by $F_R(T)$ the set of all random fixed points of a random mapping T. If k is a positive integer and $u \in F_R(T^k)$, then u is a random k-periodic points of a random mapping T. By $T^n(\omega, x)$ we denote the nth iterate $T(\omega, T(\omega, T(\dots, T(\omega, x))))$ of T, where $T^0 = I$, $I : \Omega \times X \to X$ is defined by $I(\omega, x) = x$.

Lemma 1.2 (see [3, 22]). *Let (X, d) be a Polish space and $(\Omega, \mathcal{A}, P)$ a measure space. Let $T : \Omega \times X \to X$ be a continuous random mapping. If $y : \Omega \to X$ is measurable, then $T(\omega, y(\omega)) : \Omega \to X$ is measurable.*

Lemma 1.3 (see [3, 4]). *Let (X, d) be a Polish space and $(\Omega, \mathcal{A}, P)$ a measure space. If $\{y_n(\omega)\}$ is a sequence of measurable mappings in X and $\lim_{n\to\infty} y_n(\omega) = y(\omega) \in X$ a.e., then $y : \Omega \to X$ is measurable.*

Lemma 1.4 (cf. [23]). *Let (X, d) be a Polish space and $(\Omega, \mathcal{A}, P)$ a measure space. Let $F : \Omega \to 2^X$ be a set-valued mapping. Then,*

(1) *F is measurable if and only if Graph $F = \{(\omega, x) : x \in F(\omega)\}$ is $\mathcal{A} \times \mathcal{B}_X$ measurable;*

(2) *if F is measurable and $F(\omega)$ is closed a.e., then there exists a measurable selection of F.*

Lemma 1.5 (see [18]). *Let (X, d) be a metric space and $\phi : X \to \mathbb{R}$ a functional. Then the relation $\preceq$ on X defined by*

$$x \preceq y \Longleftrightarrow d(x, y) \leq \phi(x) - \phi(y), \quad x, y \in X, \tag{1.3}$$

is a partial ordering.

By Lemma 1.5, if $\preceq$ is the partial ordering induced by ϕ, then $x \preceq y$ implies $\phi(x) \geq \phi(y)$. If (X, d) is a Polish space and $\preceq$ is the partial ordering induced by ϕ, then (X, d, ϕ) is called an ordered Polish space. If $x_0, y_0 \in X$ and $x_0 \preceq y_0$, then $[x_0, y_0] = \{x \in X : x_0 \preceq x \preceq y_0\}$ is called an order interval in X.

Definition 1.6 (cf. [19]). Let (X, d, ϕ) be an ordered Polish space and $(\Omega, \mathcal{A}, P)$ a measure space. Let $T : \Omega \times X \to X$ is a random mapping. T is is said to be increasing if

$$x \preceq y \Longrightarrow T(\omega, x) \preceq T(\omega, y), \quad \forall \omega \in \Omega \text{ a.e.}; \tag{1.4}$$

T is said to be decreasing if

$$x \preceq y \Longrightarrow T(\omega, x) \succeq T(\omega, y), \quad \forall \omega \in \Omega \text{ a.e.}; \tag{1.5}$$

a random mapping $S : \Omega \times X \times X \to X$ is said to be mixed monotone if

$$x_1 \preceq y_1, \qquad y_2 \preceq x_2 \Longrightarrow S(\omega, x_1, x_2) \preceq S(\omega, y_1, y_2), \quad \forall \omega \in \Omega \text{ a.e.} \tag{1.6}$$

It is evident that, if $S : \Omega \times X \times X \to X$ is mixed monotone, then $S(\cdot, \cdot, x) : \Omega \times X \to X$ is increasing and $S(\cdot, x, \cdot) : \Omega \times X \to X$ is decreasing, for every fixed $x \in X$.

2. Measurability of Order Continuous Random Mappings

Definition 2.1. Let (X, d, ϕ) be an ordered Polish space and $(\Omega, \mathcal{A}, P)$ a measure space. Let $T : \Omega \times X \to X$ be a random mapping. T is said to be order continuous if for every monotone sequence $\{x_n\}$,

$$x_n \longrightarrow x \Longrightarrow T(\omega, x_n) \longrightarrow T(\omega, x), \quad \forall \omega \in \Omega \text{ a.e.} \tag{2.1}$$

T is is said to be order contractive if there exists $\alpha(\omega) \in [0, 1)$ such that

$$x \preceq y \Longrightarrow d(T(\omega, x), T(\omega, y)) \leq \alpha(\omega) d(x, y), \quad \forall \omega \in \Omega \text{ a.e.} \tag{2.2}$$

It is evident that continuity implies order continuity. If $T : \Omega \times X \to X$ is order contractive, then T is order continuous. A mixed monotone random mapping $S : \Omega \times X \times X \to X$ is said to be order continuous if and only if for monotone sequences $\{x_n\}$ and $\{y_n\}$,

$$x_n \longrightarrow x, y_n \longrightarrow y \Longrightarrow T(\omega, x_n, y_n) \longrightarrow T(\omega, x, y), \quad \forall \omega \in \Omega \text{ a.e.} \tag{2.3}$$

Example 2.2. Let $\Omega = [1,2]$ and $X = \mathbb{R}^2$. Let $\phi : X \to \mathbb{R}$ and $T : \Omega \times X \to X$ be defined by

$$\phi((x_1, x_2)) = -(x_1 + x_2), \qquad T(\omega, (x_1, x_2)) = \begin{cases} (0,0), & \text{if } x_1 x_2 \le 0; \\ (\omega, \omega), & \text{if } x_1 x_2 > 0. \end{cases} \tag{2.4}$$

It is easy to check that T is order continuous, but T is not continuous at $(0,0)$.

Now we prove the following theorem which plays an important role in the sequel.

Theorem 2.3. *Let (X, d, ϕ) be an ordered Polish space and $(\Omega, \mathcal{A}, P)$ a measure space, where ϕ is continuous. Let $T : \Omega \times X \to X$ be an order continuous random mapping. If $y : \Omega \to X$ is measurable, then $T(\omega, y(\omega)) : \Omega \to X$ is measurable.*

Proof. Let $E(\omega) = \{x \in X : x \preceq y(\omega)\}$, $H(\omega) = \{x \in X : y(\omega) \preceq x\}$, and $Q_\varepsilon(\omega) = \{x \in X : d(x, y(\omega)) \le \varepsilon\}$, where $\varepsilon > 0$. Clearly, $E(\omega)$, $H(\omega)$, and $Q_\varepsilon(\omega)$ are all nonempty subsets of X for all $\omega \in \Omega$. Since d is continuous, $Q_\varepsilon(\omega)$ is closed for all $\omega \in \Omega$. Let $\{x_n\}_{n=1}^{\infty} \subset E(\omega)$ and $x_n \to x_0 (n \to \infty)$. Then, from $x_n \preceq y(\omega)$, we have

$$d(x_n, y(\omega)) \le \phi(x_n) - \phi(y(\omega)). \tag{2.5}$$

Since ϕ is continuous, we have $d(x_0, y(\omega)) \le \phi(x_0) - \phi(y(\omega))$, that is, $x_0 \preceq y(\omega)$. This shows that $x_0 \in E(\omega)$, and so $E(\omega)$ is closed for all $\omega \in \Omega$. Similarly, $H(\omega)$ is closed for all $\omega \in \Omega$. We claim that

$$E, H, Q_\varepsilon : \Omega \longrightarrow 2^X \text{ are all measurable.} \tag{2.6}$$

In fact, if $B_x = \{y \in X : x \preceq y\}$, then B_x is a closed subset of X. Let G be an open subset of X, $W = X \backslash G$, and $E_{-1}(W) = \{\omega \in \Omega : E(\omega) \subset W\}$. Then, we have

$$\begin{aligned} E_{-1}(W) &= \{\omega \in \Omega : x \preceq y(\omega),\ x \in W\} = \bigcap_{x \in W} \{\omega \in \Omega : x \preceq y(\omega)\} \\ &= \bigcap_{x \in W} y^{-1}(B_x) = y^{-1}\left(\bigcap_{x \in W} B_x\right). \end{aligned} \tag{2.7}$$

Since y is measurable and $\bigcap_{x \in W} B_x$ is closed, $E_{-1}(W)$ is measurable. From $E^{-1}(G) = \Omega \setminus E_{-1}(W)$, we see that $E^{-1}(G)$ is measurable. Hence, E is measurable. Similarly, H is measurable. Now we prove that Q_ε is measurable. Since d is continuous and y is measurable, $d(x, y(\omega)) : \Omega \times X \to \mathbb{R}$ is measurable. Note that

$$\operatorname{Graph} Q_\varepsilon = \{(\omega, x) : x \in Q_\varepsilon(\omega)\} = \{(\omega, x) : d(x, y(\omega)) \le \varepsilon\} \tag{2.8}$$

is $\mathcal{A} \times \mathcal{B}_X$ measurable. Using Lemma 1.4(1), we obtain that Q_ε is measurable. Therefore, (2.6) holds. Let $F_1(\omega) = \{x \in X : x \preceq y(\omega), d(x, y(\omega)) \le 1\}$. Then, $F_1(\omega) = E(\omega) \cap Q_1(\omega)$ is nonempty and closed for all $\omega \in \Omega$. By (2.6), F_1 is measurable. By Lemma 1.4(2), we can take $y_1(\omega) \in F_1(\omega)$, where $y_1 : \Omega \to X$ is measurable. For $n = 2, 3, \ldots$, let

$$F_n(\omega) = \left\{ x \in X : y_{n-1}(\omega) \preceq x \preceq y(\omega), d\left(x, y(\omega)\right) \le \frac{1}{n} \right\}. \tag{2.9}$$

Then, $F_n(\omega)$ is nonempty and closed for all $\omega \in \Omega$. When y_{n-1} is measurable, from (2.6), we obtain that F_n is measurable. Using Lemma 1.4(2), we can take $y_n(\omega) \in F_n(\omega)$, where $y_n : \Omega \to X$ is measurable. By induction, there exists a measurable sequence $\{y_n(\omega)\}$ such that

$$y_1(\omega) \preceq y_2(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \preceq y(\omega), \qquad y_n(\omega) \longrightarrow y(\omega)(n \longrightarrow \infty), \quad \forall \omega \in \Omega. \tag{2.10}$$

Set $Y = \overline{\bigcup_{n=1}^{\infty} \{y_n(\omega) : \omega \in \Omega\} \cup \{y(\omega) : \omega \in \Omega\}}$. Then Y is a Polish subspace of X. Since $T : \Omega \times X \to X$ is order continuous, $T : \Omega \times Y \to X$ is continuous. By (2.10), we have

$$T\left(\omega, y_n(\omega)\right) \longrightarrow T\left(\omega, y(\omega)\right)(n \longrightarrow \infty), \quad \forall \omega \in \Omega \text{ a.e.} \tag{2.11}$$

By Lemma 1.2, $T(\omega, y_n(\omega))$ is measurable for all n. Thus, from (2.11) and Lemma 1.3 it follows that $T(\omega, y(\omega))$ is measurable. This completes the Proof. □

Remark 2.4. Theorem 2.3 is a generalization of Lemma 1.2.

3. Random Periodic Points and Fixed Points for Increasing Random Mappings

Theorem 3.1. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega \times [x_0, y_0] \to X$ be an order continuous and increasing random mapping with $x_0 \preceq T^k(\omega, x_0)$ and $T^k(\omega, y_0) \preceq y_0$ for $\omega \in \Omega$ a.e., where k is a positive integer. Then there exist a minimum random k-periodic point $u(\omega)$ and a maximum random k-periodic point $v(\omega)$ in $[x_0, y_0]$ such that $u(\omega) \preceq z(\omega) \preceq v(\omega)$ a.e., for all $z \in F_R(T^k)$.*

Proof. Without loss of generality, we may assume that $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $T(\omega, \cdot)$ is order continuous for all $\omega \in \Omega_0$, and $x_0 \preceq T^k(\omega, x_0)$, $T^k(\omega, y_0) \preceq y_0$ for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$, $S = T^k$, $x_n(\omega) = S^n(\omega, x_0)$, and $y_n(\omega) = S^n(\omega, y_0)$. Since $x_0 \preceq S(\omega, x_0)$, $S(\omega, y_0) \preceq y_0$, and T is increasing, we have

$$x_0 \preceq x_1(\omega) \preceq \cdots \preceq x_n(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \preceq y_1(\omega) \preceq y_0. \tag{3.1}$$

Then, it follows from (3.1) that

$$\phi(x_0) \ge \phi(x_1(\omega)) \ge \cdots \ge \phi(x_n(\omega)) \ge \cdots \ge \phi\left(y_n(\omega)\right) \ge \cdots \ge \phi\left(y_1(\omega)\right) \ge \phi\left(y_0\right). \tag{3.2}$$

From (3.2) we see that $\{\phi(x_n(\omega))\}$ and $\{\phi(y_n(\omega))\}$ are two convergent sequences of numbers. For every $\varepsilon > 0$ there exists a positive integer N such that

$$\begin{aligned} &d(x_n(\omega), x_m(\omega)) \le \phi(x_n(\omega)) - \phi(x_m(\omega)) < \varepsilon, \quad \forall m > n > N; \\ &d(y_m(\omega), y_n(\omega)) \le \phi(y_m(\omega)) - \phi(y_n(\omega)) < \varepsilon, \quad \forall m > n > N. \end{aligned} \tag{3.3}$$

This shows that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are two Cauchy sequences in X. The completeness of X implies that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are all convergent. Define $u(\omega)$ and $v(\omega)$ by

$$u(\omega) = \begin{cases} \lim_{n\to\infty} x_n(\omega), & \text{if } \omega \in \Omega_0, \\ x_0, & \text{if } \omega \in \Omega \setminus \Omega_0; \end{cases} \qquad v(\omega) = \begin{cases} \lim_{n\to\infty} y_n(\omega), & \text{if } \omega \in \Omega_0, \\ y_0, & \text{if } \omega \in \Omega \setminus \Omega_0. \end{cases} \tag{3.4}$$

Since T is order continuous, S is order continuous. Then, we have

$$\begin{aligned} S(\omega, u(\omega)) &= \lim_{n\to\infty} S(\omega, x_n(\omega)) = \lim_{n\to\infty} x_{n+1}(\omega) = u(\omega), \quad \forall \omega \in \Omega_0; \\ S(\omega, v(\omega)) &= \lim_{n\to\infty} S(\omega, y_n(\omega)) = \lim_{n\to\infty} y_{n+1}(\omega) = v(\omega), \quad \forall \omega \in \Omega_0. \end{aligned} \tag{3.5}$$

Note that $P(\Omega \setminus \Omega_0) = 0$. By Theorem 2.3, $x_n(\omega)$ and $y_n(\omega)$ are all measurable. By Lemma 1.3, $u(\omega)$ and $v(\omega)$ are all measurable. Therefore, from (3.5) we see that $u(\omega)$ and $v(\omega)$ are all random fixed points of S, that is, $u, v \in F_R(S) = F_R(T^k)$. Since ϕ is continuous, we have, for $\omega \in \Omega$ a.e.,

$$\begin{aligned} d(x_0, u(\omega)) &= \lim_{n\to\infty} d(x_0, x_n(\omega)) \le \lim_{n\to\infty} [\phi(x_0) - \phi(x_n(\omega))] = \phi(x_0) - \phi(u(\omega)); \\ d(v(\omega), y_0) &= \lim_{n\to\infty} d(y_n(\omega), y_0) \le \lim_{n\to\infty} [\phi(y_n(\omega)) - \phi(y_0)] = \phi(v(\omega)) - \phi(y_0); \\ d(u(\omega), v(\omega)) &= \lim_{n\to\infty} d(x_n(\omega), y_n(\omega)) \le \lim_{n\to\infty} [\phi(x_n(\omega)) - \phi(y_n(\omega))] = \phi(u(\omega)) - \phi(v(\omega)). \end{aligned} \tag{3.6}$$

This shows that $x_0 \preceq u(\omega) \preceq v(\omega) \preceq y_0$ a.e. If $z \in F_R(T^k) = F_R(S)$, then we have $x_n(\omega) \preceq z(\omega) \preceq y_n(\omega)$ a.e., for all n. Thus, for $\omega \in \Omega$ a.e.,

$$\begin{aligned} d(u(\omega), z(\omega)) &= \lim_{n\to\infty} d(x_n(\omega), z(\omega)) \le \lim_{n\to\infty} [\phi(x_n(\omega)) - \phi(z(\omega))] = \phi(u(\omega)) - \phi(z(\omega)); \\ d(z(\omega), v(\omega)) &= \lim_{n\to\infty} d(z(\omega), y_n(\omega)) \le \lim_{n\to\infty} [\phi(z(\omega)) - \phi(y_n(\omega))] = \phi(z(\omega)) - \phi(v(\omega)). \end{aligned} \tag{3.7}$$

This shows that $u(\omega) \preceq z(\omega) \preceq v(\omega)$ a.e., which is the desired conclusion. □

Corollary 3.2. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega \times [x_0, y_0] \to X$ be an order continuous and increasing random mapping with $x_0 \preceq T(\omega, x_0)$ and $T(\omega, y_0) \preceq y_0$ for $\omega \in \Omega$ a.e.. Then there exist a minimum random fixed point $u(\omega)$ and a maximum random fixed point $v(\omega)$ in $[x_0, y_0]$ such that $u(\omega) \preceq z(\omega) \preceq v(\omega)$ a.e., for all $z \in F_R(T)$.*

Proof. It is obtained by taking $k = 1$ in Theorem 3.1. □

Corollary 3.3. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega\times[x_0, y_0] \to X$ be a increasing random mapping with $x_0 \preceq T^k(\omega, x_0)$ and $T^k(\omega, y_0) \preceq y_0$ for $\omega \in \Omega$ a.e., where k is a positive integer. If T is an order contraction mapping, then there exists a unique random fixed point $u(\omega)$ in $[x_0, y_0]$.*

Proof. From order contraction of T it follows that T is order continuous. By Theorem 3.1, there exist a minimum random k-periodic point $u(\omega)$ and a maximum random k-periodic point $v(\omega)$ in $[x_0, y_0]$. Since T is an order contraction mapping, for $\omega \in \Omega$ a.e., we have

$$d(u(\omega), v(\omega)) = d\Big(T^k(\omega, u(\omega)), T^k(\omega, v(\omega))\Big) \leq [\alpha(\omega)]^k d(u(\omega), v(\omega)), \tag{3.8}$$

where $\alpha(\omega) \in [0, 1)$. This shows that $u(\omega) = v(\omega)$ a.e., namely, there is a unique $u \in F_R(T^k)$. Let $T(\omega, u(\omega)) = z(\omega)$. Then we have $z(\omega) \in [x_0, y_0]$ a.e. and

$$z(\omega) = T\Big(\omega, T^k(\omega, u(\omega))\Big) = T^{k+1}(\omega, u(\omega)) = T^k(\omega, z(\omega)), \tag{3.9}$$

that is, $z \in F_R(T^k)$. Hence, we have $u = z$. This shows that $u \in F_R(T)$. If $y \in F_R(T)$ and $y(\omega) \in [x_0, y_0]$ a.e., then $y \in F_R(T^k)$, and so $y = u$, that is, there is a unique $u \in F_R(T)$. This completes the proof. □

4. Random Periodic Points and Fixed Points for Decreasing Random Mappings

Theorem 4.1. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega\times[x_0, y_0] \to X$ be an order continuous and decreasing random mapping with $x_0 \preceq T(\omega, y_0)$ and $T(\omega, x_0) \preceq y_0$ for $\omega \in \Omega$ a.e. Then there exists a random 2-periodic point u in $[x_0, y_0]$ such that $T(\omega, u(\omega)) \in [x_0, y_0]$ a.e.*

Proof. Without loss of generality, we may assume that $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $T(\omega, \cdot)$ is order continuous for all $\omega \in \Omega_0$ and $x_0 \preceq T(\omega, y_0)$, $T(\omega, x_0) \preceq y_0$ for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$, $x_n(\omega) = T(\omega, y_{n-1}(\omega))$, and $y_n(\omega) = T(\omega, x_{n-1}(\omega))$, $(n = 1, 2, \ldots)$. Since T is decreasing, we have

$$x_0 \preceq x_1(\omega) \preceq \cdots \preceq x_n(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \preceq y_1(\omega) \preceq y_0. \tag{4.1}$$

Then, from (4.1) it follows that

$$\phi(x_0) \geq \phi(x_1(\omega)) \geq \cdots \geq \phi(x_n(\omega)) \geq \cdots \geq \phi\big(y_n(\omega)\big) \geq \cdots \geq \phi\big(y_1(\omega)\big) \geq \phi\big(y_0\big). \tag{4.2}$$

From (4.2) we see that $\{\phi(x_n(\omega))\}$ and $\{\phi(y_n(\omega))\}$ are two convergent sequences of numbers. For every $\varepsilon > 0$ there exists a positive integer N such that

$$d(x_n(\omega), x_m(\omega)) \leq \phi(x_n(\omega)) - \phi(x_m(\omega)) < \varepsilon, \quad \forall m > n > N;$$
$$d(y_m(\omega), y_n(\omega)) \leq \phi(y_m(\omega)) - \phi(y_n(\omega)) < \varepsilon, \quad \forall m > n > N. \tag{4.3}$$

This shows that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are two Cauchy sequences in X. By the completeness of X we see that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are all convergent. Define $u(\omega)$ and $v(\omega)$ by (3.4). Since T is order continuous, we have

$$T(\omega, u(\omega)) = \lim_{n\to\infty} T(\omega, x_{n-1}(\omega)) = \lim_{n\to\infty} y_n(\omega) = v(\omega), \quad \forall \omega \in \Omega_0;$$
$$T(\omega, v(\omega)) = \lim_{n\to\infty} T(\omega, y_{n-1}(\omega)) = \lim_{n\to\infty} x_n(\omega) = u(\omega), \quad \forall \omega \in \Omega_0. \tag{4.4}$$

By the continuity of ϕ, we have, for $\omega \in \Omega_0$,

$$d(x_0, u(\omega)) = \lim_{n\to\infty} d(x_0, x_n(\omega)) \leq \lim_{n\to\infty} [\phi(x_0) - \phi(x_n(\omega))] = \phi(x_0) - \phi(u(\omega));$$
$$d(v(\omega), y_0) = \lim_{n\to\infty} d(y_n(\omega), y_0) \leq \lim_{n\to\infty} [\phi(y_n(\omega)) - \phi(y_0)] = \phi(v(\omega)) - \phi(y_0). \tag{4.5}$$

Since $P(\Omega\backslash\Omega_0) = 0$, we have $u(\omega), v(\omega) \in [x_0, y_0]$ a.e.. By Theorem 2.3, $x_n(\omega)$ and $y_n(\omega)$ are all measurable. By Lemma 1.3, $u(\omega)$ and $v(\omega)$ are all measurable. Therefore, from (4.4) we have

$$T^2(\omega, u(\omega)) = T(\omega, v(\omega)) = u(\omega), \quad \forall \omega \in \Omega \text{ a.e.} \tag{4.6}$$

This shows that $u \in F_R(T^2)$, which is the desired conclusion. □

Corollary 4.2. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega \times [x_0, y_0] \to X$ be a decreasing random mapping with $x_0 \preceq T(\omega, y_0)$ and $T(\omega, x_0) \preceq y_0$ for $\omega \in \Omega$ a.e. If T is an order contraction mapping, then there exists a unique random fixed point $u(\omega)$ in $[x_0, y_0]$.*

Proof. Since T is an order contraction mapping, T is order continuous. By Theorem 4.1, there exists a random 2-periodic point u in $[x_0, y_0]$ such that $T(\omega, u(\omega)) = v(\omega) \in [x_0, y_0]$ a.e. We claim that $u(\omega) = v(\omega)$ a.e. In fact that, from (4.1) we have $u(\omega) \preceq v(\omega)$ a.e. If $u(\omega) \neq v(\omega)$ a.e., then there exists $\alpha(\omega) \in [0, 1)$ such that

$$d(u(\omega), v(\omega)) = d(T(\omega, v(\omega)), T(\omega, u(\omega))) \leq \alpha(\omega) d(v(\omega), u(\omega))$$
$$< d(v(\omega), u(\omega)), \quad \forall \omega \in \Omega \text{ a.e.}, \tag{4.7}$$

which is a contradiction. Hence, $u \in F_R(T)$. If $y \in F_R(T)$ and $y(\omega) \in [x_0, y_0]$ a.e., then we have

$$x_n(\omega) \preceq y(\omega) \preceq y_n(\omega), \quad \forall \omega \in \Omega \text{ a.e.}, \tag{4.8}$$

where $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are the iterations in the proof of Theorem 4.1. It is easy to check that $u(\omega) \preceq y(\omega) \preceq v(\omega)$, for all $\omega \in \Omega$ a.e. But $u = v$, and so we have $y = u$. This completes the proof. □

Theorem 4.3. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous and $\phi(X)$ is bounded. Let $T : \Omega \times [x_0, y_0] \to X$ be an order continuous and decreasing random mapping with $y_0 \preceq T(\omega, x_0)$ and $T(\omega, y_0) \preceq x_0$ for $\omega \in \Omega$ a.e. Then there exists a random 2-periodic point u in X.*

Proof. Without loss of generality, we may assume that $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $T(\omega, \cdot)$ is order continuous for all $\omega \in \Omega_0$, and $y_0 \preceq T(\omega, x_0)$, $T(\omega, y_0) \preceq x_0$ for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$, $x_n(\omega) = T(\omega, y_{n-1}(\omega))$, and $y_n(\omega) = T(\omega, x_{n-1}(\omega))$, $(n = 1, 2, \ldots)$. Since T is decreasing, we have

$$\cdots \preceq x_n(\omega) \preceq \cdots \preceq x_1(\omega) \preceq x_0 \preceq y_0 \preceq y_1(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \tag{4.9}$$

Then, it follows from (4.9) that

$$\cdots \geq \phi(x_n(\omega)) \geq \cdots \geq \phi(x_1(\omega)) \geq \phi(x_0) \geq \phi(y_0) \geq \phi(y_1(\omega)) \geq \cdots \geq \phi(y_n(\omega)) \geq \cdots \tag{4.10}$$

This shows that $\{\phi(x_n(\omega))\}$ and $\{\phi(y_n(\omega))\}$ are two convergent sequences of numbers by the boundedness of $\phi(X)$. For every $\varepsilon > 0$ there exists a positive integer N such that

$$d(x_n(\omega), x_m(\omega)) \leq \phi(x_n(\omega)) - \phi(x_m(\omega)) < \varepsilon, \quad \forall n > m > N. \tag{4.11}$$

This shows that $\{x_n(\omega)\}$ is a Cauchy sequence in X. The completeness of X implies that $\{x_n(\omega)\}$ is convergent. Similarly, $\{y_n(\omega)\}$ is convergent. Define $u(\omega)$ and $v(\omega)$ by (3.4). Since T is order continuous, we have

$$\begin{aligned} T(\omega, u(\omega)) &= \lim_{n\to\infty} T(\omega, x_{n-1}(\omega)) = \lim_{n\to\infty} y_n(\omega) = v(\omega), \quad \forall \omega \in \Omega_0; \\ T(\omega, v(\omega)) &= \lim_{n\to\infty} T\big(\omega, y_{n-1}(\omega)\big) = \lim_{n\to\infty} x_n(\omega) = u(\omega), \quad \forall \omega \in \Omega_0. \end{aligned} \tag{4.12}$$

Since $P(\Omega \setminus \Omega_0) = 0$, by Theorem 2.3, $x_n(\omega)$ and $y_n(\omega)$ are all measurable; by Lemma 1.3, $u(\omega)$ and $v(\omega)$ are all measurable. Therefore, from (4.12) we have

$$T^2(\omega, u(\omega)) = T(\omega, v(\omega)) = u(\omega), \quad \forall \omega \in \Omega \text{a.e.} \tag{4.13}$$

This shows that $u \in F_R(T^2)$, which is the desired conclusion. □

5. Coupled Random Periodic Point and Fixed Point Theorems

Theorem 5.1. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous. Let $T : \Omega \times [x_0, y_0] \times [x_0, y_0] \to X$ be an order continuous and mixed monotone random mapping with $x_0 \preceq T^k(\omega, x_0, y_0)$ and $T^k(\omega, y_0, x_0) \preceq y_0$ for $\omega \in \Omega$ a.e., where k is a positive integer. Then there exists a coupled* *random k-periodic point* (u, v) *such that* $T^k(\omega, u(\omega), v(\omega)) = u(\omega)$, $T^k(\omega, v(\omega), u(\omega)) = v(\omega)$, *and* $[u(\omega), v(\omega)] \subset [x_0, y_0]$ *a.e. If* (u_1, v_1) *is a coupled random k-periodic point such that* $[u_1(\omega), v_1(\omega)] \subset [x_0, y_0]$ *a.e., then* $[u_1(\omega), v_1(\omega)] \subset [u(\omega), v(\omega)]$ *a.e.*

Proof. Without loss of generality, we may assume that $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $T(\omega, \cdot, \cdot)$ is order continuous for all $\omega \in \Omega_0$ and $x_0 \preceq T^k(\omega, x_0, y_0)$, $T^k(\omega, y_0, x_0) \preceq y_0$ for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$, $S = T^k$, $x_n(\omega) = S^n(\omega, x_{n-1}(\omega), y_{n-1}(\omega))$, and $y_n(\omega) = S^n(\omega, y_{n-1}(\omega), x_{n-1}(\omega))$, $(n = 1, 2, \ldots)$. Since T is a mixed monotone mapping, we have

$$x_0 \preceq x_1(\omega) = S(\omega, x_0, y_0) \preceq S(\omega, y_0, y_0) \preceq S(\omega, y_0, x_0) = y_1(\omega) \preceq y_0. \tag{5.1}$$

By induction, we have

$$x_0 \preceq x_1(\omega) \preceq \cdots \preceq x_n(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \preceq y_1(\omega) \preceq y_0. \tag{5.2}$$

Thus, from (5.2) it follows that

$$\phi(x_0) \geq \phi(x_1(\omega)) \geq \cdots \geq \phi(x_n(\omega)) \geq \cdots \geq \phi(y_n(\omega)) \geq \cdots \geq \phi(y_1(\omega)) \geq \phi(y_0). \tag{5.3}$$

This shows that $\{\phi(x_n(\omega))\}$ and $\{\phi(y_n(\omega))\}$ are two convergent sequences of numbers. In a similar way to the proof of Theorem 3.1, we can check that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are two Cauchy sequences in X. The completeness of X implies that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are all convergent. Define $u(\omega)$ and $v(\omega)$ by (3.4). Since ϕ is continuous, it is easy to prove that $x_n(\omega) \preceq u(\omega) \preceq v(\omega) \preceq y_n(\omega)$ for all n. Since T is order continuous, S is order continuous. Then, we have

$$\begin{aligned} S(\omega, u(\omega), v(\omega)) &= \lim_{n\to\infty} S(\omega, x_n(\omega), y_n(\omega)) = \lim_{n\to\infty} x_{n+1}(\omega) = u(\omega), \quad \forall \omega \in \Omega_0; \\ S(\omega, v(\omega), u(\omega)) &= \lim_{n\to\infty} S(\omega, y_n(\omega), x_n(\omega)) = \lim_{n\to\infty} y_{n+1}(\omega) = v(\omega), \quad \forall \omega \in \Omega_0. \end{aligned} \tag{5.4}$$

Note that $P(\Omega \setminus \Omega_0) = 0$. By Theorem 2.3, $x_n(\omega)$ and $y_n(\omega)$ are all measurable. By Lemma 1.3, $u(\omega)$ and $v(\omega)$ are all measurable. Therefore, from (5.4) we see that (u, v) is a coupled random fixed point of S, that is, it is a coupled random k-periodic point of T. If (u_1, v_1) is a coupled random k-periodic point such that $[u_1(\omega), v_1(\omega)] \subset [x_0, y_0]$ a.e., then, by mixed monotonicity of T, we have $x_1(\omega) = T(\omega, x_0, y_0) \preceq T(\omega, u_1(\omega), v_1(\omega)) = u_1(\omega)$ a.e. and $v_1(\omega) = T(\omega, v_1(\omega), u_1(\omega)) \preceq T(\omega, y_0, x_0) = y_1(\omega)$ a.e. Then, by induction, we have

$$x_n(\omega) \preceq u_1(\omega) \text{ a.e.}, \qquad v_1(\omega) \preceq y_n(\omega) \text{ a.e.}, \quad \forall n. \tag{5.5}$$

Since T is order continuous, we have $[u_1(\omega), v_1(\omega)] \subset [u(\omega), v(\omega)]$ a.e.. This completes the proof. □

Corollary 5.2. *Let* (X, d, ϕ) *be an ordered Polish space, where* ϕ *is continuous. Let* $T : \Omega \times [x_0, y_0] \times [x_0, y_0] \to X$ *be an order continuous and mixed monotone random mapping with* $x_0 \preceq T(\omega, x_0, y_0)$ *and* $T(\omega, y_0, x_0) \preceq y_0$ *for* $\omega \in \Omega$ *a.e. Then there exists a coupled random fixed point* (u, v) *such that*

$T(\omega, u(\omega), v(\omega)) = u(\omega)$, $T(\omega, v(\omega), u(\omega)) = v(\omega)$ and $[u(\omega), v(\omega)] \subset [x_0, y_0]$ a.e. If (u_1, v_1) is also a coupled random fixed point such that $[u_1(\omega), v_1(\omega)] \subset [x_0, y_0]$ a.e., then $[u_1(\omega), v_1(\omega)] \subset [u(\omega), v(\omega)]$ a.e.

Proof. It is obtained by taking $k = 1$ in Theorem 5.1. □

Theorem 5.3. *Let (X, d, ϕ) be an ordered Polish space, where ϕ is continuous and $\phi(X)$ is bounded. Let $T : \Omega \times [x_0, y_0] \times [x_0, y_0] \to X$ be an order continuous and mixed monotone random mapping with $T(\omega, x_0, y_0) \preceq x_0$ and $y_0 \preceq T(\omega, y_0, x_0)$ for $\omega \in \Omega$ a.e., where $x_0 \neq y_0$. Then there exists a coupled random fixed point $(u(\omega), v(\omega))$ such that $T(\omega, u(\omega), v(\omega)) = u(\omega)$, $T(\omega, v(\omega), u(\omega)) = v(\omega)$, and $[x_0, y_0] \subset [u(\omega), v(\omega)]$ a.e. If (u_1, v_1) is also a coupled random fixed point such that $[x_0, y_0] \subset [u_1(\omega), v_1(\omega)]$ a.e., then $[u(\omega), v(\omega)] \subset [u_1(\omega), v_1(\omega)]$ a.e.*

Proof. Without loss of generality, we may assume that $\Omega_0 \subset \Omega$, $P(\Omega_0) = 1$, $T(\omega, \cdot, \cdot)$ is order continuous for all $\omega \in \Omega_0$ and $T(\omega, x_0, y_0) \preceq x_0$, $y_0 \preceq T(\omega, y_0, x_0)$ for all $\omega \in \Omega_0$. Let $\omega \in \Omega_0$, $x_n(\omega) = T(\omega, x_{n-1}(\omega), y_{n-1}(\omega))$, and $y_n(\omega) = T(\omega, y_{n-1}(\omega), x_{n-1}(\omega))$, $(n = 1, 2, \ldots)$. Then,

$$x_1(\omega) = T\big(\omega, x_0, y_0\big) \preceq x_0 \preceq y_0 \preceq T\big(\omega, y_0, x_0\big) = y_1(\omega). \tag{5.6}$$

Since T is a mixed monotone mapping, we have $x_2(\omega) = T(\omega, x_1(\omega), y_1(\omega)) \preceq T(\omega, x_0, y_0) = x_1(\omega)$, and $y_1(\omega) = T(\omega, y_0, x_0) \preceq T(\omega, y_1(\omega), x_1(\omega)) = y_2(\omega)$. By induction, we have

$$\cdots \preceq x_n(\omega) \preceq \cdots \preceq x_1(\omega) \preceq x_0 \preceq y_0 \preceq y_1(\omega) \preceq \cdots \preceq y_n(\omega) \preceq \cdots \tag{5.7}$$

Thus, from (5.7) it follows that

$$\cdots \geq \phi(x_n(\omega)) \geq \cdots \geq \phi(x_1(\omega)) \geq \phi(x_0) \geq \phi\big(y_0\big) \geq \phi\big(y_1(\omega)\big) \geq \cdots \geq \phi\big(y_n(\omega)\big) \geq \cdots \tag{5.8}$$

This shows that $\{\phi(x_n(\omega))\}$ and $\{\phi(y_n(\omega))\}$ are two convergent sequences of numbers by the boundedness of $\phi(X)$. In a similar way to the proof of Theorem 4.3, we can check that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are two Cauchy sequences in X. The completeness of X implies that $\{x_n(\omega)\}$ and $\{y_n(\omega)\}$ are all convergent. Define $u(\omega)$ and $v(\omega)$ by (3.4). Since ϕ is continuous, it is easy to prove that $u(\omega) \preceq x_n(\omega) \preceq x_0$ and $y_0 \preceq y_n(\omega) \preceq v(\omega)$ for all n. Since T is order continuous, we have

$$\begin{aligned}
T(\omega, u(\omega), v(\omega)) &= \lim_{n\to\infty} T\big(\omega, x_n(\omega), y_n(\omega)\big) = \lim_{n\to\infty} x_{n+1}(\omega) = u(\omega), \quad \forall \omega \in \Omega_0; \\
T(\omega, v(\omega), u(\omega)) &= \lim_{n\to\infty} T\big(\omega, y_n(\omega), x_n(\omega)\big) = \lim_{n\to\infty} y_{n+1}(\omega) = v(\omega), \quad \forall \omega \in \Omega_0.
\end{aligned} \tag{5.9}$$

Note that $P(\Omega \backslash \Omega_0) = 0$. By Theorem 2.3, $x_n(\omega)$ and $y_n(\omega)$ are all measurable. By Lemma 1.3, $u(\omega)$ and $v(\omega)$ are all measurable. Therefore, from (5.9) we see that (u, v) is a coupled random fixed point of T. If (u_1, v_1) is a coupled random point of T with $[x_0, y_0] \subset (u_1(\omega), v_1(\omega))$ a.e., then, by mixed monotonicity of T, we have $u_1(\omega) = T(\omega, u_1(\omega), v_1(\omega)) \preceq T(\omega, x_0, y_0) = x_1(\omega)$ a.e., and $v_1(\omega) = T(\omega, v_1(\omega), u_1(\omega)) \succeq T(\omega, y_0, x_0) = y_1(\omega)$ a.e., namely, $[x_1(\omega), y_1(\omega)] \subset [u_1(\omega), v_1(\omega)]$ a.e. By induction, we have

$$u_1(\omega) \preceq x_n(\omega) \text{ a.e.}, \qquad y_n(\omega) \preceq v_1(\omega) \text{ a.e.}, \quad \forall n. \tag{5.10}$$

Since T is order continuous, we have $[u(\omega), v(\omega)] \subset [u_1(\omega), v_1(\omega)]$ a.e. This completes the proof. □

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