Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 852030, 12 pages
doi:10.1155/2010/852030

Research Article

Two New Iterative Methods for a Countable Family of Nonexpansive Mappings in Hilbert Spaces

Shuang Wang[1] and Changsong Hu[2]

[1] *School of Mathematical Sciences, Yancheng Teachers University, Yancheng, Jiangsu 224051, China*
[2] *Department of Mathematics, Hubei Normal University, Huangshi 435002, China*

Correspondence should be addressed to Shuang Wang, wangshuang19841119@163.com

Received 6 August 2010; Accepted 5 October 2010

Academic Editor: Tomonari Suzuki

We consider two new iterative methods for a countable family of nonexpansive mappings in Hilbert spaces. We proved that the proposed algorithms strongly converge to a common fixed point of a countable family of nonexpansive mappings which solves the corresponding variational inequality. Our results improve and extend the corresponding ones announced by many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx-Ty\| \le \|x-y\|$, for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T. A mapping $F : H \to H$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \le k\|x - y\|, \quad \forall x, y \in H. \tag{1.1}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in H. \tag{1.2}$$

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Ax, x\rangle \ge \tilde{\gamma}\|x\|^2, \quad \forall x \in H. \tag{1.3}$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in F(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.4}$$

where b is a given point in H.

Remark 1.1. From the definition of A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and $\tilde{\gamma}$-strongly monotone operator.

Construction of fixed points of nonlinear mappings is an important and active research area. In particular, iterative algorithms for finding fixed points of nonexpansive mappings have received vast investigation (cf. [1, 2]) since these algorithms find applications in variety of applied areas of inverse problem, partial differential equations, image recovery, and signal processing; see [3–8]. One classical way to find the fixed point of a nonexpansive mapping T is to use a contraction to approximate it. More precisely, take $t \in (0,1)$ and define a contraction $T_t : C \to C$ by $T_t x = tu + (1-t)Tx$, where $u \in C$ is a fixed point. Banach's Contraction Mapping Principle guarantees that T_t has a unique fixed point x_t in C, that is,

$$x_t = tu + (1-t)Tx_t, \quad u \in C. \tag{1.5}$$

The strong convergence of the path x_t has been studied by Browder [9] and Halpern [10] in a Hilbert space.

Recently, Yao et al. [11] considered the following algorithms:

$$x_t = TP_C[(1-t)x_t], \tag{1.6}$$

and for $x_0 \in C$ arbitrarily,

$$\begin{aligned} y_n &= P_C[(1-\alpha_n)x_n], \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n T y_n, \quad n \geq 0. \end{aligned} \tag{1.7}$$

They proved that if $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying appropriate conditions, then the $\{x_t\}$ defined by (1.6) and $\{x_n\}$ defined by (1.7) converge strongly to a fixed point of T.

On the other hand, Yamada [12] introduced the following hybrid iterative method for solving the variational inequality:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \quad n \geq 0, \tag{1.8}$$

where F is a k-Lipschitzian and η-strongly monotone operator with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$. Then he proved that $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of variational inequality $\langle F\tilde{x}, x - \tilde{x}\rangle \geq 0$, $x \in F(T)$.

In this paper, motivated and inspired by the above results, we introduce two new algorithms (3.3) and (3.13) for a countable family of nonexpansive mappings in Hilbert spaces. We prove that the proposed algorithms strongly converge to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$ which solves the variational inequality: $\langle Fx^*, x^* - u\rangle \leq 0$, $u \in \bigcap_{n=1}^{\infty} F(T_n)$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.1}$$

The mapping P_C is called the metric projection of H onto C. It is well know that P_C is a nonexpansive mapping. In a real Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y \rangle, \quad \forall x, y \in H. \tag{2.2}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.1 (see [13]). *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$, if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 2.2 (see [14]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in Banach space E and $\{\gamma_n\}$ a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \tag{2.3}$$

Suppose that $x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n$, $n \geq 0$ and $\limsup_{n \to \infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \to \infty}\|z_n - x_n\| = 0$.

Lemma 2.3 (see [15, 16]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n \delta_n + \gamma_n, \quad n \geq 0, \tag{2.4}$$

where $\{\lambda_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy the following conditions: (i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$, (ii) $\limsup_{n \to \infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n \delta_n < \infty$, (iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. *Then* $\lim_{n \to \infty} s_n = 0$.

Lemma 2.4 (see [17, Lemma 3.2]). *Let C be a nonempty closed convex subset of a Banach space E. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty. \tag{2.5}$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n \to \infty} T_n y$, for all $y \in C$. Then $\lim_{n \to \infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0$.

Lemma 2.5. *Let F be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $0 < \eta \le k$ and $0 < t < \eta/k^2$. Then $S = (I - tF) : H \to H$ is a contraction with contraction coefficient $\tau_t = \sqrt{1 - t(2\eta - tk^2)}$.*

Proof. From (1.1), (1.2), and (2.2), we have

$$\begin{aligned} \|Sx - Sy\|^2 &= \|(x - y) - t(Fx - Fy)\|^2 \\ &= \|x - y\|^2 + t^2\|Fx - Fy\|^2 - 2t\langle Fx - Fy, x - y\rangle \\ &\le \|x - y\|^2 + t^2k^2\|x - y\|^2 - 2t\eta\|x - y\|^2 \\ &= \left[1 - t\left(2\eta - tk^2\right)\right]\|x - y\|^2, \end{aligned} \tag{2.6}$$

for all $x, y \in H$. From $0 < \eta \le k$ and $0 < t < \eta/k^2$, we have

$$\|Sx - Sy\| \le \tau_t\|x - y\|, \tag{2.7}$$

where $\tau_t = \sqrt{1 - t(2\eta - tk^2)}$. Hence S is a contraction with contraction coefficient τ_t. □

3. Main Results

Let F be a k-Lipschitzian and η-strongly monotone operator on H with $0 < \eta \le k$ and $T : C \to C$ a nonexpansive mapping. Let $t \in (0, \eta/k^2)$ and $\tau_t = \sqrt{1 - t(2\eta - tk^2)}$; consider a mapping S_t on C defined by

$$S_t x = TP_C[(I - tF)x], \quad x \in C. \tag{3.1}$$

It is easy to see that S_t is a contraction. Indeed, from Lemma 2.5, we have

$$\begin{aligned} \|S_t x - S_t y\| &\le \|TP_C[(I - tF)x] - TP_C(I - tF)y\| \\ &\le \|(I - tF)x - (I - tF)y\| \\ &\le \tau_t\|x - y\|, \end{aligned} \tag{3.2}$$

for all $x, y \in C$. Hence it has a unique fixed point, denoted x_t, which uniquely solves the fixed point equation

$$x_t = TP_C[(I - tF)x_t], \quad x_t \in C. \tag{3.3}$$

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \ne \emptyset$. Let F be a k-Lipschitzian and η-strongly monotone* *operator on H with $0 < \eta \leq k$. For each $t \in (0, \eta/k^2)$, let the net $\{x_t\}$ be generated by* (3.3). *Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point x^* of T which solves the variational inequality:*

$$\langle Fx^*, x^* - u\rangle \leq 0, \quad u \in F(T). \tag{3.4}$$

Proof. We first show the uniqueness of a solution of the variational inequality (3.4), which is indeed a consequence of the strong monotonicity of F. Suppose $x^* \in F(T)$ and $\tilde{x} \in F(T)$ both are solutions to (3.4); then

$$\begin{aligned} &\langle Fx^*, x^* - \tilde{x}\rangle \leq 0, \\ &\langle F\tilde{x}, \tilde{x} - x^*\rangle \leq 0. \end{aligned} \tag{3.5}$$

Adding up (3.5) gets

$$\langle Fx^* - F\tilde{x}, x^* - \tilde{x}\rangle \leq 0. \tag{3.6}$$

The strong monotonicity of F implies that $x^* = \tilde{x}$ and the uniqueness is proved. Below we use $x^* \in F(T)$ to denote the unique solution of (3.4).

Next, we prove that $\{x_t\}$ is bounded. Take $u \in F(T)$; from (3.3) and using Lemma 2.5, we have

$$\begin{aligned} \|x_t - u\| &= \|TP_C[(I - tF)x_t] - TP_C u\| \\ &\leq \|(I - tF)x_t - u\| \\ &\leq \|(I - tF)x_t - (I - tF)u - tFu\| \\ &\leq \|(I - tF)x_t - (I - tF)u\| + t\|Fu\| \\ &\leq \tau_t\|x_t - u\| + t\|Fu\|, \end{aligned} \tag{3.7}$$

that is,

$$\|x_t - u\| \leq \frac{t}{1 - \tau_t}\|Fu\|. \tag{3.8}$$

Observe that

$$\lim_{t \to 0^+} \frac{t}{1 - \tau_t} = \frac{1}{\eta}. \tag{3.9}$$

From $t \to 0$, we may assume, without loss of generality, that $t \leq \eta/k^2 - \epsilon$. Thus, we have that $t/(1 - \tau_t)$ is continuous, for all $t \in [0,\ \eta/k^2 - \epsilon]$. Therefore, we obtain

$$\sup\left\{\frac{t}{1 - \tau_t} \ : \ t \in \left(0, \frac{\eta}{k^2} - \epsilon\right]\right\} < +\infty. \tag{3.10}$$

From (3.8) and (3.10), we have that $\{x_t\}$ is bounded and so is $\{Fx_t\}$.

On the other hand, from (3.3), we obtain

$$\|x_t - Tx_t\| = \|TP_C[(I - tF)x_t] - TP_C x_t\| \le \|(I - tF)x_t - x_t\| = t\|Fx_t\| \longrightarrow 0 \quad (t \longrightarrow 0). \tag{3.11}$$

To prove that $x_t \to x^*$. For a given $u \in F(T)$, by (2.2) and using Lemma 2.5, we have

$$\begin{aligned}
\|x_t - u\|^2 &= \|TP_C[(I - tF)x_t] - TP_C u\|^2 \\
&\le \|(I - tF)x_t - (I - tF)u - tFu\|^2 \\
&\le {\tau_t}^2\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\langle (I - tF)u - (I - tF)x_t, Fu\rangle \\
&\le \tau_t\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\langle u - x_t, Fu\rangle + 2t^2\langle Fx_t - Fu, Fu\rangle \\
&\le \tau_t\|x_t - u\|^2 + t^2\|Fu\|^2 + 2t\langle u - x_t, Fu\rangle + 2t^2 k\|x_t - u\|\|Fu\|.
\end{aligned} \tag{3.12}$$

Therefore,

$$\|x_t - u\|^2 \le \frac{t^2}{1 - \tau_t}\|Fu\|^2 + \frac{2t}{1 - \tau_t}\langle u - x_t, Fu\rangle + \frac{2t^2 k}{1 - \tau_t}\|x_t - u\|\|Fu\|. \tag{3.13}$$

From $\tau_t = \sqrt{1 - t(2\eta - tk^2)}$, we have $\lim_{t\to 0}(t^2/(1 - \tau_t)) = 0$ and $\lim_{t\to 0}(2t^2 k/(1 - \tau_t)) = 0$. Observe that, if $x_t \rightharpoonup u$, we have $\lim_{t\to 0}(2t/(1 - \tau_t))\langle u - x_t, Fu\rangle = 0$.

Since $\{x_t\}$ is bounded, we see that if $\{t_n\}$ is a sequence in $(0, \eta/k^2 - \epsilon]$ such that $t_n \to 0$ and $x_{t_n} \rightharpoonup \tilde{x}$, then by (3.13), we see $x_{t_n} \to \tilde{x}$. Moreover, by (3.11) and using Lemma 2.1, we have $\tilde{x} \in F(T)$. We next prove that $\tilde{x}$ solves the variational inequality (3.4). From (3.3) and $u \in F(T)$, we have

$$\begin{aligned}
\|x_t - u\|^2 &\le \|(I - tF)x_t - u\|^2 \\
&= \|x_t - u\|^2 + t^2\|Fx_t\|^2 - 2t\langle Fx_t, x_t - u\rangle,
\end{aligned} \tag{3.14}$$

that is,

$$\langle Fx_t, x_t - u\rangle \le \frac{t}{2}\|Fx_t\|^2. \tag{3.15}$$

Now replacing t in (3.15) with t_n and letting $n \to \infty$, we have

$$\langle F\tilde{x}, \tilde{x} - u\rangle \le 0. \tag{3.16}$$

That is $\tilde{x} \in F(T)$ is a solution of (3.4); hence $\tilde{x} = x^*$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ (as $t \to 0$) equals x^*. Therefore, $x_t \to x^*$ as $t \to 0$. □

Setting $F = A$ in Theorem 3.1, we can obtain the following result.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $0 < \tilde{\gamma} \leq \|A\|$. For each $t \in (0, \tilde{\gamma}/\|A\|^2)$, let the net $\{x_t\}$ be generated by $x_t = TP_C[(I - tA)x_t]$. Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point x^* of T which solves the variational inequality:*

$$\langle Ax^*, x^* - u \rangle \leq 0, \quad u \in F(T). \tag{3.17}$$

Setting $F = I$, the identity mapping, in Theorem 3.1, we can obtain the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. For each $t \in (0,1)$, let the net $\{x_t\}$ be generated by (1.6). Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to a fixed point x^* of T which solves the variational inequality:*

$$\langle x^*, x^* - u \rangle \leq 0, \quad u \in F(T). \tag{3.18}$$

Remark 3.4. The Corollary 3.3 complements the results of Theorem 3.1 in Yao et al. [11], that is, x^* is the solution of the variational inequality: $\langle x^*, x^* - u \rangle \leq 0,\ u \in F(T)$.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let F be a k-Lipschitzian and η-strongly monotone operator on H with $0 < \eta \leq k$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$ and satisfy the conditions:*

(A1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(A2) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Suppose that $\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let T be a mapping of C into itself defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by

$$\begin{aligned} y_n &= P_C[(I - \alpha_n F)x_n], \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T_n y_n, \quad n \geq 1. \end{aligned} \tag{3.19}$$

Then the sequence $\{x_n\}$ strongly converges to a $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$ which solves the variational inequality:

$$\langle Fx^*, x^* - u \rangle \leq 0, \quad u \in \bigcap_{n=1}^{\infty} F(T_n). \tag{3.20}$$

Proof. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. From $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $0 < \alpha_n \le \eta/k^2 - \epsilon$ for all n. In fact, let $u \in \bigcap_{n=1}^{\infty} F(T_n)$, from (3.19) and using Lemma 2.5, we have

$$\begin{aligned}\|y_n - u\| &= \|P_C[(I-\alpha_n F)x_n] - P_C u\| \\ &\le \|(I-\alpha_n F)x_n - (I-\alpha_n F)u - \alpha_n F u\| \\ &\le \tau_{\alpha_n}\|x_n - u\| + \alpha_n\|Fu\|,\end{aligned} \tag{3.21}$$

where $\tau_{\alpha_n} = \sqrt{1-\alpha_n(2\eta - \alpha_n k^2)}$. Then from (3.19) and (3.21), we obtain

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1-\beta_n)(x_n - u) + \beta_n(T_n y_n - u)\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n[\tau_{\alpha_n}\|x_n - u\| + \alpha_n\|Fu\|] \\ &\le [1-\beta_n(1-\tau_{\alpha_n})]\|x_n - u\| + \beta_n\alpha_n\|Fu\| \\ &\le \max\left\{\|x_n - u\|, \frac{\alpha_n\|Fu\|}{1-\tau_{\alpha_n}}\right\}.\end{aligned} \tag{3.22}$$

By induction, we have

$$\|x_n - u\| \le \max\{\|x_1 - u\|, M_1\|Fu\|\}, \tag{3.23}$$

where $M_1 = \sup\{\alpha_n/(1-\tau_{\alpha_n}) : 0 < \alpha_n \le \eta/k^2 - \epsilon\} < +\infty$. Therefore, $\{x_n\}$ is bounded. We also obtain that $\{y_n\}$, $\{T_n y_n\}$, and $\{Fx_n\}$ are bounded. Without loss of generality, we may assume that $\{x_n\}$, $\{y_n\}$, $\{T_n y_n\}$, and $\{Fx_n\} \subset B$, where B is a bounded set of C.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this end, define a sequence $\{z_n\}$ by $z_n = T_n y_n$. It follows that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|T_{n+1}y_{n+1} - T_n y_n\| \\ &\le \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_n y_n\| \\ &\le \|y_{n+1} - y_n\| + \|T_{n+1}y_n - T_n y_n\| \\ &\le \|(I-\alpha_{n+1}F)x_{n+1} - (I-\alpha_n F)x_n\| + \|T_{n+1}y_n - T_n y_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_{n+1}\|Fx_{n+1}\| + \alpha_n\|Fx_n\| + \sup\{\|T_{n+1}z - T_n z\| : z \in B\}.\end{aligned} \tag{3.24}$$

Thus, we have

$$\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le \alpha_{n+1}\|Fx_{n+1}\| + \alpha_n\|Fx_n\| + \sup\{\|T_{n+1}z - T_n z\| : z \in B\}. \tag{3.25}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and (3.25), we have

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{3.26}$$

By (3.26), (A2), and using Lemma 2.2, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Therefore,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|z_n - x_n\| = 0. \tag{3.27}$$

Step 3. We claim that $\lim_{n\to\infty}\|x_n - T_nx_n\| = 0$. Observe that

$$\begin{aligned}
\|x_n - T_nx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_nx_n\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_nx_n\| + \beta_n\left\|T_ny_n - T_nx_n\right\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_nx_n\| + \beta_n\left\|y_n - x_n\right\| \\
&\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - T_nx_n\| + \alpha_n\|Fx_n\|,
\end{aligned} \tag{3.28}$$

that is,

$$\|x_n - T_nx_n\| \le \frac{1}{\beta_n}(\|x_{n+1} - x_n\| + \alpha_n\|Fx_n\|) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{3.29}$$

Step 4. We claim that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. Observe that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - T_nx_n\| + \|T_nx_n - Tx_n\| \\
&\le \|x_n - T_nx_n\| + \sup\{\|T_nz - Tz\| : z \in B\}.
\end{aligned} \tag{3.30}$$

Hence, from Step 3 and using Lemma 2.4, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.31}$$

Step 5. We claim that $\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle \le 0$, where $x^* = \lim_{t\to 0} x_t$ and x_t is defined by (3.3). Since x_n is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to ω. From Step 4, we obtain $Tx_{n_k} \rightharpoonup \omega$. From Lemma 2.1, we have $\omega \in F(T)$. Hence, by Theorem 3.1, we have

$$\limsup_{n\to\infty}\langle Fx^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle Fx^*, x^* - x_{n_k}\rangle = \langle Fx^*, x^* - \omega\rangle \le 0. \tag{3.32}$$

Step 6. We claim that $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$. From (3.19), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \left\|T_n y_n - x^*\right\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \left\|y_n - x^*\right\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \|(I-\alpha_n F)x_n - (I-\alpha_n F)x^* - \alpha_n F x^*\|^2 \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 \\
&\quad + \beta_n \left[\tau_{\alpha_n}^2 \|x_n - x^*\|^2 + \alpha_n^2 \|Fx^*\| + 2\alpha_n \langle (I-\alpha_n F)x^* - (I-\alpha_n F)x_n, Fx^*\rangle\right] \\
&\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n \tau_{\alpha_n} \|x_n - x^*\|^2 + \beta_n \alpha_n^2 \|Fx^*\|^2 + 2\alpha_n \beta_n \langle x^* - x_n, Fx^*\rangle \\
&\quad + 2\beta_n \alpha_n^2 \langle Fx_n - Fx^*, Fx^*\rangle \\
&\leq \left[1 - \beta_n(1-\tau_{\alpha_n})\right]\|x_n - x^*\|^2 + \beta_n \alpha_n^2 \|Fx^*\|^2 + 2\alpha_n \beta_n \langle x^* - x_n, Fx^*\rangle \\
&\quad + 2\beta_n \alpha_n^2 k \|x_n - x^*\| \|Fx^*\| \\
&\leq \left[1 - \beta_n(1-\tau_{\alpha_n})\right]\|x_n - x^*\|^2 + \beta_n \alpha_n^2 M_2 + 2\alpha_n \beta_n \langle x^* - x_n, Fx^*\rangle + 2\beta_n \alpha_n^2 M_2 \\
&\leq \left[1 - \beta_n(1-\tau_{\alpha_n})\right]\|x_n - x^*\|^2 + \beta_n(1-\tau_{\alpha_n})\left\{\frac{3\alpha_n^2 M_2}{1-\tau_{\alpha_n}} + 2M_1 \langle x^* - x_n, Fx^*\rangle\right\} \\
&= (1-\lambda_n)\|x_n - x^*\|^2 + \lambda_n \delta_n,
\end{aligned} \tag{3.33}$$

where $M_2 = \sup\{\|Fx^*\|^2, k\|x_n - x^*\|\|Fx^*\|\}$, $\lambda_n = \beta_n(1-\tau_{\alpha_n})$, and $\delta_n = 3\alpha_n^2 M_2/(1-\tau_{\alpha_n}) + 2M_1\langle x^* - x_n, Fx^*\rangle$. It is easy to see that $\lambda_n \to 0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $\limsup_{n\to\infty} \delta_n \leq 0$. Hence, by Lemma 2.3, the sequence $\{x_n\}$ converges strongly to $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$. From $x^* = \lim_{t\to 0} x_t$ and Theorem 3.1, we have that x^* is the unique solution of the variational inequality: $\langle Fx^*, x^* - u\rangle \leq 0,\ u \in \bigcap_{n=1}^{\infty} F(T_n)$. □

Remark 3.6. From Remark 3.1 of Peng and Yao [18], we obtain that $\{W_n\}$ is a sequence of nonexpansive mappings satisfying condition $\sum_{n=1}^{\infty} \sup\{\|W_{n+1}z - W_n z\| : z \in B\} < \infty$ for any bounded subset B of H. Moreover, let W be the W-mapping; we know that $Wy = \lim_{n\to\infty} W_n y$ for all $y \in C$ and that $F(W) = \bigcap_{n=1}^{\infty} F(W_n)$. If we replace $\{T_n\}$ by $\{W_n\}$ in the recursion formula (3.19), we can obtain the corresponding results of the so-called W-mapping.

Setting $F = A$ and $T_n = T$ in Theorem 3.5, we can obtain the following result.

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let A be a strongly positive bounded linear operator with coefficient $0 < \widetilde{\gamma} \leq \|A\|$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$ and satisfy the conditions (A1) and (A2). For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by*

$$\begin{aligned}
y_n &= P_C[(I-\alpha_n A)x_n], \\
x_{n+1} &= (1-\beta_n)x_n + \beta_n T y_n, \quad n \geq 1.
\end{aligned} \tag{3.34}$$

Then the sequence $\{x_n\}$ strongly converges to a fixed point x^* of T which solves the variational inequality:

$$\langle Ax^*, x^* - u \rangle \le 0, \quad u \in F(T). \tag{3.35}$$

Setting $F = I$ and $T_n = T$ in Theorem 3.5, we can obtain the following result.

Corollary 3.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $(0,1)$ and satisfy the conditions (A1) and (A2). For given $x_1 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated by* (1.7)*. Then the sequence $\{x_n\}$ strongly converges to a fixed point x^* of T which solves the variational inequality:*

$$\langle x^*, x^* - u \rangle \le 0, \quad u \in F(T). \tag{3.36}$$

Remark 3.9. The Corollary 3.8 complements the results of Theorem 3.2 in Yao et al. [11], that is, x^* is the solution of the variational inequality: $\langle x^*, x^* - u \rangle \le 0$, $u \in F(T)$.

Acknowledgment

This paper is supported by the National Science Foundation of China under Grant (10771175).

References

[1] S. Reich, "Almost convergence and nonlinear ergodic theorems," *Journal of Approximation Theory*, vol. 24, no. 4, pp. 269–272, 1978.

[2] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, vol. 83 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker Inc., New York, NY, USA, 1984.

[3] C. Byrne, "A unified treatment of some iterative algorithms in signal processing and image reconstruction," *Inverse Problems*, vol. 20, no. 1, pp. 103–120, 2004.

[4] D. Youla, "Mathematical theory of image restoration by the method of convex projection," in *Image Recovery Theory and Applications*, H. Stark, Ed., pp. 29–77, Academic Press, Orlando, Fla, USA, 1987.

[5] C. I. Podilchuk and R. J. Mammone, "Image recovery by convex projections using a least-squares constraint," *Journal of the Optical Society of America*, vol. 7, no. 3, pp. 517–512, 1990.

[6] P. L. Combettes, "On the numerical robustness of the parallel projection method in signal synthesis," *IEEE Signal Processing Letters*, vol. 8, no. 2, pp. 45–47, 2001.

[7] H. W. Engl and A. Leitão, "A Mann iterative regularization method for elliptic Cauchy problems," *Numerical Functional Analysis and Optimization*, vol. 22, no. 7-8, pp. 861–884, 2001.

[8] P. L. Combettes, "The convex feasibility problem in image recovery," in *Advances in Imaging and Electron Physics*, P. Hawkes, Ed., vol. 95, pp. 155–270, Academic Press, New York, NY, USA, 1996.

[9] F. E. Browder, "Fixed-point theorems for noncompact mappings in Hilbert space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 53, pp. 1272–1276, 1965.

[10] B. Halpern, "Fixed points of nonexpanding maps," *Bulletin of the American Mathematical Society*, vol. 73, pp. 957–961, 1967.

[11] Y. Yao, Y. C. Liou, and G. Marino, "Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 279058, 7 pages, 2009.

[12] I. Yamada, “The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings,” in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000)*, D. Butnariu, Y. Censor, and S. Reich, Eds., vol. 8 of *Stud. Comput. Math.*, pp. 473–504, North-Holland, Amsterdam, The Netherlands, 2001.

[13] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

[14] T. Suzuki, “Strong convergence of Krasnoselskii and Mann’s type sequences for one-parameter nonexpansive semigroups without Bochner integrals,” *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[15] L. S. Liu, “Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces,” *Journal of Mathematical Analysis and Applications*, vol. 194, no. 1, pp. 114–125, 1995.

[16] H.-K. Xu, “Iterative algorithms for nonlinear operators,” *Journal of the London Mathematical Society*, vol. 66, no. 1, pp. 240–256, 2002.

[17] K. Aoyama, Y. Kimura, W. Takahashi, and M. Toyoda, “Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 67, no. 8, pp. 2350–2360, 2007.

[18] J.-W. Peng and J.-C. Yao, “A viscosity approximation scheme for system of equilibrium problems, nonexpansive mappings and monotone mappings,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 12, pp. 6001–6010, 2009.