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Research Article

Strong Convergence of a Generalized Iterative Method for Semigroups of Nonexpansive Mappings in Hilbert Spaces

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Using δ-strongly accretive and λ-strictly pseudocontractive mapping, we introduce a general iterative method for finding a common fixed point of a semigroup of non-expansive mappings in a Hilbert space, with respect to a sequence of left regular means defined on an appropriate space of bounded real-valued functions of the semigroup. We prove the strong convergence of the proposed iterative algorithm to the unique solution of a variational inequality.

1. Introduction

Let H be a real Hilbert space. A mapping T of H into itself is called non-expansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in H$. By $\mathrm{Fix}(T)$, we denote the set of fixed points of T (i.e., $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$).

Mann [1] introduced an iteration procedure for approximation of fixed points of a non-expansive mapping T on a Hilbert space as follows. Let $x_0 \in H$ and

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \qquad n \geq 0, \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. See also [2].

On the other hand, Moudafi [3] introduced the viscosity approximation method for fixed point of non-expansive mappings (see [4] for further developments in both Hilbert and Banach spaces). Let f be a contraction on a Hilbert space H (i.e., $\|fx - fy\| \leq \alpha\|x - y\|$, for all $x, y \in H$ and $0 \le \alpha < 1$). Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \qquad n \ge 0, \tag{1.2}$$

where α_n is sequence in $(0, 1)$. It is proved in [3, 4] that, under appropriate condition imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.2) converges strongly to the unique solution x^* in $\mathrm{Fix}(T)$ of the variational inequality:

$$\left\langle (I - f)x^*, x - x^* \right\rangle \ge 0, \quad x \in \mathrm{Fix}(T). \tag{1.3}$$

Assume that A is strongly positive, that is, there is a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2, \qquad \forall x \in H. \tag{1.4}$$

In [4] (see also [5]), it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \qquad n \ge 0, \tag{1.5}$$

converges strongly to the unique solution of the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.6}$$

provided that the sequence $\{\alpha_n\}$ satisfies certain conditions. Marino and Xu [6] combined the iterative (1.5) with the viscosity approximation method (1.2) and considered the following general iterative methods:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \ge 0, \tag{1.7}$$

where $0 < \gamma < \overline{\gamma}/\alpha$. They proved that if $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(C_1) $\alpha_n \to 0$,

(C_2) $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(C_3) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$,

then, the sequence $\{x_n\}$ generated by (1.7) converges strongly, as $n \to \infty$, to the unique solution of the variational inequality:

$$\left\langle (A - \gamma f)x^*, x - x^* \right\rangle \ge 0, \quad \forall x \in \mathrm{Fix}(T), \tag{1.8}$$

which is the optimality condition for minimization problem

$$\min_{x\in \mathrm{Fix}(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.9}$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$, for all $x \in H$).

Let E^* be the topological dual of a Banach space E. The value of $j \in E^*$ at $x \in E$ will be denoted by $\langle x, j\rangle$ or $j(x)$. With each $x \in E$, we associate the set

$$J(x) = \left\{ j \in E^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2 \right\}. \tag{1.10}$$

Using the Hahn-Banach theorem, it is immediately clear that $J(x) \neq \phi$ for each $x \in E$. The multivalued mapping J from E into E^* is said to be the (normalized) duality mapping. A Banach space E is said to be smooth if the duality mapping J is single valued. As it is well known, the duality mapping is the identity when E is a Hilbert space; see [7].

Let δ and λ be two positive real numbers such that $\delta, \lambda < 1$. Recall that a mapping F with domain $D(F)$ and range $R(F)$ in E is called δ-strongly accretive if, for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \geq \delta \|x-y\|^2. \tag{1.11}$$

Recall also that a mapping F is called λ-strictly pseudo-contractive if, for each $x, y \in D(F)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Fx - Fy, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \|(x-y) - (Fx - Fy)\|^2. \tag{1.12}$$

It is easy to see that (1.12) can be rewritten as

$$\langle (I-F)x - (I-F)y, j(x-y)\rangle \geq \lambda \|(I-F)x - (I-F)y\|^2, \tag{1.13}$$

see [8].

In this paper, motivated and inspired by Atsushiba and Takahashi [9], Lau et al. [10], Marino and Xu [6] and Xu [4, 11], we introduce the iterative below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) T_{\mu_n}(x_n), \qquad n \geq 0, \tag{1.14}$$

where F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, f is a contraction on a Hilbert space H with coefficient $0 < \alpha < 1$, γ is a positive real number such that $\gamma < 1 - \sqrt{(1-\delta)/\lambda}/\alpha$, and $\varphi = \{T_t : t \in S\}$ is a non-expansive semigroup on H such that the set $\mathrm{Fix}(\varphi)$ of common fixed point of φ is nonempty, X is a subspace of $B(S)$ such that $1 \in X$ and the mapping $t \to \langle T_t(x), y\rangle$ is an element of X for each $x, y \in H$, and $\{\mu_n\}$ is a sequence of means on X. Our purpose in this paper is to introduce this general iterative algorithm for approximating a common fixed points of semigroups of non-expansive mappings which solves some variational inequality. We will prove that if $\{\mu_n\}$ is left regular and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions (C_1) and (C_2), then $\{x_n\}$ converges strongly to $x^* \in \mathrm{Fix}(\varphi)$, which solves the variational inequality:

$$\left\langle (F-\gamma f)x^*, x-x^*\right\rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{1.15}$$

Various applications to the additive semigroup of nonnegative real numbers and commuting pairs of non-expansive mappings are also presented. It is worth mentioning that we obtain our result without assuming condition (C_3).

2. Preliminaries

Let S be a semigroup and let $B(S)$ be the space of all bounded real-valued functions defined on S with supremum norm. For $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ by

$$\left(l_s f\right)(t) = f(st), \quad \left(r_s f\right)(t) = f(ts), \quad \forall t \in S. \tag{2.1}$$

Let X be a subspace of $B(S)$ containing 1, and let X^* be its dual. An element μ in X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (resp., right invariant), that is, $l_s(X) \subset X$ (resp., $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is said to be left invariant (right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp., right) amenable if X has a left (resp., right) invariant mean. X is amenable if X is both left and right amenable. As it is well known, $B(S)$ is amenable when S is a commutative semigroup; see [12]. A net $\{\mu_\alpha\}$ of means on X is said to be left regular if

$$\lim_{\alpha}\left\|l_s{}^*\mu_\alpha - \mu_\alpha\right\| = 0, \tag{2.2}$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty closed and convex subset of a reflexive Banach space E. A family $\varphi = \{T_t : t \in S\}$ of mapping from C into itself is said to be a non-expansive semigroup on C if T_t is non-expansive and $T_{ts} = T_t T_s$ for each $t, s \in S$. We denote by $\mathrm{Fix}(\varphi)$ the set of common fixed points of φ, that is,

$$\mathrm{Fix}(\varphi) = \bigcap_{t\in S}\{x \in C : T_t x = x\}. \tag{2.3}$$

The open ball of radius r centered at 0 is denoted by B_r. For subset D of E, by $\overline{\mathrm{co}}D$, we denote the closed convex hull of D. Weak convergence is denoted by $\rightharpoonup$, and strong convergence is denoted by $\rightarrow$.

Lemma 2.1 (see [12, 13]). *Let f be a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact, and let X be a subspace of $B(S)$ containing all functions $t \to \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle, \tag{2.4}$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X then

$$\int f(t) d\mu(t) \in \overline{\text{co}}\{f(t) : t \in S\}. \tag{2.5}$$

One can write f_μ by $\int f(t) d\mu(t)$.

Lemma 2.2 (see [13]). *Let C be a closed convex subset of a Hilbert space H, $\varphi = \{T_t : t \in S\}$ a semigroup from C into C such that $\text{Fix}(\varphi) \neq \emptyset$, the mapping $t \to \langle T_t x, y \rangle$ an element of X for each $x \in C$ and $y \in H$, and μ a mean on X. If one writes $T_\mu(x)$ instead of $\int T_t x d\mu(t)$, then the following holds.*

(i) *T_μ is non-expansive mapping from C into C.*

(ii) *$T_\mu(x) = x$ for each $x \in \text{Fix}(\varphi)$.*

(iii) *$T_\mu(x) \in \overline{\text{co}}\{T_t x : t \in S\}$ for each $x \in C$.*

(iv) *If μ is left invariant, then T_μ is a non-expansive retraction from C onto $\text{Fix}(\varphi)$.*

Let C be a nonempty subset of a normed space E, and let $x \in E$. An element $y_0 \in C$ is said to be the best approximation to x if

$$\|x - y_0\| = d(x, C), \tag{2.6}$$

where $d(x, C) = \inf_{y \in C} \|x - y\|$. The number $d(x, C)$ is called the distance from x to C or the error in approximating x by C. The (possibly empty) set of all best approximation from x to C is denoted by

$$P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}. \tag{2.7}$$

This defines a mapping P_C from X into 2^C and is called metric (the nearest point) projection onto C.

Lemma 2.3 (see [7]). *Let C be a nonempty convex subset of a smooth Banach space E and let $x \in X$ and $y \in C$. Then, the following is equivalent.*

(i) *y is the best approximation to x.*

(ii) *y is a solution of the variational inequality*

$$\langle y - z, J(x - y) \rangle \geq 0, \quad \forall z \in C. \tag{2.8}$$

Let C be a nonempty subset of a Banach space E and $T : C \to E$ a mapping. Then T is said to be demiclosed at $v \in E$ if, for any sequence $\{x_n\}$ in C, the following implication holds:

$$x_n \rightharpoonup u \in C, \quad Tx_n \longrightarrow v, \quad \text{imply } Tu = v. \tag{2.9}$$

Lemma 2.4 (see [14]). *Let C be a nonempty closed convex subset of a Hilbert space H and suppose that $T : C \to H$ is non-expansive. Then, the mapping $I - T$ is demiclosed at zero.*

The following lemma is well known.

Lemma 2.5. *Let H be a real Hilbert space. Then, for all $x, y \in H$*

(i) $\|x - y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$,

(ii) $\|x - y\|^2 \geq \|x\|^2 + 2\langle y, x\rangle$.

Lemma 2.6 (see [11]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - b_n)a_n + b_n c_n, \qquad n \geq 0, \tag{2.10}$$

where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers satisfying the following conditions:

(i) $\{b_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} b_n = \infty$,

(ii) *either* $\limsup_{n\to\infty} c_n \leq 0$ *or* $\sum_{n=0}^{\infty} |b_n c_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

The following lemma will be frequently used throughout the paper. For the sake of completeness, we include its proof.

Lemma 2.7. *Let E be a real smooth Banach space and $F : E \to E$ a mapping.*

(i) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then, $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.*

(ii) *If F is δ-strongly accretive and λ-strictly pseudo-contractive with $\delta + \lambda > 1$, then, for any fixed number $\tau \in (0,1)$, $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda}\,)$.*

Proof. (i) From (1.11) and (1.13), we obtain

$$\lambda\left\|(I-F)x - (I-F)y\right\|^2 \leq \left\|x - y\right\|^2 - \langle Fx - Fy, J(x-y)\rangle \leq (1-\delta)\left\|x-y\right\|^2. \tag{2.11}$$

Because $\delta + \lambda > 1 \Leftrightarrow \sqrt{(1-\delta)/\lambda} \in (0,1)$, we have

$$\left\|(I-F)x - (I-F)y\right\| \leq \sqrt{\frac{1-\delta}{\lambda}}\left\|x-y\right\|, \tag{2.12}$$

and, therefore, $I - F$ is contractive with constant$\sqrt{(1-\delta)/\lambda}$.

(ii) Because $I - F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$, for each fixed number $\tau \in (0,1)$, we have

$$\begin{aligned}
\|x - y - \tau(F(x) - F(y))\| &= \|(1-\tau)(x-y) + \tau[(I-F)x - (I-F)y]\| \\
&\leq (1-\tau)\|x-y\| + \tau\|(I-F)x - (I-F)y\| \\
&\leq (1-\tau)\|x-y\| + \tau\sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \\
&= \left(1 - \tau\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x-y\|.
\end{aligned} \tag{2.13}$$

This shows that $I - \tau F$ is contractive with constant $1 - \tau(1 - \sqrt{(1-\delta)/\lambda})$. □

Throughout this paper, F will denote a δ-strongly accretive and λ-strictly pseudo-contractive mapping with $\delta + \lambda > 1$, and f is a contraction with coefficient $0 < \alpha < 1$ on a Hilbert space H. We will also always use γ to mean a number in $(0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$.

3. Strong Convergence Theorem

The following is our main result.

Theorem 3.1. *Let $\varphi = \{T_t : t \in S\}$ be a non-expansive semigroup on a real Hilbert space H such that* $\text{Fix}(\varphi) \neq \emptyset$*. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \to \langle T_t x, y\rangle$ is an element of X for each $x, y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X, and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\alpha_n \to 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$. Let $x_0 \in H$ and $\{x_n\}$ be generated by the iteration algorithm* (1.14)*. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \text{Fix}(\varphi)$, which is a unique solution of the variational inequality* (1.15)*. Equivalently, one has*

$$P_{\text{Fix}(\varphi)}(I - F + \gamma f)x^* = x^*. \tag{3.1}$$

Proof. First, we claim that $\{x_n\}$ is bounded. Let $p \in \text{Fix}(\varphi)$; by Lemmas 2.2 and 2.7 we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - p\| \\
&= \|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)p - \alpha_n F(p)\| \\
&\leq \alpha_n\|\gamma f(x_n) - F(p)\| + \|(I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)p\| \\
&\leq \alpha_n\|\gamma f(x_n) - \gamma f(p)\| \\
&\quad + \alpha_n\|\gamma f(p) - F(p)\| + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|T\mu_n(x_n) - p\| \\
&\leq \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \gamma\alpha\right)\right)\|x_n - p\| + \alpha_n\|\gamma f(p) - F(p)\|
\end{aligned}$$

$$
\begin{aligned}
&= \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)\right)\|x_n-p\| \\
&\quad+\frac{\alpha_n\left(1-\sqrt{(1-\delta)/\lambda}-\gamma\alpha\right)}{\left(1-\gamma\alpha-\sqrt{(1-\delta)/\lambda}\right)}\|\gamma f(p)-F(p)\| \\
&\le \max\left\{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)^{-1}\|\gamma f(p)-F(p)\|,\|x_n-p\|\right\}.
\end{aligned}
\tag{3.2}
$$

By induction,

$$
\|x_n-p\| \le \max\left\{\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)^{-1}\|\gamma f(p)-F(p)\|,\|x_0-p\|\right\} = M_0. \tag{3.3}
$$

Therefore, $\{x_n\}$ is bounded and so is $\{f(x_n)\}$.

Set $D=\{y\in H : \|y-p\|\le M_0\}$. We remark that D is φ-invariant bounded closed convex set and $\{x_n\}\subset D$. Now we claim that

$$
\limsup_{n\to\infty}\sup_{y\in D}\left\|T_{\mu_n}(y)-T_t\left(T_{\mu_n}(y)\right)\right\| = 0, \quad \forall t\in S. \tag{3.4}
$$

Let $\epsilon>0$. By [15, Theorem 1.2], there exists $\delta>0$ such that

$$
\overline{\text{co}}F_\delta(T_t;D)+B_\delta\subset F_\epsilon(T_t;D), \quad \forall t\in S. \tag{3.5}
$$

Also by [15, Corollary 1.1], there exists a natural number N such that

$$
\left\|\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}(y)-T_t\left(\frac{1}{N+1}\sum_{i=0}^{N}T_{t^i s}(y)\right)\right\|\le\delta, \tag{3.6}
$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l_{t^i}^* \mu_n\| \le \delta/(M_0 + \|p\|)$ for $n \ge n_0$ and $i = 1, 2, \ldots, N$. Then we have

$$\begin{aligned}
&\sup_{y\in D}\left\| T_{\mu_n}(y) - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y)\,d\mu_n(s)\right\| \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left| \langle T_{\mu_n}(y), z\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y)\,d\mu_n(s), z\right\rangle\right| \\
&= \sup_{y\in D}\sup_{\|z\|=1}\left| \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_s(y), z\rangle - \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{t^i s}(y), z\rangle\right| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T_s(y), z\rangle - (l_{t^i}^*\mu_n)_s\langle T_s(y), z\rangle\right| \\
&\le \max_{i=0,1,2,\ldots,N}\|\mu_n - l_{t^i}^*\mu_n\|(M_0 + \|p\|) \le \delta, \quad \forall n \ge n_0.
\end{aligned} \tag{3.7}$$

By Lemma 2.2 we have

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y)\,d\mu_n(s) \in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s(y)) : s \in S\right\}. \tag{3.8}$$

It follows from (3.5), (3.6), (3.7), and (3.8) that

$$T_{\mu_n}(y) \in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}(y) : s \in S\right\} + B_\delta \subset \overline{\mathrm{co}}F_\delta(T_t; D) + B_\delta \subset F_\epsilon(T_t; D), \tag{3.9}$$

for all $y \in D$ and $n \ge n_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D}\|T_t(T_{\mu_n}(y)) - T_{\mu_n}(y)\| \le \epsilon. \tag{3.10}$$

Since $\epsilon > 0$ is arbitrary, we get (3.4). In this stage, we will show that

$$\lim_{n\to\infty}\|x_n - T_t(x_n)\| = 0, \quad \forall t \in S. \tag{3.11}$$

Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$, which satisfies (3.5). Take

$$L_0 = \left[\left(1 + \gamma\alpha + \sqrt{\frac{1-\delta}{\lambda}}\right)M_0 + \|\gamma f(p) - F(p)\|\right]. \tag{3.12}$$

From $\lim_{n\to\infty}\alpha_n = 0$ and (3.4) there exists $n_0 \in \mathbb{N}$ such that $\alpha_n \leq \delta/L_0$ and $T_{\mu_n}(x_n) \in F_\delta(T_t)$, for all $n \geq n_0$. By Lemma 2.7, we have

$$\begin{aligned}
&\alpha_n\|\gamma f(x_n) - FT_{\mu_n}(x_n)\| \\
&\quad\leq \alpha_n\left(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - F(T_{\mu_n}(x_n))\|\right) \\
&\quad\leq \alpha_n(\gamma\alpha\|x_n - p\| + \|\gamma f(p) - F(p)\|) \\
&\qquad + \alpha_n\left(\|(I-F)p - (I-F)T_{\mu_n}(x_n)\| + \|p - T_{\mu_n}(x_n)\|\right) \\
&\quad\leq \alpha_n\left(1 + \sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)\|x_n - p\| + \alpha_n\|\gamma f(p) - F(p)\| \\
&\quad\leq \alpha_n\left[\left(1 + \sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)M_0 + \|\gamma f(p) - F(p)\|\right] \\
&\quad\leq \alpha_n L_0 \leq \delta,
\end{aligned} \tag{3.13}$$

for all $n \geq n_0$. Therefore, we have

$$x_{n+1} = T_{\mu_n}(x_n) + \alpha_n\left[\gamma f(x_n) + F(T_{\mu_n}(x_n))\right] \in F_\delta(T) + B_\delta \subset F_\epsilon(T_t), \tag{3.14}$$

for all $n \geq n_0$. This shows that

$$\|x_n - T_t(x_n)\| \leq \epsilon, \quad \forall n \geq n_0. \tag{3.15}$$

Since $\epsilon > 0$ is arbitrary, we get (3.11).

Let $Q = P_{\mathrm{Fix}(\varphi)}$. Then $Q(I - F - \gamma f)$ is a contraction of H into itself. In fact, we see that

$$\begin{aligned}
&\|Q(I - F + \gamma \mathrm{f})(x) - Q(I - F + \gamma f)(y)\| \\
&\quad\leq \|(I - F + \gamma f)(x) - (I - F + \gamma f)(y)\| \\
&\quad\leq \|(I - F)(x) - (I - F)(y)\| + \gamma\|f(x) - f(y)\| \\
&\quad\leq \left(\sqrt{\frac{1-\delta}{\lambda}} + \gamma\alpha\right)\|x - y\|,
\end{aligned} \tag{3.16}$$

and hence $Q(I - F - \gamma f)$ is a contraction due to $(\sqrt{(1-\delta)/\lambda} + \gamma\alpha) \in (0,1)$.

Therefore, by Banach contraction principal, $P_{\mathrm{Fix}(\varphi)}(\gamma f + I - F)$ has a unique fixed point x^*. Then using Lemma 2.3, x^* is the unique solution of the variational inequality

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{3.17}$$

We show that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle \le 0. \tag{3.18}$$

Indeed, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle = \lim_{k\to\infty}\langle \gamma f(x^*) - F(x^*), x_{n_k} - x^*\rangle. \tag{3.19}$$

Because $\{x_n\}$ is bounded, we may assume that $x_n \rightharpoonup z$. In terms of Lemma 2.4 and (3.11), we conclude that $z \in \mathrm{Fix}(\varphi)$. Therefore,

$$\limsup_{n\to\infty}\langle \gamma f(x^*) - F(x^*), x_n - x^*\rangle = \langle \gamma f(x^*) - F(x^*), z - x^*\rangle \le 0. \tag{3.20}$$

Finally, we prove that $x_n \to x^*$ as $n \to \infty$. By Lemmas 2.5 and 2.7 we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T_{\mu_n}(x_n) - x^*\right\|^2 \\
&\quad = \left\|\alpha_n\gamma f(x_n) - \alpha_n F(x^*) + (I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)x^*\right\|^2 \\
&\quad = \left\|(I - \alpha_n F)T_{\mu_n}(x_n) - (I - \alpha_n F)x^*\right\|^2 + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
&\quad \le \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle \\
&\quad \le \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - x^*\|^2 + 2\alpha_n\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\qquad + 2\alpha_n\langle \gamma f(x^*) - F(x^*), x_{n+1} - x^*\rangle.
\end{aligned} \tag{3.21}$$

On the other hand

$$\begin{aligned}
&\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^*\rangle \\
&\quad \le \gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\
&\quad \le \gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 \\
&\qquad + \gamma\alpha\|x_n - x^*\|\sqrt{2\left|\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^*\rangle\right|}\sqrt{\alpha_n}.
\end{aligned} \tag{3.22}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, we can take a constant $G_0 > 0$ such that

$$\gamma\alpha\|x_n - x^*\|\sqrt{2\left|\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^* \rangle\right|} < G_0, \quad \forall n \in \mathbb{N}. \tag{3.23}$$

So from the above, we reach the following:

$$\langle \gamma f(x_n) - \gamma f(x^*), x_{n+1} - x^* \rangle \leq \gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 + G_0\sqrt{\alpha_n}. \tag{3.24}$$

Substituting (3.24) in (3.21), we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leq \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2\|x_n - x^*\|^2 + 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - x^*\|^2 \\
&\qquad + 2\alpha_n G_0\sqrt{\alpha_n} + 2\alpha_n\langle \gamma f(x_n) - F(x^*), x_{n+1} - x^* \rangle \\
&\quad = \left(1 - 2\alpha_n\left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right)\|x_n - x^*\|^2 \\
&\qquad + \alpha_n\left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - x^*\|^2 + 2G_0\sqrt{\alpha_n} + 2\langle \gamma f(x^*) - F(x^*), x_n - x^* \rangle\right].
\end{aligned} \tag{3.25}$$

It follows that

$$\|x_{n+1} - x^*\|^2 \leq \left(1 - \alpha_n\left[2\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha\gamma\right) + 2\alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right)\|x_n - x^*\|^2 + \alpha_n\beta_n, \tag{3.26}$$

where

$$\beta_n = \left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\|x_n - x^*\|^2 + 2G_0\sqrt{\alpha_n} + 2\langle \gamma f(x^*) - F(x^*), x_n - x^* \rangle\right]. \tag{3.27}$$

Since $\{x_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, by (3.18), we get

$$\limsup_{n\to\infty} \beta_n \leq 0. \tag{3.28}$$

Consequently, applying Lemma 2.6, to (3.26), we conclude that $x_n \to x^*$. □

Corollary 3.2. *Let X, φ, $\{\mu_n\}$, and $\{\alpha_n\}$ be as in Theorem 3.1. Suppose that A a strongly positive bounded linear operator on H with coefficient $\overline{\gamma} > 1/2$ and $0 < \zeta < (1 - \sqrt{2 - 2\overline{\gamma}})/\alpha$. Let $\{x_n\}$ be defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \zeta f(x_n) + (I - \alpha_n A) T_{\mu_n}(x_n), \quad n \geq 0. \tag{3.29}$$

Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(\varphi)$, which is a unique solution of the variational inequality

$$\langle (A - \zeta f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{3.30}$$

Proof. Because A is strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$, we have

$$\langle Ax - Ay, x - y \rangle \geq \overline{\gamma} \|x - y\|^2. \tag{3.31}$$

Therefore, A is $\overline{\gamma}$-strongly accretive. On the other hand,

$$\begin{aligned}
&\|(I - A)x - (I - A)y\|^2 \\
&\quad = \langle (x - y) - (Ax - Ay), (x - y) - (Ax - Ay) \rangle \\
&\quad = \langle x - y, x - y \rangle - 2\langle Ax - Ay, x - y \rangle + \langle Ax - Ay, Ax - Ay \rangle \\
&\quad \leq \|x - y\|^2 - 2\langle Ax - Ay, x - y \rangle + \|A\| \|x - y\|^2.
\end{aligned} \tag{3.32}$$

Since A is strongly positive if and only if $(1/\|A\|)A$ is strongly positive, we may assume, with no loss of generality, that $\|A\| = 1$, so that

$$\langle Ax - Ay, x - y \rangle \leq \|x - y\|^2 - \frac{1}{2} \|(I - A)x - (I - A)y\|^2. \tag{3.33}$$

This shows that A is $1/2$-strictly pseudo-contractive. Now apply Theorem 3.1 to conclude the result. □

Corollary 3.3. *Let X, φ, $\{\mu_n\}$ and $\{\alpha_n\}$ be as in Theorem 3.1. Suppose $u, x_0 \in H$ and define a sequence $\{x_n\}$ by the iterative algorithm*

$$x_{n+1} = \alpha_n u + (I - \alpha_n F) T_{\mu_n}(x_n), \quad n \geq 0. \tag{3.34}$$

Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to a $x^* \in \mathrm{Fix}(\varphi)$, which is a unique solution of the variational inequality

$$\langle F x^* - u, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(\varphi). \tag{3.35}$$

Proof. It is sufficient to take $f = u$ and$\gamma = 1$ in Theorem 3.1. □

4. Some Application

Corollary 4.1. *Let S and T be non-expansive mappings on a Hilbert space H with $ST = TS$ such that* $\mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. *Let $\{\alpha_n\}$ be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$, $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$ and define a sequence $\{x_n\}$ by the iterative algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^i (x_n), \qquad n \geq 0. \tag{4.1}$$

Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$ which solves the variational inequality:

$$\left\langle (F - \gamma f)x^*, x - x^* \right\rangle \geq 0, \qquad \forall x \in \mathrm{Fix}(S) \cap \mathrm{Fix}(T). \tag{4.2}$$

Proof. Let $T(i,j) = S^i T^j$ for each $i, j \in \mathbb{N} \cup \{0\}$. Then $\{T(i,j) : i, j \in \mathbb{N} \cup \{0\}\}$ is a semigroup of non-expansive mappings on H. Now, for each $n \in \mathbb{N}$ and $i, j \in B((\mathbb{N} \cup \{0\})^2)$, we define $\mu_n(f) = (1/n^2)\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(i,j)$. Then, $\{\mu_n\}$ is regular sequence of means [16]. Next, for each $x \in H$ and $n \in \mathbb{N}$, we have

$$T_{\mu_n}(x) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j (x). \tag{4.3}$$

Therefore, applying Theorem 3.1, the result follows. □

Corollary 4.2. *Let $\varphi = \{T_t : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of non-expansive mappings on a Hilbert space H such that* $\mathrm{Fix}(\varphi) \neq \emptyset$. *Let α_n be a sequence in $(0,1)$ satisfying conditions $\alpha_n \to 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. Let $x_0 \in H$ and $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. Let $\{x_n\}$ be a sequence defined by the iterative algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)\frac{1}{t_n}\int_0^{t_n} T_s(x_n)\,ds, \quad n \geq 0, \tag{4.4}$$

where $\{t_n\}$ is an increasing sequence in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$. Then, $\{x_n\}$ converges strongly, as $n \to \infty$, to $x^* \in \mathrm{Fix}(\varphi)$, which solves the variational inequality

$$\left\langle (F - \gamma f)x^*, x - x^* \right\rangle \geq 0, \qquad \forall x \in \mathrm{Fix}(\varphi). \tag{4.5}$$

Proof. For $n \in \mathbb{N}$, we define $\mu_n(f) = (1/t_n)\int_0^{t_n} f(t)\,dt$ for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ denotes the space of all real-valued bounded continuous functions on R^+ with supremum norm. Then, $\{\mu_n\}$ is regular sequence of means [16]. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = (1/t_n)\int_0^{t_n} T_s(x)\,ds$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.3. *Let* $\varphi = \{T_t : t \in \mathbb{R}^+\}$ *be a strongly continuous semigroup of non-expansive mappings on a Hilbert space* H *such that* $\mathrm{Fix}(\varphi) \neq \emptyset$. *Let* α_n *be a sequence in* $(0,1)$ *satisfying conditions* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Let* $x_0 \in H$ *and* $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. *Let* $\{x_n\}$ *be a sequence defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) r_n \int_0^{\infty} \exp(-r_n s) T_s x_n ds, \qquad n \geq 0, \tag{4.6}$$

where $\{r_n\}$ *is an decreasing sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} r_n = 0$. *Then* $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $x^* \in \mathrm{Fix}(\varphi)$, *which solves the variational inequality*

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in \mathrm{Fix}(\varphi). \tag{4.7}$$

Proof. For $n \in \mathbb{N}$, we define $\mu_n(f) = r_n \int_0^{\infty} \exp(-r_n t) f(t) dt$ for each $f \in C(\mathbb{R}_+)$. Then $\{\mu_n\}$ is regular sequence of means [16]. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = r_n \int_0^{\infty} \exp(-r_n t) T_t(x) dt$. Now, apply Theorem 3.1 to conclude the result. □

Corollary 4.4. *Let* T *be a non-expansive mapping on a Hilbert space* H *such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let* α_n *be a sequence in* $(0,1)$ *satisfying conditions* $\alpha_n \to 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and let* $Q = \{q_{n,m}\}$ *be a strongly regular matrix. Let* $x_0 \in H$ *and* $\gamma \in (0, 1 - \sqrt{(1-\delta)/\lambda}/\alpha)$. *Let* $\{x_n\}$ *be a sequence defined by the iterative algorithm*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F) \sum_{m=0}^{\infty} q_{n,m} T^m x_n, \qquad n \geq 0. \tag{4.8}$$

Then, $\{x_n\}$ *converges strongly, as* $n \to \infty$, *to* $x^* \in \mathrm{Fix}(T)$ *which solves the variational inequality*

$$\langle (F - \gamma f)x^*, x - x^* \rangle \geq 0, \qquad \forall x \in \mathrm{Fix}(T). \tag{4.9}$$

Proof. For each $n \in \mathbb{N}$, we define

$$\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m), \tag{4.10}$$

for each $f \in B(\mathbb{N} \cup \{0\})$. Since Q is a strongly regular matrix, for each m, we have $q_{n,m} \to 0$, as $n \to \infty$; see [17]. Then, it is easy to see that $\{\mu_n\}$ is regular sequence of means. Furthermore, for each $x \in H$, we have $T_{\mu_n}(x) = \sum_{m=0}^{\infty} q_{n,m} T^m(x)$. Now, apply Theorem 3.1 to conclude the result. □

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