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Research Article

Iterative Algorithms with Variable Coefficients for Multivalued Generalized Φ-Hemicontractive Mappings without Generalized Lipschitz Assumption

Ci-Shui Ge

Department of Mathematics and Physics, Anhui University of Architecture, Hefei, Anhui 230022, China

Correspondence should be addressed to Ci-Shui Ge, gecishui@sohu.com

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We introduce and study some new Ishikawa-type iterative algorithms with variable coefficients for multivalued generalized Φ-hemicontractive mappings. Several new fixed-point theorems for multivalued generalized Φ-hemicontractive mappings without generalized Lipschitz assumption are established in p-uniformly smooth real Banach spaces. A result for multivalued generalized Φ-hemicontractive mappings with bounded range is obtained in uniformly smooth real Banach spaces. As applications, several theorems for multivalued generalized Φ-hemiaccretive mapping equations are given.

1. Introduction

Let X be a real Banach space and X^* the dual space of X. $\langle *, *\rangle$ denotes the generalized duality pairing between X and X^*. J is the normalized duality mapping from X to 2^{X^*} given by $J(x)$

$$J(x) := \{f \in X^* : \langle x, f\rangle = \|f\| \cdot \|x\|, \ \|f\| = \|x\|\}, \quad x \in X. \tag{1.1}$$

Let D be a nonempty convex subset of X and $CB(D)$ the family of all nonempty bounded closed subsets of D. $H(\cdot,\cdot)$ denotes the Hausdorff metric on $CB(D)$ defined by

$$H(A,B) := \max\left\{\sup_{y\in B}\inf_{x\in A}\|x-y\|, \ \sup_{y\in B}\inf_{x\in A}\|x-y\|\right\}, \quad A,B \in CB(D). \tag{1.2}$$

We use $F(T)$ to denote the fixed-point set of T, that is, $F(T) := \{x : x \in Tx\}$. $\mathbb{N}$ denotes the set of nonnegative integers.

Recall that a mapping $T : D \to D$ is called to be a generalized Lipschitz mapping [1], if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(1 + \|x - y\|), \quad \forall x, y \in D. \tag{1.3}$$

Similarly, a multivalued mapping $T : D \to CB(D)$ is said to be a generalized Lipschitz mapping, if there exists a constant $L > 0$ such that

$$H(Tx, Ty) \leq L(1 + \|x - y\|), \quad \forall x, y \in D. \tag{1.4}$$

A multivalued mapping $T : D \to 2^D$ is said to be a bounded mapping if for any bounded subset A of D,

$$T(A) := \{x : x \in T(y),\ \exists y \in A\} \tag{1.5}$$

is a bounded subset of D.

Clearly, every mapping with bounded range is a generalized Lipschitz mapping[1, Example]. Furthermore, every generalized Lipschitz mapping is a bounded mapping. The following example shows that the class of generalized Lipschitz mappings is a proper subset of the class of bounded mappings.

Example 1.1. Take $D = (0, \infty)$ and define $T : D \to D$ by

$$Tx = \exp(x) + x\ \mathrm{sgn}(\sin x), \tag{1.6}$$

where $\mathrm{sgn}(\cdot)$ denotes sign function. Then, T is a bounded mapping but not a generalized Lipschitz mapping.

Definition 1.2 (see [2]). Let D be a nonempty subset of X. $T : D \to 2^D$ is said to be a multivalued Φ-hemicontractive mapping if the fixed point set $F(T)$ of T is nonempty, and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for each $x \in D$ and $x^* \in F(T)$, there exists a $j(x - x^*) \in J(x - x^*)$ such that

$$\langle u - x^*, j(x - x^*)\rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|) \cdot \|x - x^*\|, \tag{1.7}$$

for all $u \in Tx$.

T is said to be a multivalued Φ-hemiaccretive mapping if $I - T$ is a multivalued Φ-hemicontractive mapping.

Definition 1.3. Let D be a nonempty subset of X. $T{:}D \to 2^D$ is said to be a multivalued generalized Φ-hemicontractive mapping if the fixed point set $F(T)$ of T is nonempty, and there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that for each $x \in D$ and $x^* \in F(T)$, there exists a $j(x - x^*) \in J(x - x^*)$ such that

$$\left\langle u - x^*, j(x - x^*)\right\rangle \le \|x - x^*\|^2 - \Phi(\|x - x^*\|), \tag{1.8}$$

for all $u \in Tx$.

T is said to be a multivalued generalized Φ-hemiaccretive mapping if $I - T$ is a multivalued generalized Φ-hemicontractive mapping.

The following example shows that the class of Φ-hemicontractive mappings is a proper subset of the class of generalized Φ-hemicontractive mappings.

Example 1.4. Let $X = \mathbb{R}^2$ with the Euclidean norm $\|\cdot\|$, where $\mathbb{R}$ denotes the set of the real numbers. Define $T : X \to X$ by

$$Tx = \frac{\|x\|^2}{1 + \|x\|^2} x. \tag{1.9}$$

Thus, $F(T) = \{(0,0)\} \neq \emptyset$. It is easy to verify that T is a generalized Φ-hemicontractive mapping with $\Phi(t) = t^2/(1 + t^2)$. However, T is not Φ-hemicontractive. Indeed, if there exists a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for each $x \in X$ and $x^* = (0,0) \in F(T)$,

$$\langle Tx - x^*, J(x - x^*)\rangle \le \|x - x^*\|^2 - \phi(\|x - x^*\|) \cdot \|x - x^*\|, \tag{1.10}$$

then we get $\phi(t) \le t/(1 + t^2)$ for all $t \in (0,\infty)$. Thus, $\lim_{t\to\infty}\phi(t) = 0$. This is in contradiction with the hypotheses that $\phi(t)$ is strictly increasing and $\phi(0) = 0$.

In the last twenty years or so, numerous papers have been written on the existence and convergence of fixed points for nonlinear mappings, and strong and weak convergence theorems have been obtained by using some well-known iterative algorithms (see, e.g., [1–9] and the references therein).

For multivalued ϕ-hemicontractive mappings, Hirano and Huang [2] obtained the following result.

Theorem HH (See [2, Theorem 1]). *Let E be a uniformly smooth Banach space and $T : E \to 2^E$ be a multivalued ϕ- hemicontractive operator with bounded range. Suppose$\{a_n\}$, $\{b_n\}$, $\{c_n\}$and$\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$are real sequences in $[0,1)$ satisfying the following conditions:*

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$, for all $n \in \mathbb{N}$,

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} b'_n = \lim_{n\to\infty} c_n = 0$,

(iii) $\sum_{n=1}^{\infty} b_n = \infty$,

(iv) $c_n = o(b_n)$.

For arbitrary $x_1, u_1, v_1 \in E$, define the sequence $\{x_n\}_{n=1}^{\infty}$ by

$$\begin{aligned} x_{n+1} &= a_n\ x_n + b_n\ \eta_n + c_n\ u_n, \quad \exists \eta_n \in Ty_n,\ n \in \mathbb{N}, \\ y_n &= a_n'\ x_n + b_n'\ \xi_n + c_n'\ v_n, \quad \exists \xi_n \in Tx_n, n \in \mathbb{N}, \end{aligned} \tag{1.11}$$

where $\{u_n\}_{n=1}^{\infty}$, $\{v_n\}_{n=1}^{\infty}$ are arbitrary bounded sequences in E. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of T.

Further, for general multivalued generalized Φ-hemicontractive mappings, C. E. Chidume and C. O. Chidume [1] gave the following interesting result.

Theorem CC (see [1, Theorem 3.8]). *Let E be a uniformly smooth real Banach space. Let $F(T) := \{x \in E : x \in Tx\} \neq \emptyset$. Suppose $T : E \to 2^E$ is a multivalued generalized Lipschitz and generalized Φ-hemicontractive mapping. Let $\{a_n\}$, $\{b_n\}$and$\{c_n\}$be real sequences in$[0,1)$ satisfying the following conditions: (i) $a_n + b_n + c_n = 1$, (ii) $\sum(b_n + c_n) = \infty$, (iii) $\sum c_n < \infty$, and (iv) $\lim b_n = 0$. Let $\{x_n\}$ be generated iteratively from arbitrary $x_0 \in E$ by*

$$x_{n+1} = a_n x_n + b_n \eta_n + c_n u_n, \quad \exists \eta_n \in Tx_n\ n \geq 0, \tag{1.12}$$

where $\{u_n\}$ is an arbitray bounded sequence in E. Then, there exists $\gamma_0 \in \mathbb{R}$ such that if $b_n + c_n \leq \gamma_0$ for all $n \geq 0$, the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.

Remark 1.5. (1) Theorem CC [1, Theorem 3.8] is a multivalued version of Theorem 3.2 of [1]. Theorem 3.2 of [1] was obtained directly from Theorem 3.1 of [1]. However, it seems that there exists a gap in the proof of Theorem 3.1 in [1]. Indeed, the following inequality in the proof of Theorem 3.1 in [1].

$$a_0 \sum_{j=0}^{n} \alpha_j \leq \sum_{j=0}^{n} \left(\|x_j - x^*\|^2 - \|x_{j+1} - x^*\|^2 \right) + M \sum_{j=0}^{n} c_j < \infty \tag{$*$}$$

was obtained by using implicitly the following conditions:

$$\|x_j - x^*\| \leq 2\Phi^{-1}(a_0), \qquad \|x_{j+1} - x^*\| > 2\Phi^{-1}(a_0), \qquad j = 0, 1, \ldots, n. \tag{1.13}$$

Thus, $(*)$ is dubious in the remainder of [1, Theorem 3.1]. Hence, Theorem 3.1 of [1] is dubious, as is Theorem CC [1, Theorem 3.8].

(2) The real number γ_0 in Theorem CC is not easy to get.

It is our purpose in this paper to try to obtain some fixed-point theorems for multivalued generalized Φ-hemicontractive mappings without generalized Lipschitz assumption as in Theorem CC. Motivated and inspired by [1, 2, 5, 7], we introduce and study some new Ishikawa-type iterative algorithms with variable coefficients for multivalued generalized Φ-hemicontractive mappings. Our results improve essentially the corresponding results of [1] in the framework of p-uniformly smooth real Banach spaces and the corresponding results of [2] in uniformly smooth real Banach spaces.

2. Preliminaries

Let X be a real Banach space of dimension $\dim X \geq 2$. The modulus of smoothness of X is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\tau) := \sup\left\{2^{-1}\left(\|x+y\| + \|x-y\|\right) - 1 : \|x\| = 1, \|y\| \leq \tau\right\}, \quad \tau > 0. \tag{2.1}$$

The function $\rho_X(\tau)$ is convex, continuous, and increasing, and $\rho_X(0) = 0$.

The space X is called uniformly smooth if and only if

$$\lim_{\tau \to 0^+} \frac{\rho_X(\tau)}{\tau} = 0. \tag{2.2}$$

The space X is called p-uniformly smooth if and only if there exist a constant C_p and a real number $1 < p \leq 2$, such that

$$\rho_X(\tau) \leq C_p \tau^p. \tag{2.3}$$

Typical examples of uniformly smooth spaces are the Lebesgue L_p, the sequence ℓ_p, and Sobolev W_p^m spaces for $1 < p < \infty$. In particular, for $1 < p \leq 2$, these spaces are p-uniformly smooth and for $2 \leq p < \infty$, they are 2-uniformly smooth.

It is well known that if X is uniformly smooth, then the normalized duality mapping J is single-valued and uniformly continuous on any bounded subset of X.

Lemma 2.1 (see [3, 9]). *If X is a uniformly smooth Banach space, then for all $x, y \in X$ with $\|x\| \leq R$, $\|y\| \leq R$,*

$$\begin{gathered} \langle x - y, Jx - Jy \rangle \leq 2L_F R^2 \rho_X\left(\frac{4\|x-y\|}{R}\right), \\ \|Jx - Jy\| \leq 8Rh_X\left(\frac{16L_F\|x-y\|}{R}\right), \end{gathered} \tag{2.4}$$

where $h_X(\tau) := \rho_X(\tau)/\tau$, L_F is the Figiel s constant, $1 < L_F < 1.7$.

Lemma 2.2 (see [1]). *Let X be a real Banach space and J be the normalized duality mapping. Then, for any given $x, y \in X$, we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y). \tag{2.5}$$

Lemma 2.3 (see [8]). *Let $\{\alpha_n\}_{n\geq1}$, $\{\beta_n\}_{n\geq1}$ and $\{\gamma_n\}_{n\geq1}$ be nonnegative sequences satisfying*

$$\alpha_{n+1} \leq (1+\gamma_n)\alpha_n + \beta_n, \quad n \geq 1, \quad \sum_{n=1}^{\infty}\beta_n < \infty, \quad \sum_{n=1}^{\infty}\gamma_n < \infty. \tag{2.6}$$

Then, $\lim_{n\to\infty}\alpha_n$ exists. Moreover, if $\liminf_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 2.4 (see [4]). *Let $f, g : \mathbb{N} \to [0, \infty)$ be sequences and suppose that*

$$g(n) \leq 1, \quad \forall n \in \mathbb{N}, \quad g(n) \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad \sum_{n=1}^{\infty} g(n) = \infty. \tag{2.7}$$

Then,

$$\sum_{n=1}^{\infty} f(n) < \infty \Rightarrow f = o(g), \quad \text{as } n \longrightarrow \infty. \tag{2.8}$$

The converse is false.

3. Main Results and Their Proofs

Theorem 3.1. *Let X be a p-uniformly smooth real Banach space and D a nonempty convex subset of X. Suppose $T : D \to 2^D$ is a multivalued generalized Φ-hemicontractive and bounded mapping. For any given $x_0, u_0, v_0 \in D$, let $\{x_n\}$ be the sequence generated by the following Ishikawa-type iterative algorithm with variable coefficients:*

$$\begin{aligned} y_n &= \widehat{a}_n \ x_n + \widehat{b}_n \ \xi_n + \widehat{c}_n \ v_n, \quad \exists \xi_n \in T x_n, \\ x_{n+1} &= \widehat{\alpha}_n \ x_n + \widehat{\beta}_n \ \eta_n + \widehat{\gamma}_n \ u_n, \quad \exists \eta_n \in T y_n, \end{aligned} \quad n \in \mathbb{N}, \tag{3.1}$$

where $\{u_n\}$ and $\{v_n\}$ are arbitrary bounded sequences in D,

$$\begin{aligned} &\widehat{a}_n = 1 - \widehat{b}_n - \widehat{c}_n, \quad \widehat{b}_n = \frac{b_n}{r_n^2}, \quad \widehat{c}_n = \frac{c_n}{r_n^2}, \quad r_n = 2 + \|x_n\| + \|\xi_n\| + \|v_n\|, \\ &\widehat{\alpha}_n = 1 - \widehat{\beta}_n - \widehat{\gamma}_n, \quad \widehat{\beta}_n = \frac{\beta_n}{R_n^2}, \quad \widehat{\gamma}_n = \frac{\gamma_n}{R_n^2}, \quad R_n = r_n + \|\eta_n\| + \|u_n\|, \end{aligned} \tag{3.2}$$

$\{\beta_n\}$, $\{\gamma_n\}$, $\{b_n\}$ and $\{c_n\}$ are four sequences in $[0, 1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n^p < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty, \quad b_n \leq O(\beta_n), \quad c_n \leq O(\beta_n). \tag{3.3}$$

Then, $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. Since T is generalized Φ-hemicontractive, then the fixed-point set $F(T)$ of T is nonempty and there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that for each $x \in D$ and $x^* \in F(T)$, the following inequality holds:

$$\langle \xi - x^*, J(x - x^*) \rangle \leq \|x - x^*\|^2 - \Phi(\|x - x^*\|), \quad \forall \xi \in Tx. \tag{3.4}$$

If $z \in F(T)$, that is, $z \in Tz$, then, by (3.4), we have

$$\|z - x^*\|^2 = \langle z - x^*, J(z - x^*) \rangle \leq \|z - x^*\|^2 - \Phi(\|z - x^*\|). \tag{3.5}$$

So, T has a unique fixed point, say x^*.

From (3.1) and (3.2), we have $\|x_n - x^*\| \le r_n + \|x^*\|$, $\|y_n - x^*\| \le r_n + \|x^*\|$, $\|\xi_n - x^*\| \le r_n + \|x^*\|$, $\|\eta_n - x^*\| \le R_n + \|x^*\|$ and $\|x_{n+1} - x^*\| \le R_n + \|x^*\|$.

By Lemma 2.4 and (3.3), we know $\gamma_n = o(\beta_n)$. Since D is a convex subset of X and $T : \mathrm{D} \to 2^D$, it follows from (3.1), (3.2), and (3.3) that

$$\begin{aligned}
\|x_{n+1} - y_n\| &= \|(x_{n+1} - x^*) - (y_n - x^*)\| \\
&= \Big\| \Big(1 - \widehat{\beta}_n - \widehat{\gamma}_n\Big)(x_n - x^*) + \widehat{\beta}_n \ (\eta_n - x^*) + \widehat{\gamma}_n \ (u_n - x^*) \\
&\quad - \Big(1 - \widehat{b}_n - \widehat{c}_n\Big)(x_n - x^*) + \widehat{b}_n \ (\xi_n - x^*) + \widehat{c}_n \ (v_n - x^*) \Big\| \\
&\le \frac{O(\beta_n)}{r_n^2}(r_n + \|x^*\|) + \frac{O(\beta_n)}{R_n^2}(R_n + \|x^*\|) \\
&\le \frac{O(\beta_n)}{r_n} \longrightarrow 0 \quad (n \longrightarrow \infty).
\end{aligned} \tag{3.6}$$

From (3.6) and $\|y_n - x^*\| \le r_n + \|x^*\|$, we have $\|x_{n+1} - x^*\| \le r_n + \|x^*\| + (O(\beta_n)/r_n)$.

Considering $1 < p \le 2$ and $r_n \ge 2$, by Lemma 2.1, we have

$$\begin{aligned}
\|J(x_{n+1} - x^*) - J(y_n - x^*)\| &\le 8\left(r_n + \|x^*\| + \frac{O(\beta_n)}{r_n}\right) C_p \cdot \left(\frac{16 L_F \|x_{n+1} - y_n\|}{r_n + \|x^*\| + O(\beta_n)/r_n}\right)^{p-1} \\
&\le \left(r_n + \|x^*\| + \frac{O(\beta_n)}{r_n}\right)^{2-p} \frac{O\left(\beta_n^{p-1}\right)}{r_n^{p-1}} \\
&\le \left(r_n^2 + r_n\|x^*\| + O(\beta_n)\right)^{2-p} \frac{O\left(\beta_n^{p-1}\right)}{r_n} \\
&\le r_n \cdot O\left(\beta_n^{p-1}\right).
\end{aligned} \tag{3.7}$$

By (3.1), (3.2), (3.3) and Lemma 2.2, we have

$$\begin{aligned}
\|y_n - x^*\|^2 &\le \left\| \widehat{a}_n \ (x_n - x^*) + \widehat{b}_n(\eta_n - x^*) + \widehat{c}_n \ (v_n - x^*) \right\|^2 \\
&\le \widehat{a}_n^2 \ \|x_n - x^*\|^2 + 2\left\langle \widehat{b}_n(\xi_n - x^*) + \widehat{c}_n(v_n - x^*), J(y_n - x^*) \right\rangle \\
&\le \|x_n - x^*\|^2 + O(\beta_n).
\end{aligned} \tag{3.8}$$

From (3.1), (3.2), (3.7), and (3.8) and Lemma 2.2, it can be concluded that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\widehat{\alpha}_n(x_n-x^*)+\widehat{\beta}_n(\eta_n-x^*)+\widehat{\gamma}_n(u_n-x^*)\right\|^2 \\
&\leq \widehat{\alpha}_n^2\|x_n-x^*\|^2+2\widehat{\beta}_n\langle \eta_n-x^*, J(x_{n+1}-x^*)-J(y_n-x^*)\rangle \\
&\quad +2\widehat{\beta}_n\langle \eta_n-x^*, J(y_n-x^*)\rangle + 2\widehat{\gamma}_n\langle u_n-x^*, J(x_{n+1}-x^*)\rangle \\
&\leq \widehat{\alpha}_n^2\|x_n-x^*\|^2+2\widehat{\beta}_n\|\eta_n-x^*\|\cdot\|J(x_{n+1}-x^*)-J(y_n-x^*)\| \\
&\quad +2\widehat{\beta}_n\left(\|y_n-x^*\|^2-\Phi(\|y_n-x^*\|)\right) + 2\widehat{\gamma}_n\|u_n-x^*\|\cdot\|J(x_{n+1}-x^*)\| \\
&\leq \left(1-\widehat{\beta}_n-\widehat{\gamma}_n\right)^2\|x_n-x^*\|^2+2\widehat{\beta}_n(R_n+\|x^*\|)\cdot r_n\cdot O\left(\beta_n^{p-1}\right) \\
&\quad +2\widehat{\beta}_n\left(\|x_n-x^*\|^2+O(\beta_n)\right)-2\widehat{\beta}_n\Phi(\|y_n-x^*\|)+2\widehat{\gamma}_n\cdot(R_n+\|x^*\|)^2 \\
&\leq \|x_n-x^*\|^2+\left(\widehat{\beta}_n+\widehat{\gamma}_n\right)^2\|x_n-x^*\|^2+O\left(\beta_n^p\right)+O\left(\beta_n^2\right)+O(\gamma_n) \\
&\quad -2\widehat{\beta}_n\Phi(\|y_n-x^*\|) \\
&\leq \|x_n-x^*\|^2+O\left(\beta_n^2\right)\|x_n-x^*\|^2+O\left(\beta_n^p\right)+O(\gamma_n)-2\widehat{\beta}_n\Phi(\|y_n-x^*\|).
\end{aligned} \tag{3.9}$$

From (3.3) and (3.9), we have

$$\|x_{n+1}-x^*\|^2 \leq \left(1+O\left(\beta_n^2\right)\right)\|x_n-x^*\|^2+O\left(\beta_n^p\right)+O(\gamma_n). \tag{3.10}$$

Thus, by (3.3), (3.10) and Lemma 2.3, we have $\{\|x_n-x^*\|\}$ bounded. It implies the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. Since T is a bounded mapping, we have $T\{x_n\}$ and $T\{y_n\}$ bounded. Since $\eta_n \in Ty_n$ and $\xi_n \in Tx_n$, $\{R_n\}$ is bounded. Let its bound be $R > 0$. From (3.9), there exists a number $M > 0$ such that

$$\|x_{n+1}-x^*\|^2 \leq \left(1+M\beta_n^2\right)\|x_n-x^*\|^2+M\left(\beta_n^p+\gamma_n\right)-\frac{2\beta_n}{R^2}\Phi(\|y_n-x^*\|). \tag{3.11}$$

Next, we will show

$$\liminf_{n\to\infty}\Phi(\|y_n-x^*\|)=0. \tag{3.12}$$

If it is not true, then there exist a $n_0 \in \mathbb{N}$ and a positive constant m_0 such that for any positive integer $n \geq n_0$

$$\Phi(\|y_n-x^*\|) \geq m_0. \tag{3.13}$$

In view of (3.11) and (3.13), for any positive integer $n \geq n_0$, we have

$$\|x_{n+1} - x^*\|^2 \leq \left(1 + M\beta_n^2\right)\|x_n - x^*\|^2 + M\left(\beta_n^p + \gamma_n\right) - \frac{2m_0\beta_n}{R^2}. \tag{3.14}$$

Taking $n = n_0, n_0 + 1, \dots, k$ in (3.14) above, we have

$$\begin{aligned}\sum_{n=n_0}^{k}\|x_{n+1} - x^*\|^2 \leq \sum_{n=n_0}^{k}\|x_n - x^*\|^2 + \sum_{n=n_0}^{k} M\beta_n^2 (R + \|x^*\|)^2 \\ + \sum_{n=n_0}^{k} M\left(\beta_n^p + \gamma_n\right) - \sum_{n=n_0}^{k}\frac{2m_0\beta_n}{R^2}.\end{aligned} \tag{3.15}$$

So,

$$\frac{2m_0}{R^2}\sum_{n=n_0}^{k}\beta_n \leq M(R + \|x^*\|)^2 \sum_{n=n_0}^{k}\beta_n^2 + M\left(\sum_{n=n_0}^{k}\beta_n^p + \sum_{n=n_0}^{k}\gamma_n\right). \tag{3.16}$$

This leads to a contradiction as $k \to \infty$. Hence, $\liminf_{n\to\infty} \Phi(\|y_n - x^*\|) = 0$.

By the definition of Φ and (3.12), there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $\{y_{n_i}\} \to x^*$ as $i \to \infty$. Thus, by (3.6), we have $\liminf_{n\to\infty} \|x_n - x^*\| = 0$. Further, Using Lemma 2.3 and (3.11), we obtain $\lim_{n\to\infty}\|x_n - x^*\| = 0$. It means that $\{x_n\}$ converges strongly to the unique fixed point of T. The proof is finished. □

From Theorem 3.1, we can obtain the following theorems.

Theorem 3.2. *Let X be a p-uniformly smooth Banach space, D be a nonempty convex subset of X, and $T : D \to 2^D$ a multivalued generalized Φ-hemicontractive and bounded mapping. For any given $x_0, u_0 \in D$, let $\{x_n\}$ be the sequence generated by the following Mann-type iterative algorithm with variable coefficients:*

$$x_{n+1} = \widehat{\alpha}_n x_n + \widehat{\beta}_n \eta_n + \widehat{\gamma}_n u_n, \quad \exists\ \eta_n \in T x_n,\ n \in \mathbb{N}, \tag{3.17}$$

where $\{u_n\}$ is an arbitrary bounded sequence in D,

$$\widehat{\alpha}_n = 1 - \widehat{\beta}_n - \widehat{\gamma}_n, \quad \widehat{\beta}_n = \frac{\beta_n}{R_n^2}, \quad \widehat{\gamma}_n = \frac{\gamma_n}{R_n^2}, \quad R_n = 2 + \|x_n\| + \left\|\eta_n\right\| + \|u_n\|, \tag{3.18}$$

$\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty}\beta_n = \infty, \quad \sum_{n=0}^{\infty}\beta_n^p < \infty, \quad \sum_{n=0}^{\infty}\gamma_n < \infty. \tag{3.19}$$

Then, $\{x_n\}$ converges strongly to the unique fixed point of T.

Remark 3.3. Theorems 3.1 and 3.2 improve Theorem CC [1, Theorem 3.8] in p-uniformly smooth real Banach spaces since the class of multivalued generalized Lipschitz mappings is a proper subset of the class of bounded mappings and the number γ_0 in Theorem CC [1, Theorem 3.8] is dropped off.

In uniformly smooth real Banach spaces, we have the following theorem.

Theorem 3.4. *Let X be a uniformly smooth real Banach space and D a nonempty convex subset of X. Suppose $T : D \to 2^D$ is a multivalued generalized Φ-hemicontractive mapping with bounded range. For any given $x_0, u_0, v_0 \in D$, let $\{x_n\}$ be the sequence generated by the following Ishikawa-type iterative algorithm with variable coefficients:*

$$\begin{aligned} y_n &= \widehat{a}_n\ x_n + \widehat{b}_n\ \xi_n + \widehat{c}_n\ v_n, \quad \exists \xi_n \in Tx_n, \\ x_{n+1} &= \widehat{\alpha}_n\ x_n + \widehat{\beta}_n\ \eta_n + \widehat{\gamma}_n\ u_n, \quad \exists \eta_n \in Ty_n, \end{aligned} \quad n \in \mathbb{N}, \tag{3.20}$$

where $\{u_n\}$ and $\{v_n\}$ are arbitrary bounded sequences in D,

$$\begin{aligned} \widehat{a}_n &= 1 - \widehat{b}_n - \widehat{c}_n, \quad \widehat{b}_n = \frac{b_n}{r_n^2}, \quad \widehat{c}_n = \frac{c_n}{r_n^2}, \quad r_n = 2 + \|x_n\| + \|\xi_n\| + \|v_n\|, \\ \widehat{\alpha}_n &= 1 - \widehat{\beta}_n - \widehat{\gamma}_n, \quad \widehat{\beta}_n = \frac{\beta_n}{R_n^2}, \quad \widehat{\gamma}_n = \frac{\gamma_n}{R_n^2}, \quad R_n = r_n + \|\eta_n\| + \|u_n\|, \end{aligned} \tag{3.21}$$

$\{\beta_n\}$, $\{\gamma_n\}$, $\{b_n\}$ and $\{c_n\}$ are four sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n^2 < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty, \quad b_n \le O(\beta_n), \quad c_n \le O(\beta_n). \tag{3.22}$$

Then, $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. From Theorem 3.1, T has a unique fixed point, say x^*. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by the algorithm (3.20). Since T has a bounded range, we set

$$\begin{aligned} d := \sup\{\|\xi - \eta\| : x, y \in D,\ \xi \in Tx,\ \eta \in Ty\} + \sup\{\|u_n - x^*\|,\ n \in \mathbb{N}\} \\ + \sup\{\|v_n - x^*\|\ , n \in \mathbb{N}\}. \end{aligned} \tag{3.23}$$

Obviously, $d < \infty$. Next, we will prove that for $n \ge 0$, $\|x_n - x^*\| \le d + \|x_0 - x^*\|$. In fact, for $n = 0$, the above inequality holds. Assume the inequality is true for $n = k$. Then, for $n = k+1$, there exists a $\eta_k \in Ty_k$ such that

$$\begin{aligned} \|x_{k+1} - x^*\| &\le \widehat{\alpha}_k \|x_n - x^*\| + \widehat{\beta}_k \|\eta_k - x^*\| + \widehat{\gamma}_k \|u_k - x^*\| \\ &\le \widehat{\alpha}_k (d + \|x_0 - x^*\|) + \widehat{\beta}_k d + \widehat{\gamma}_k d \\ &\le d + \|x_0 - x^*\|. \end{aligned} \tag{3.24}$$

By induction, we have the sequence $\{x_n\}$ bounded. Similarly, we have the sequence $\{y_n\}$ also bounded.

From the proof of Theorem 3.1, we have $\|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Since X is a real uniformly smooth Banach space, so that the normalized duality mapping J is single valued and uniformly continuous on any bounded subset of X, thus

$$d_n := \left\|J(x_{n+1} - x^*) - J(y_n - x^*)\right\| \longrightarrow 0 \tag{3.25}$$

as $n \to \infty$.

Next, following the reasoning in the proof of Theorem 3.1, we deduce the conclusion of Theorem 3.4. □

Remark 3.5. In view of Example 1.4, the class of Φ-hemicontractive mappings is a proper subset of the class of generalized Φ-hemicontractive mappings. Hence, Theorem 3.4 improves essentially the result of [2, Theorem 2].

As applications, we give the following theorems.

Theorem 3.6. *Let X be a p-uniformly smooth Banach space $T : X \to 2^X$, a multivalued generalized Φ-hemiaccretive and bounded mapping. For any given $f \in X$, define $S : X \to 2^X$ by $Sx := x - Tx + f$ for all $x \in X$. For any given $x_0, u_0, v_0 \in X$, let $\{x_n\}$ be the Ishikawa-type iterative sequence with variable coefficients, defined by*

$$\begin{aligned} y_n &= \widehat{a}_n\, x_n + \widehat{b}_n\, \xi_n + \widehat{c}_n\, v_n, \quad \exists \xi_n \in Sx_n, \\ x_{n+1} &= \widehat{\alpha}_n\, x_n + \widehat{\beta}_n\, \eta_n + \widehat{\gamma}_n\, u_n, \quad \exists \eta_n \in Sy_n, \end{aligned} \quad n \in \mathbb{N}, \tag{3.26}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in X,

$$\begin{aligned} \widehat{a}_n &= 1 - \widehat{b}_n - \widehat{c}_n, \quad \widehat{b}_n = \frac{b_n}{r_n^2}, \quad \widehat{c}_n = \frac{c_n}{r_n^2}, \quad r_n = 2 + \|x_n\| + \|\xi_n\| + \|v_n\|, \\ \widehat{\alpha}_n &= 1 - \widehat{\beta}_n - \widehat{\gamma}_n, \quad \widehat{\beta}_n = \frac{\beta_n}{R_n^2}, \quad \widehat{\gamma}_n = \frac{\gamma_n}{R_n^2}, \quad R_n = r_n + \left\|\eta_n\right\| + \|u_n\|, \end{aligned} \tag{3.27}$$

$\{\beta_n\}$, $\{\gamma_n\}$, $\{b_n\}$, and $\{c_n\}$ are four sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n^p < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty, \; b_n \le O(\beta_n), \quad c_n \le O(\beta_n). \tag{3.28}$$

Then, $\{x_n\}$ converges strongly to the unique solution of the generalized Φ-hemiaccretive mapping equation $f \in Tx$.

Theorem 3.7. *Let X be a uniformly smooth Banach space and $T : X \to 2^X$ a generalized Φ-hemiaccretive with bounded range. For any given $f \in X$, define $S : X \to 2^X$ by $Sx := x - Tx + f$ for all $x \in X$. For any given $x_0, u_0, v_0 \in X$, let $\{x_n\}$ be the Ishikawa-type iterative sequence with variable coefficients, defined by*

$$\begin{aligned} y_n &= \widehat{a}_n\, x_n + \widehat{b}_n\, \xi_n + \widehat{c}_n\, v_n, \quad \exists \xi_n \in Sx_n, \\ x_{n+1} &= \widehat{\alpha}_n\, x_n + \widehat{\beta}_n\, \eta_n + \widehat{\gamma}_n\, u_n, \quad \exists \eta_n \in Sy_n, \end{aligned} \quad n = 0, 1, 2, \ldots, \tag{3.29}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in X,

$$\begin{aligned} \widehat{a}_n &= 1 - \widehat{b}_n - \widehat{c}_n, \quad \widehat{b}_n = \frac{b_n}{r_n^2}, \quad \widehat{c}_n = \frac{c_n}{r_n^2}, \quad r_n = 2 + \|x_n\| + \|\xi_n\| + \|v_n\|, \\ \widehat{\alpha}_n &= 1 - \widehat{\beta}_n - \widehat{\gamma}_n, \quad \widehat{\beta}_n = \frac{\beta_n}{R_n^2}, \quad \widehat{\gamma}_n = \frac{\gamma_n}{R_n^2}, \quad R_n = r_n + \|\eta_n\| + \|u_n\|, \end{aligned} \tag{3.30}$$

$\{\beta_n\}$, $\{\gamma_n\}$, $\{b_n\}$ and $\{c_n\}$ are four sequences in $[0,1]$ satisfying the following conditions:

$$\sum_{n=0}^{\infty} \beta_n = \infty, \quad \sum_{n=0}^{\infty} \beta_n^2 < \infty, \quad \sum_{n=0}^{\infty} \gamma_n < \infty, \quad b_n \le O(\beta_n), \quad c_n \le O(\beta_n). \tag{3.31}$$

Then, $\{x_n\}$ converges strongly to the unique solution of the generalized Φ-hemiaccretive mapping equation $f \in Tx$.

Remark 3.8. (1) Theorem 3.6 improves some recent results, for example, [1, Theorem 3.7] and [2, Theorem 2] in p-uniformly smooth real Banach spaces since the multivalued generalized Φ-hemiaccretive mapping within the equation has no generalized Lipschitz assumption.

(2) In view of Example 1.4, the class of Φ-hemicontractive mappings is a proper subset of the class of generalized Φ-hemicontractive mappings. Hence, Theorem 3.7 improves essentially the result of [2, Theorem 2] in uniformly smooth real Banach spaces.

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