Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2010, Article ID 978121, 15 pages
doi:10.1155/2010/978121

Research Article

Common Fixed Point Results in Metric-Type Spaces

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Received 16 October 2010; Accepted 8 December 2010

Academic Editor: Tomonari Suzuki

Several fixed point and common fixed point theorems are obtained in the setting of metric-type spaces introduced by M. A. Khamsi in 2010.

1. Introduction

Symmetric spaces were introduced in 1931 by Wilson [1], as metric-like spaces lacking the triangle inequality. Several fixed point results in such spaces were obtained, for example, in [2–4]. A new impulse to the theory of such spaces was given by Huang and Zhang [5] when they reintroduced cone metric spaces replacing the set of real numbers by a cone in a Banach space, as the codomain of a metric (such spaces were known earlier under the name of K-metric spaces, see [6]). Namely, it was observed in [7] that if $d(x, y)$ is a cone metric on the set X (in the sense of [5]), then $D(x, y) = \|d(x, y)\|$ is symmetric with some special properties, particularly in the case when the underlying cone is normal. The space (X, D) was then called the symmetric space associated with cone metric space (X, d).

The last observation also led Khamsi [8] to introduce a new type of spaces which he called metric-type spaces, satisfying basic properties of the associated space (X, D), $D = \|d\|$. Some fixed point results were obtained in metric-type spaces in the papers [7–10].

In this paper we prove several other fixed point and common fixed point results in metric-type spaces. In particular, metric-type versions of very well-known results of Hardy-Rogers, Ćirić, Das-Naik, Fisher, and others are obtained.

2. Preliminaries

Let X be a nonempty set. Suppose that a mapping $D : X \times X \to [0, +\infty)$ satisfies the following:

(s1) $D(x, y) = 0$ if and only if $x = y$;

(s2) $D(x, y) = D(y, x)$, for all $x, y \in X$.

Then D is called a *symmetric* on X, and (X, D) is called a *symmetric space* [1].

Let E be a real Banach space. A nonempty subset $P \neq \{0\}$ of E is called a *cone* if P is closed, if $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$, and if $P \cap (-P) = \{0\}$. Given a cone $P \subset E$, we define the partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$.

Let X be a nonempty set. Suppose that a mapping $d : X \times X \to E$ satisfies the following:

(co 1) $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(co 2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(co 3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a *cone metric* on X and (X, d) is called a *cone metric space* [5].

If (X, d) is a cone metric space, the function $D(x, y) = \|d(x, y\|$ is easily seen to be a symmetric on X [7, 8]. Following [7], the space (X, D) will then be called *associated symmetric space* with the cone metric space (X, d). If the underlying cone P of (X, d) is normal (i.e., if, for some $k \geq 1$, $0 \preceq x \preceq y$ always implies $\|x\| \leq k\|y\|$), the symmetric D satisfies some additional properties. This led M.A. Khamsi to introduce a new type of spaces which he called metric type spaces. We will use the following version of his definition.

Definition 2.1 (see [8]). Let X be a nonempty set, let $K \geq 1$ be a real number, and let the function $D : X \times X \to \mathbb{R}$ satisfy the following properties:

(a) $D(x, y) = 0$ if and only if $x = y$;

(b) $D(x, y) = D(y, x)$ for all $x, y \in X$;

(c) $D(x, z) \leq K(D(x, y) + D(y, z))$ for all $x, y, z \in X$.

Then (X, D, K) is called a *metric-type space*.

Obviously, for $K = 1$, metric-type space is simply a metric space.

A metric type space may satisfy some of the following additional properties:

(d) $D(x, z) \leq K(D(x, y_1) + D(y_1, y_2) + \cdots + D(y_n, z))$ for arbitrary points $x, y_1, y_2, \ldots, y_n, z \in X$;

(e) function D is continuous in two variables; that is,

$$x_n \longrightarrow x, \quad y_n \longrightarrow y \text{ (in } (X, D, K)) \text{ imply } D(x_n, y_n) \longrightarrow D(x, y). \tag{2.1}$$

(The last condition is in the theory of symmetric spaces usually called "property (H_E)".)

Condition (d) was used instead of (c) in the original definition of a metric-type space by Khamsi [8]. Both conditions (d) and (e) are satisfied by the symmetric $D(x,y) = \|d(x,y)\|$ which is associated with a cone metric d (with a normal cone) (see [7–9]).

Note that the weaker version of property (e):

(e′) $x_n \to x$ and $y_n \to x$ (in (X,D,K)) imply that $D(x_n,y_n) \to 0$

is satisfied in an arbitrary metric type space. It can also be proved easily that the limit of a sequence in a metric type space is unique. Indeed, if $x_n \to x$ and $x_n \to y$ (in (X,D,K)) and $D(x,y) = \varepsilon > 0$, then

$$0 \leq D(x,y) \leq K(D(x,x_n) + D(x_n,y)) < K\left(\frac{\varepsilon}{2K} + \frac{\varepsilon}{2K}\right) = \varepsilon \tag{2.2}$$

for sufficiently large n, which is impossible.

The notions such as *convergent sequence*, *Cauchy sequence*, and *complete space* are defined in an obvious way.

We prove in this paper several versions of fixed point and common fixed point results in metric type spaces. We start with versions of classical Banach, Kannan and Zamfirescu results then proceed with Hardy-Rogers-type theorems, and with quasicontractions of Ćirić and Das-Naik, and results for four mappings of Fisher and finally conclude with a result for strict contractions.

Recall also that a mapping $f : X \to X$ is said to have property P [11] if $\mathrm{Fix}(f^n) = \mathrm{Fix}(f)$ for each $n \in \mathbb{N}$, where $\mathrm{Fix}(f)$ stands for the set of fixed points of f.

A point $w \in X$ is called a *point of coincidence* of a pair of self-maps $f, g : X \to X$ and $u \in X$ is its *coincidence point* if $fu = gu = w$. Mappings f and g are *weakly compatible* if $fgu = gfu$ for each of their coincidence points u [12, 13]. The notion of *occasionally weak compatibility* is also used in some papers, but it was shown in [14] that it is actually superfluous.

3. Results

We begin with a simple, but useful lemma.

Lemma 3.1. *Let $\{y_n\}$ be a sequence in a metric type space (X,D,K) such that*

$$D(y_n, y_{n+1}) \leq \lambda D(y_{n-1}, y_n) \tag{3.1}$$

for some λ, $0 < \lambda < 1/K$, and each $n = 1, 2, \ldots$. Then $\{y_n\}$ is a Cauchy sequence in (X,D,K).

Proof. Let $m, n \in \mathbb{N}$ and $m < n$. Applying the triangle-type inequality (c) to triples $\{y_m, y_{m+1}, y_n\}$, $\{y_{m+1}, y_{m+2}, y_n\}, \ldots, \{y_{n-2}, y_{n-1}, y_n\}$ we obtain

$$
\begin{aligned}
D(y_m, y_n) &\le K\big(D(y_m, y_{m+1}) + D(y_{m+1}, y_n)\big) \\
&\le KD(y_m, y_{m+1}) + K^2\big(D(y_{m+1}, y_{m+2}) + D(y_{m+2}, y_n)\big) \\
&\le \cdots \le KD(y_m, y_{m+1}) + K^2 D(y_{m+1}, y_{m+2}) + \cdots \\
&\quad + K^{n-m-1}\big(D(y_{n-2}, y_{n-1}) + D(y_{n-1}, y_n)\big) \\
&\le KD(y_m, y_{m+1}) + K^2 D(y_{m+1}, y_{m+2}) + \cdots \\
&\quad + K^{n-m-1} D(y_{n-2}, y_{n-1}) + K^{n-m} D(y_{n-1}, y_n).
\end{aligned}
\tag{3.2}
$$

Now (3.1) and $K\lambda < 1$ imply that

$$
\begin{aligned}
D(y_m, y_n) &\le \left(K\lambda^m + K^2\lambda^{m+1} + \cdots + K^{n-m}\lambda^{n-1}\right) D(y_0, y_1) \\
&= K\lambda^m \left(1 + (K\lambda) + \cdots + (K\lambda)^{n-m-1}\right) D(y_0, y_1) \\
&\le \frac{K\lambda^m}{1 - K\lambda} D(y_0, y_1) \longrightarrow 0 \quad \text{when } m \longrightarrow \infty.
\end{aligned}
\tag{3.3}
$$

It follows that $\{y_n\}$ is a Cauchy sequence. □

Remark 3.2. If, instead of triangle-type inequality (c), we use stronger condition (d), then a weaker condition $0 < \lambda < 1$ can be used in the previous lemma to obtain the same conclusion. The proof is similar.

Next is the simplest: Banach-type version of a fixed point result for contractive mappings in a metric type space.

Theorem 3.3. *Let (X, D, K) be a complete metric type space, and let $f : X \to X$ be a map such that for some λ, $0 < \lambda < 1/K$,*

$$
D(fx, fy) \le \lambda D(x, y) \tag{3.4}
$$

holds for all $x, y \in X$. Then f has a unique fixed point z, and for every $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to z.

Proof. Take an arbitrary $x_0 \in X$ and denote $y_n = f^n x_0$. Then

$$
D(y_n, y_{n+1}) = D(fy_{n-1}, fy_n) \le \lambda D(y_{n-1}, y_n) \tag{3.5}
$$

for each $n = 1, 2 \ldots$. Lemma 3.1 implies that $\{y_n\}$ is a Cauchy sequence, and since (X, D, K) is complete, there exists $z \in X$ such that $y_n \to z$ when $n \to \infty$. Then

$$
D(fz, z) \le K\big(D(fz, fy_n) + D(y_{n+1}, z)\big) \le K\big(\lambda D(z, y_n) + D(y_{n+1}, z)\big) \longrightarrow 0, \tag{3.6}
$$

when $n \to \infty$. Hence, $D(fz, z) = 0$ and z is a fixed point of f.

If z_1 is another fixed point of f, then $D(z, z_1) = D(fz, fz_1) \leq \lambda D(z, z_1)$ which is possible only if $z = z_1$. □

Remark 3.4. In a standard way we prove that the following estimate holds for the sequence $\{f^n x_0\}$:

$$D(f^m x_0, z) \leq \frac{K^2 \lambda^m}{1 - K\lambda} D(x_0, f x_0) \tag{3.7}$$

for each $m \in \mathbb{N}$. Indeed, for $m < n$,

$$D(f^m x_0, z) \leq K\left(D(f^m x_0, f^n x_0) + D(f^n x_0, z)\right) \leq \frac{K^2 \lambda^m}{1 - K\lambda} D(x_0, f x_0) + K D(f^n x_0, z), \tag{3.8}$$

and passing to the limit when $n \to \infty$, we obtain estimate (3.7).

Note that continuity of function D (property (e)) was not used.

The first part of the following result was obtained, under the additional assumption of boundedness of the orbit, in [8, Theorem 3.3].

Theorem 3.5. *Let (X, D, K) be a complete metric type space. Let $f : X \to X$ be a map such that for every $n \in \mathbb{N}$ there is $\lambda_n \in (0, 1)$ such that $D(f^n x, f^n y) \leq \lambda_n D(x, y)$ for all $x, y \in X$ and let $\lim_{n\to\infty} \lambda_n = 0$. Then f has a unique fixed point z. Moreover, f has the property P.*

Proof. Take λ such that $0 < \lambda < 1/K$. Since $\lambda_n \to 0$, $n \to \infty$, there exists $n_0 \in \mathbb{N}$ such that $\lambda_n < \lambda$ for each $n \geq n_0$. Then $D(f^n x, f^n y) \leq \lambda D(x, y)$ for all $x, y \in X$ whenever $n \geq n_0$. In other words, for any $m \geq n_0$, $g = f^m$ satisfies $D(gx, gy) \leq \lambda D(x, y)$ for all $x, y \in X$. Theorem 3.3 implies that g has a unique fixed point, say z. Then $f^m z = z$, implying that $f^{m+1} z = f^m(fz) = fz$ and fz is a fixed point of $g = f^m$. Since the fixed point of g is unique, it follows that $fz = z$ and z is also a fixed point of f.

From the given condition we get that $D(fx, f^2 x) = D(fx, ffx) \leq \lambda_1 D(x, fx)$ for some $\lambda_1 < 1$ and each $x \in X$. This property, together with $\mathrm{Fix}(f) \neq \emptyset$, implies, in the same way as in [11, Theorem 1.1], that f has the property P. □

Remark 3.6. If, in addition to the assumptions of previous theorem, we suppose that the series $\sum_{n=1}^{\infty} \lambda_n$ converges and that D satisfies property (d), we can prove that, for each $x \in X$, the respective Picard sequence $\{f^n x\}$ converges to the fixed point z.

Indeed, let $m, n \in \mathbb{N}$ and $n > m$. Then

$$\begin{aligned} D(f^m x, f^n x) &\leq K\left(D\left(f^m x, f^{m+1} x\right) + \cdots + D\left(f^{n-1} x f^n x\right)\right) \\ &= K\left(D(f^m x, f^m f x) + \cdots + D\left(f^{n-1} x, f^{n-1} f x\right)\right) \\ &\leq K(\lambda_m + \cdots + \lambda_{n-1}) D(x, fx) \longrightarrow 0, \end{aligned} \tag{3.9}$$

when $m \to \infty$ (due to the convergence of the given series). So, $\{f^n x\}$ is a Cauchy sequence and it is convergent. For m chosen in the proof of Theorem 3.5 such that $f^m = g$, it is $g^n = f^{mn}$

and $g^n x \to z$ when $n \to \infty$, but $\{f^{mn}x\}$ is a subsequence of $\{f^n x\}$ which is convergent; hence, the latter converges to z.

The next is a common fixed point theorem of Hardy-Rogers type (see, e.g., [15]) in metric type spaces.

Theorem 3.7. *Let (X, D, K) be a metric type space, and let $f, g : X \to X$ be two mappings such that $fX \subset gX$ and one of these subsets of X is complete. Suppose that there exist nonnegative coefficients a_i, $i = 1, \ldots, 5$ such that*

$$2Ka_1 + (K+1)(a_2 + a_3) + \left(K^2 + K\right)(a_4 + a_5) < 2 \tag{3.10}$$

and that for all $x, y \in X$

$$D(fx, fy) \leq a_1 D(gx, gy) + a_2 D(gx, fx) + a_3 D(gy, fy) + a_4 D(gx, fy) + a_5 D(gy, fx) \tag{3.11}$$

holds. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Note that condition (3.10) is satisfied, for example, when $\sum_{i=1}^{5} a_i < 1/K^2$. Note also that when $K = 1$ it reduces to the standard Hardy-Rogers condition in metric spaces.

Proof. Suppose, for example, that gX is complete. Take an arbitrary $x_0 \in X$ and, using that $fX \subset gX$, construct a Jungck sequence $\{y_n\}$ defined by $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. Let us prove that this is a Cauchy sequence. Indeed, using (3.11), we get that

$$\begin{aligned} D(y_n, y_{n+1}) &= D(fx_n, fx_{n+1}) \leq a_1 D(gx_n, gx_{n+1}) + a_2 D(gx_n, fx_n) \\ &\quad + a_3 D(gx_{n+1}, fx_{n+1}) + a_4 D(gx_n, fx_{n+1}) + a_5 D(gx_{n+1}, fx_n) \\ &= a_1 D(y_{n-1}, y_n) + a_2 \mathrm{D}(y_{n-1}, y_n) + a_3 D(y_n, y_{n+1}) \\ &\quad + a_4 D(y_{n-1}, y_{n+1}) + a_5 \cdot 0 \\ &\leq (a_1 + a_2) D(y_{n-1}, y_n) + a_3 D(y_n, y_{n+1}) \\ &\quad + a_4 K(D(y_{n-1}, y_n) + D(y_n, y_{n+1})) \\ &= (a_1 + a_2 + Ka_4) D(y_{n-1}, y_n) + (a_3 + Ka_4) D(y_n, y_{n+1}). \end{aligned} \tag{3.12}$$

Similarly, we conclude that

$$D(y_{n+1}, \mathrm{y}_n) = D(fx_{n+1}, fx_n) \leq (a_1 + a_3 + Ka_5) D(y_{n-1}, y_n) + (a_2 + Ka_5) D(y_n, y_{n+1}). \tag{3.13}$$

Adding the last two inequalities, we get that

$$2D(y_n, y_{n+1}) \leq (2a_1 + a_2 + a_3 + Ka_4 + Ka_5) D(y_{n-1}, y_n) + (a_2 + a_3 + Ka_4 + Ka_5) D(y_n, y_{n+1}), \tag{3.14}$$

that is,

$$D(y_n, y_{n+1}) \leq \frac{2a_1 + a_2 + a_3 + Ka_4 + Ka_5}{2 - a_2 - a_3 - Ka_4 - Ka_5} D(y_{n-1}, y_n) = \lambda D(y_{n-1}, y_n). \tag{3.15}$$

The assumption (3.10) implies that

$$\begin{gathered} 2Ka_1 + Ka_2 + Ka_3 + K^2(a_4 + a_5) < 2 - a_2 - a_3 - Ka_4 - Ka_5, \\ \lambda = \frac{2a_1 + a_2 + a_3 + Ka_4 + Ka_5}{2 - a_2 - a_3 - Ka_4 - Ka_5} < \frac{1}{K}. \end{gathered} \tag{3.16}$$

Lemma 3.1 implies that $\{y_n\}$ is a Cauchy sequence in gX and so there is $z \in X$ such that $fx_n = gx_{n+1} \to gz$ when $n \to \infty$. We will prove that $fz = gz$.

Using (3.11) we conclude that

$$\begin{aligned} D(fx_n, fz) &\leq a_1 D(gx_n, gz) + a_2 D(gx_n, fx_n) + a_3 D(gz, fz) \\ &\quad + a_4 D(gx_n, fz) + a_5 D(gz, fx_n) \\ &\leq a_1 D(gx_n, gz) + a_2 D(gx_n, fx_n) + a_3 K(D(gz, fx_n) + D(fx_n, fz)) \\ &\quad + a_4 K(D(gx_n, fx_n) + D(fx_n, fz)) + a_5 D(gz, fx_n) \\ &= a_1 D(gx_n, gz) + (a_2 + Ka_4) D(gx_n, fx_n) \\ &\quad + (Ka_3 + a_5) D(gz, fx_n) + K(a_3 + a_4) D(fx_n, fz). \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} D(fz, fx_n) &\leq a_1 D(gx_n, gz) + (Ka_2 + a_4) D(gz, fx_n) \\ &\quad + K(a_2 + a_5) D(fx_n, fz) + (a_3 + Ka_5) D(fx_n, gx_n). \end{aligned} \tag{3.18}$$

Adding up, one concludes that

$$\begin{aligned} &(2 - K(a_2 + a_3 + a_4 + a_5)) D(fx_n, \mathrm{f}z) \\ &\quad \leq 2a_1 D(gx_n, gz) + (a_2 + a_3 + K(a_4 + a_5)) D(fx_n, gx_n) \\ &\qquad + (K(a_2 + a_3) + a_4 + a_5) D(fx_n, gz). \end{aligned} \tag{3.19}$$

The right-hand side of the last inequality tends to 0 when $n \to \infty$. Since $K(a_2 + a_3 + a_4 + a_5) < 2Ka_1 + (K+1)(a_2 + a_3) + (K^2 + K)(a_4 + a_5) < 2$ (because of (3.10)), it is $2 - K(a_2 + a_3 + a_4 + a_5) > 0$, and so also the left-hand side tends to 0, and $fx_n \to fz$. Since the limit of a sequence is unique, it follows that $fz = gz = w$ and f and g have a point of coincidence w.

Suppose that $w_1 = fz_1 = gz_1$ is another point of coincidence for f and g. Then (3.11) implies that

$$\begin{aligned} D(w, w_1) &= D(fz, fz_1) \\ &\le a_1 D(gz, gz_1) + a_2 D(gz, fz) + a_3 D(gz_1, fz_1) \\ &\quad + a_4 D(gz, fz_1) + a_5 D(gz_1, fz) \\ &= a_1 D(w, w_1) + a_2 \cdot 0 + a_3 \cdot 0 + a_4 D(w, w_1) + a_5 D(w_1, w) \\ &= (a_1 + a_4 + a_5) D(w, w_1). \end{aligned} \tag{3.20}$$

Since $a_1 + a_4 + a_5 < 1$ (because of (3.10)), the last relation is possible only if $w = w_1$. So, the point of coincidence is unique.

If (f, g) is weakly compatible, then [13, Proposition 1.12] implies that f and g have a unique common fixed point. □

Taking special values for constants a_i, we obtain as special cases Theorem 3.3 as well as metric type versions of some other well-known theorems (Kannan, Zamfirescu, see, e.g., [15]):

Corollary 3.8. *Let (X, D, K) be a metric type space, and let $f, g : X \to X$ be two mappings such that $fX \subset gX$ and one of these subsets of X is complete. Suppose that one of the following three conditions holds:*

(1°) $D(fx, fy) \le a_1 D(gx, gy)$ *for some* $a_1 < 1/K$ *and all* $x, y \in X$;

(2°) $D(fx, fy) \le a_2(D(gx, fx) + D(gy, fy))$ *for some* $a_2 < 1/(K + 1)$ *and all* $x, y \in X$;

(3°) $D(fx, fy) \le a_4(D(gx, fy) + D(gy, fx))$ *for some* $a_4 < 1/(K^2 + K)$ *and all* $x, y \in X$.

Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Putting $g = i_X$ in Theorem 3.7, we get metric type version of Hardy-Rogers theorem (which is obviously a special case for $K = 1$).

Corollary 3.9. *Let (X, D, K) be a complete metric type space, and let $f : X \to X$ satisfy*

$$D(fx, fy) \le a_1 D(x, y) + a_2 D(x, fx) + a_3 D(y, fy) + a_4 D(x, fy) + a_5 D(y, fx) \tag{3.21}$$

for some a_i, $i = 1, \ldots, 5$ satisfying (3.10) *and for all $x, y \in X$. Then f has a unique fixed point. Moreover, f has property P.*

Proof. We have only to prove the last assertion. For arbitrary $x \in X$, we have that

$$\begin{aligned} D\left(fx, f^2x\right) &= D(fx, ffx) \\ &\leq a_1 D(x, fx) + a_2 D(x, fx) + a_3 D\left(fx, f^2x\right) + a_4 D\left(x, f^2x\right) + a_5 D(\text{f}x, fx) \\ &\leq (a_1 + a_2 + K a_4) D(x, fx) + (a_3 + K a_4) D\left(fx, f^2x\right), \end{aligned} \tag{3.22}$$

and similarly

$$D\left(f^2x, fx\right) = D(ffx, fx) \leq (a_1 + a_3 + K a_5) D(x, fx) + (a_2 + K a_5) D\left(fx, f^2x\right). \tag{3.23}$$

Adding the last two inequalities, we obtain

$$D\left(fx, f^2x\right) \leq \frac{2a_1 + a_2 + a_3 + K(a_4 + a_5)}{2 - a_2 - a_3 - K(a_4 + a_5)} D(x, fx) = \lambda D(x, fx). \tag{3.24}$$

Similarly as in the proof of Theorem 3.7, we get that $\lambda < 1/K < 1$. Now [11, Theorem 1.1] implies that f has property P. □

Remark 3.10. If the metric-type function D satisfies both properties (d) and (e), then it is easy to see that condition (3.10) in Theorem 3.7 and the last corollary can be weakened to $a_1 + a_2 + a_3 + K(a_4 + a_5) < 1$. In particular, this is the case when $D(x, y) = \|d(x, y)\|$ for a cone metric d on X (over a normal cone, see [7]).

The next is a possible metric-type variant of a common fixed point result for Ćirić and Das-Naik quasicontractions [16, 17].

Theorem 3.11. *Let (X, D, K) be a metric type space, and let $f, g : X \to X$ be two mappings such that $fX \subset gX$ and one of these subsets of X is complete. Suppose that there exists λ, $0 < \lambda < 1/K$ such that for all $x, y \in X$*

$$D(fx, fy) \leq \lambda \max M(f, g; x, y), \tag{3.25}$$

where

$$M(f, g; x, y) = \left\{ D(gx, gy), D(gx, fx), D(gy, fy), \frac{D(gx, fy)}{2K}, \frac{D(gy, fx)}{2K} \right\}. \tag{3.26}$$

Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is weakly compatible, then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and, using condition $fX \subset gX$, construct a Jungck sequence $\{y_n\}$ satisfying $y_n = fx_n = gx_{n+1}$, $n = 0, 1, 2, \ldots$. Suppose that $D(y_n, y_{n+1}) > 0$ for each n (otherwise the conclusion follows easily). Using (3.25) we conclude that

$$\begin{aligned} &D(y_{n+1}, y_n) \\ &\quad = D(fx_{n+1}, fx_n) \\ &\quad \le \lambda \max\left\{ D(gx_{n+1}, gx_n), D(gx_{n+1}, fx_{n+1}), D(gx_n, fx_n), \frac{D(gx_{n+1}, fx_n)}{2K}, \frac{D(gx_n, fx_{n+1})}{2K} \right\} \\ &\quad = \lambda \max\left\{ D(y_n, y_{n-1}), D(y_n, y_{n+1}), D(y_{n-1}, y_n), 0, \frac{D(y_{n-1}, y_{n+1})}{2K} \right\} \\ &\quad \le \lambda \max\left\{ D(y_n, y_{n-1}), \frac{1}{2}\left(D(y_{n-1}, y_n) + D(y_n, y_{n+1})\right) \right\}. \end{aligned} \tag{3.27}$$

If $D(y_n, y_{n-1}) < D(y_{n+1}, y_n)$, then $D(y_n, y_{n-1}) < (1/2)(D(y_{n-1}, y_n) + D(y_n, y_{n+1})) < D(y_n, y_{n+1})$, and it would follow from (3.27) that $D(y_{n+1}, y_n) \le \lambda D(y_{n+1}, y_n)$ which is impossible since $\lambda < 1$. (For the same reason the term $D(y_n, y_{n+1})$ was omitted in the last row of the previous series of inequalities.) Hence, $D(y_n, y_{n-1}) > D(y_{n+1}, y_n)$ and (3.27) becomes $D(y_{n+1}, y_n) \le \lambda D(y_n, y_{n-1})$. Using Lemma 3.1, we conclude that $\{y_n\}$ is a Cauchy sequence in gX. Supposing that, for example, the last subset of X is complete, we conclude that $y_n = fx_n = gx_{n+1} \to gz$ when $n \to \infty$ for some $z \in X$.

To prove that $fz = gz$, put $x = x_n$ and $y = z$ in (3.25) to get

$$D(fx_n, fz) \le \lambda \max\left\{ D(gx_n, gz), D(gx_n, fx_n), D(gz, fz), \frac{D(gx_n, fz)}{2K}, \frac{D(gz, fx_n)}{2K} \right\}. \tag{3.28}$$

Note that $fx_n \to gz$ and $gx_n \to gz$ when $n \to \infty$, implying that $D(gx_n, fx_n) \le K(D(gx_n, gz) + D(gz, fx_n)) \to 0$ when $n \to \infty$. It follows that the only possibilities are the following:

(1°) $D(fx_n, fz) \le \lambda D(gz, fz) \le \lambda K(D(gz, fx_n) + D(fx_n, fz))$; in this case $(1 - \lambda K)D(fx_n, fz) \le \lambda K D(gz, fx_n) \to 0$, and since $1 - \lambda K > 0$, it follows that $fx_n \to fz$.

(2°) $D(fx_n, fz) \le \lambda(1/2K)D(gx_n, fz) \le (\lambda/2)(D(gx_n, fx_n) + D(fx_n, fz))$; in this case, $(1 - (\lambda/2))D(fx_n, fz) \le (\lambda/2)D(gx_n, fx_n) \to 0$, so again $fx_n \to fz$, $n \to \infty$.

Since the limit of a sequence is unique, it follows that $fz = gz$.

The rest of conclusion follows as in the proof of Theorem 3.7. □

Putting $g = i_X$, we obtain the first part of the following corollary.

Corollary 3.12. *Let (X, D, K) be a complete metric type space, and let $f : X \to X$ be such that for some λ, $0 < \lambda < 1/K$, and for all $x, y \in X$,*

$$D(fx, fy) \le \lambda \max\left\{D(x,y), D(x,fx), D(y,fy), \frac{D(x,fy)}{2K}, \frac{D(y,fx)}{2K}\right\} \tag{3.29}$$

holds. Then f has a unique fixed point, say z. Moreover, the function f is continuous at point z and it has the property P.

Proof. Let $x_n \to z$ when $n \to \infty$. Then

$$\begin{aligned} D(fx_n, fz) &\le \lambda \max\left\{D(x_n,z), D(x_n,fx_n), D(z,fz), \frac{D(x_n,fz)}{2K}, \frac{D(fx_n,z)}{2K}\right\} \\ &= \lambda \max\left\{D(x_n,z), D(x_n,fx_n), \frac{D(fx_n,z)}{2K}\right\}. \end{aligned} \tag{3.30}$$

Since $D(x_n, z) \to 0$ and $D(x_n, fx_n) \le K(D(x_n, z) + D(fz, fx_n))$, the only possibility is that $D(fx, fz) \le \lambda K(D(x_n, z) + D(fz, fx_n))$, implying that $(1-\lambda K)D(fx_n, fz) \le \lambda K D(x_n, z) \to 0$, $n \to \infty$. Since $0 < \lambda K < 1$, it follows that $fx_n \to fz = z$, $n \to \infty$, and f is continuous at the point z.

We will prove that f satisfies

$$D\left(fx, f^2x\right) \le hD(x, fx) \tag{3.31}$$

for some h, $0 < h < 1$ and each $x \in X$.

Applying (3.29) to the points x and fx (for any $x \in X$), we conclude that

$$\begin{aligned} D\left(fx, f^2x\right) &\le \lambda \max\left\{D(x,fx), D(x,fx), D\left(fx,f^2x\right), \frac{D(x,f^2x)}{2K}, \frac{D(fx,fx)}{2K}\right\} \\ &= \lambda \max\left\{D(x,fx), D\left(fx,f^2x\right), \frac{D(x,f^2x)}{2K}\right\}. \end{aligned} \tag{3.32}$$

The following cases are possible:

(1°) $D(fx, f^2x) \le \lambda D(x, fx)$, and (3.31) holds with $h = \lambda$;

(2°) $D(fx, f^2x) \le \lambda D(fx, f^2x)$, which is only possible if $D(fx, f^2x) = 0$ and then (3.31) obviously holds.

(3°) $D(fx, f^2x) \le (\lambda/2K)D(x, f^2x) \le (\lambda/2K)K(D(x, fx) + D(fx, f^2x))$, implying that $(1 - (\lambda/2))D(fx, f^2x) \le (\lambda/2)D(x, fx)$ and $D(fx, f^2x) \le hD(x, fx)$, where $0 < h = \lambda/(2 - \lambda) < 1$ since $0 < \lambda < 1$.

So, relation (3.31) holds for some h, $0 < h < 1$ and each $x \in X$. Using the mentioned analogue of [11, Theorem 1.1], one obtains that f satisfies property P. □

We will now prove a generalization and an extension of Fisher's theorem on four mappings from [18] to metric type spaces. Note that, unlike in [18], we will not use the case when f and S, as well as g and T, commute, neither when S and T are continuous. Also, function D need not be continuous (i.e., we do not use property (e)).

Theorem 3.13. *Let (X, D, K) be a metric type space, and let $f, g, S, T : X \to X$ be four mappings such that $fX \subset TX$ and $gX \subset SX$, and suppose that at least one of these four subsets of X is complete. Let*

$$D(fx, gy) \leq \lambda D(Sx, Ty) \tag{3.33}$$

holds for some λ, $0 < \lambda < 1/K$ and all $x, y \in X$. Then pairs (f, S) and (g, T) have a unique common point of coincidence. If, moreover, pairs (f, S) and (g, T) are weakly compatible, then f, g, S, and T have a unique common fixed point.

Proof. Let $x_0 \in X$ be arbitrary and construct sequences $\{x_n\}$ and $\{y_n\}$ such that

$$fx_{2n-2} = Tx_{2n-1} = y_{2n-1}, \quad gx_{2n-1} = Sx_{2n} = y_{2n} \tag{3.34}$$

for $n = 1, 2, \ldots$. We will prove that condition (3.1) holds for $n = 1, 2, \ldots$. Indeed,

$$\begin{aligned} D(y_{2n+1}, y_{2n+2}) &= D(fx_{2n}, gx_{2n+1}) \leq \lambda D(Sx_{2n}, Tx_{2n+1}) = \lambda D(y_{2n}, y_{2n+1}), \\ D(y_{2n+3}, y_{2n+2}) &= D(fx_{2n+2}, gx_{2n+1}) \leq \lambda D(Sx_{2n+2}, Tx_{2n+1}) = \lambda D(y_{2n+2}, y_{2n+1}). \end{aligned} \tag{3.35}$$

Using Lemma 3.1, we conclude that $\{y_n\}$ is a Cauchy sequence. Suppose, for example, that SX is a complete subset of X. Then $y_n \to u = Sv$, $n \to \infty$, for some $v \in X$. Of course, subsequences $\{y_{2n-1}\}$ and $\{y_{2n}\}$ also converge to u. Let us prove that $fv = u$. Using (3.33), we get that

$$\begin{aligned} D(fv, u) &\leq K(D(fv, gx_{2n-1}) + D(gx_{2n-1}, u)) \\ &\leq K(\lambda D(Sv, Tx_{2n-1}) + D(gx_{2n-1}, u)) \longrightarrow K(\lambda \cdot 0 + 0) = 0. \end{aligned} \tag{3.36}$$

hence $fv = u = Sv$. Since $u \in fX \subset TX$, we get that there exists $w \in X$ such that $Tw = u$. Let us prove that also $gw = u$. Using (3.33), again we conclude that

$$\begin{aligned} D(gw, u) &\leq K(D(gw, fx_{2n}) + D(fx_{2n}, u)) \\ &\leq K(\lambda D(Sx_{2n}, Tw) + D(fx_{2n}, u)) \longrightarrow K(\lambda \cdot 0 + 0) = 0, \end{aligned} \tag{3.37}$$

implying that $gw = u = Tw$. We have proved that u is a common point of coincidence for pairs (f, S) and (g, T).

If now these pairs are weakly compatible, then for example, $fu = fSv = Sfv = Su = z_1$ and $gu = gTw = Tgw = Tu = z_2$for example, . Moreover, $D(z_1, z_2) = D(fu, gu) \leq \lambda D(Su, Tu) = \lambda D(z_1, z_2)$ and $0 < \lambda < 1$ implies that $z_1 = z_2$. So, we have that $fu = gu = Su = Tu$. It remains to prove that, for example, $u = gu$. Indeed, $D(u, gu) = D(fv, gu) \leq \lambda D(Sv, Tu) = \lambda D(u, gu)$, implying that $u = gu$. The proof that this common fixed point of $f, g, S,$ and T is unique is straightforward. □

We conclude with a metric type version of a fixed point theorem for strict contractions. The proof is similar to the respective proof, for example, for cone metric spaces in [5]. An example follows showing that additional condition of (sequential) compactness cannot be omitted.

Theorem 3.14. *Let a metric type space (X, D, K) be sequentially compact, and let D be a continuous function (satisfying property (e)). If $f : X \to X$ is a mapping such that*

$$D(fx, fy) < D(x, y), \quad \text{for } x, y \in X,\ x \neq y, \tag{3.38}$$

then f has a unique fixed point.

Proof. According to [9, Theorem 3.1], sequential compactness and compactness are equivalent in metric type spaces, and also continuity is a sequential property. The given condition (3.38) of strict continuity implies that a fixed point of f is unique (if it exists) and that both mappings f and f^2 are continuous. Let $x_0 \in X$ be an arbitrary point, and let $\{x_n\}$ be the respective Picard sequence (i.e., $x_n = f^n x_0$). If $x_n = x_{n+1}$ for some n, then x_n is a (unique) fixed point. If $x_n \neq x_{n+1}$ for each $n = 0, 1, 2, \ldots$, then

$$D_n := D(x_{n+1}, x_n) = D\left(f^{n+1}x_0, f^n x_0\right) < D\left(f^n x_0, f^{n-1}x_0\right) = D_{n-1}. \tag{3.39}$$

Hence, there exists D^*, such that $0 \leq D^* \leq D_n$ for each n and $D_n \to D^*$, $n \to \infty$. Using sequential compactness of X, choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges to some $x^* \in X$ when $i \to \infty$. The continuity of f and f^2 implies that

$$fx_{n_i} \longrightarrow fx^*, \quad f^2 x_{n_i} \to f^2 x^* \quad \text{when } i \longrightarrow \infty, \tag{3.40}$$

and the continuity of the symmetric D implies that

$$D(fx_{n_i}, x_{n_i}) \longrightarrow D(fx^*, x^*), \quad D\left(f^2 x_{n_i}, fx_{n_i}\right) \longrightarrow D\left(f^2 x^*, fx^*\right) \quad \text{when } i \longrightarrow \infty. \tag{3.41}$$

It follows that $D(fx_{n_i}, x_{n_i}) = D_{n_i} \to D^* = D(fx^*, x^*)$. It remains to prove that $fx^* = x^*$. If $fx^* \neq x^*$, then $D^* > 0$ and (3.41) implies that

$$D^* = \lim_{i \to \infty} D_{n_i+1} = \lim_{i \to \infty} D\left(f^2 x_{n_i}, fx_{n_i}\right) = D\left(f^2 x^*, fx^*\right) < D(fx^*, x^*) = D^*. \tag{3.42}$$

This is a contradiction. □

Example 3.15. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = [1, +\infty)$, and $d : X \times X \to E$ be defined by $d(x, y) = (|x - y|, |x - y|)$. Then (X, d) is a cone metric space over a normal cone with the normal constant $K = 1$ (see, e.g., [5]). The associated symmetric is in this case simply the metric $D(x, y) = \|d(x, y)\| = |x - y|\sqrt{2}$.

Let $f : X \to X$ be defined by $fx = x + 1/x$. Then

$$D(fx, fy) = \left|x - y\right|\left(1 - \frac{1}{xy}\right)\sqrt{2} < \left|x - y\right|\sqrt{2} = D(x, y) \tag{3.43}$$

for all $x, y \in X$. Hence, f satisfies condition (3.38) but it has no fixed points. Obviously, (X, D, K) is not (sequentially) compact.

Acknowledgment

The authors are thankful to the Ministry of Science and Technological Development of Serbia.

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