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Research Article

A Hybrid Method for Monotone Variational Inequalities Involving Pseudocontractions

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We use strongly pseudocontraction to regularize the following ill-posed monotone variational inequality: finding a point x^* with the property $x^* \in \mathrm{Fix}(T)$ such that $\langle (I-S)x^*, x-x^*\rangle \geq 0$, $x \in \mathrm{Fix}(T)$ where S, T are two pseudocontractive self-mappings of a closed convex subset C of a Hilbert space with the set of fixed points $\mathrm{Fix}(T) \neq \emptyset$. Assume the solution set Ω of (VI) is nonempty. In this paper, we introduce one implicit scheme which can be used to find an element $x^* \in \Omega$. Our results improve and extend a recent result of (Lu et al. 2009).

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively, and let C be a nonempty closed convex subset of H. Let $F : C \to H$ be a nonlinear mapping. A variational inequality problem, denoted $\mathrm{VI}(F,C)$, is to find a point x^* with the property

$$x^* \in C \quad \text{such that } \langle Fx^*, x-x^*\rangle \geq 0 \quad \forall x \in C. \tag{1.1}$$

If the mapping F is a monotone operator, then we say that $\mathrm{VI}(F,C)$ is monotone. It is well known that if F is Lipschitzian and strongly monotone, then for small enough $\gamma > 0$, the mapping $P_C(I-\gamma F)$ is a contraction on C and so the sequence $\{x_n\}$ of Picard iterates, given by $x_n = P_C(I-\gamma F)x_{n-1}$ $(n \geq 1)$ converges strongly to the unique solution of the $\mathrm{VI}(F,C)$. Hybrid methods for solving the variational inequality $\mathrm{VI}(F,C)$ were studied by Yamada [1], where he assumed that F is Lipschitzian and strongly monotone.

In this paper, we devote to consider the following monotone variational inequality: finding a point x^* with the property

$$x^* \in \mathrm{Fix}(T) \quad \text{such that } \langle (I - S)x^*, x - x^* \rangle \geq 0 \quad \forall x \in \mathrm{Fix}(T), \tag{1.2}$$

where $S, T : C \to C$ are two nonexpansive mappings with the set of fixed points $\mathrm{Fix}(T) = \{x \in C : Tx = x\} \neq \emptyset$. Let Ω denote the set of solutions of VI (1.2) and assume that Ω is nonempty.

We next briefly review some literatures in which the involved mappings S and T are all nonexpansive.

First, we note that Yamada's methods do not apply to VI (1.2) since the mapping $I - S$ fails, in general, to be strongly monotone, though it is Lipschitzian. As a matter of fact, the variational inequality (1.2) is, in general, ill-posed, and thus regularization is needed. Recently, Moudafi and Maingé [2] studied the VI (1.2) by regularizing the mapping $tS + (1 - t)T$ and defined $(x_{s,t})$ as the unique fixed point of the equation

$$x_{s,t} = sf(x_{s,t}) + (1 - s)[tSx_{s,t} + (1 - t)Tx_{s,t}], \quad s, t \in (0, 1). \tag{1.3}$$

Since Moudafi and Maingé's regularization depends on t, the convergence of the scheme (1.3) is more complicated. Very recently, Lu et al. [3] studied the VI (1.2) by regularizing the mapping S and defined $(x_{s,t})$ as the unique fixed point of the equation

$$x_{s,t} = s\left[tf(x_{s,t}) + (1 - t)Sx_{s,t}\right] + (1 - s)Tx_{s,t}, \quad s, t \in (0, 1). \tag{1.4}$$

Note that Lu et al.'s regularization (1.4) does no longer depend on t. Related work can also be found in [4–9].

In this paper, we will extend Lu et al.'s result to a general case. We will further study the strong convergence of the algorithm (1.4) for solving VI (1.2) under the assumption that the mappings $S, T : C \to C$ are all pseudocontractive. As far as we know, this appears to be the first time in the literature that the solutions of the monotone variational inequalities of kind (1.2) are investigated in the framework that feasible solutions are fixed points of a pseudocontractive mapping T.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Recall that a mapping $f : C \to C$ is called strongly pseudocontractive if there exists a constant $\rho \in [0, 1)$ such that $\langle f(x) - f(y), x - y \rangle \leq \rho \|x - y\|^2$, for all $x, y \in C$. A mapping $T : C \to C$ is a pseudocontraction if it satisfies the property

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in C. \tag{2.1}$$

We denote by $\mathrm{Fix}(T)$ the set of fixed points of T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Note that $\mathrm{Fix}(T)$ is always closed and convex (but may be empty). However, for VI (1.2), we always assume $\mathrm{Fix}(T) \neq \emptyset$. It is not hard to find that T is a pseudocontraction if and only if T satisfies one of the following two equivalent properties:

(a) $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y)\|^2$ for all $x, y \in C$, or

(b) $I - T$ is monotone on C: $\langle x - y, (I - T)x - (I - T)y\rangle \geq 0$ for all $x, y \in C$.

Below is the so-called demiclosedness principle for pseudocontractive mappings.

Lemma 2.1 (see [10]). *Let C be a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a Lipschitz pseudocontraction. Then,* $\mathrm{Fix}(T)$ *is a closed convex subset of C, and the mapping $I - T$ is demiclosed at 0; that is, whenever $\{x_n\} \subset C$ is such that $x_n \rightharpoonup x$ and $(I-T)x_n \to 0$, then $(I-T)x = 0$.*

We also need the following lemma.

Lemma 2.2 (see [3]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments; that is, $F(x + ty) \to F(x)$ weakly as $t \to 0$. Then, the variational inequality*

$$x^* \in C, \quad \langle Fx^*, x - x^*\rangle \geq 0, \quad \forall x \in C \tag{2.2}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle Fx, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{2.3}$$

3. Main Results

In this section, we introduce an implicit algorithm and prove this algorithm converges strongly to x^* which solves the VI (1.2). Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a strongly pseudocontraction. Let $S, T : C \to C$ be two Lipschitz pseudocontractions. For $s, t \in (0, 1)$, we define the following mapping

$$x \longmapsto W_{s,t}x := s\left[tf(x) + (1 - t)Sx\right] + (1 - s)Tx. \tag{3.1}$$

It easy to see that the mapping $W_{s,t} : C \to C$ is strongly pseudocontractive; that is, $\langle W_{s,t}x - W_{s,t}y, x - y\rangle \leq [1 - (1 - \rho)st]\|x - y\|^2$, for all $x, y \in C$. So, by Deimling [11], $W_{s,t}$ has a unique fixed point which is denoted $x_{s,t} \in C$; that is,

$$x_{s,t} = s\left[tf(x_{s,t}) + (1 - t)Sx_{s,t}\right] + (1 - s)Tx_{s,t}, \quad s, t \in (0, 1). \tag{3.2}$$

Below is our main result of this paper which displays the behavior of the net $\{x_{s,t}\}$ as $s \to 0$ and $t \to 0$ successively.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a strongly pseudocontraction. Let $S, T : C \to C$ be two Lipschitz pseudocontractions with* $\mathrm{Fix}(T) \neq \emptyset$. *Suppose that the solution set Ω of VI* (1.2) *is nonempty. Let, for each $(s, t) \in (0, 1)^2$, $\{x_{s,t}\}$ be defined implicitly by* (3.2). *Then, for each fixed $t \in (0, 1)$, the net $\{x_{s,t}\}$ converges in norm, as $s \to 0$, to a* *point* $x_t \in \mathrm{Fix}(T)$. *Moreover, as* $t \to 0$, *the net* $\{x_t\}$ *converges in norm to the unique solution* x^* *of the following VI:*

$$x^* \in \Omega, \quad \langle (I-f)x^*, x-x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{3.3}$$

Hence, for each null sequence $\{t_n\}$ *in* $(0,1)$, *there exists another null sequence* $\{s_n\}$ *in* $(0,1)$, *such that the sequence* $x_{s_n,t_n} \to x^*$ *in norm as* $n \to \infty$.

We divide our details proofs into several lemmas as follows. Throughout, we assume all conditions of Theorem 3.1 are satisfied.

Lemma 3.2. *For each fixed* $t \in (0,1)$, *the net* $\{x_{s,t}\}$ *is bounded.*

Proof. Take any $z \in \mathrm{Fix}(T)$ to derive that, for all $s,t \in (0,1)$,

$$\begin{aligned}
\|x_{s,t}-z\|^2 &= st\langle f(x_{s,t})-f(z), x_{s,t}-z\rangle + st\langle f(z)-z, x_{s,t}-z\rangle \\
&\quad + s(1-t)\langle Sx_{s,t}-Sz, x_{s,t}-z\rangle + s(1-t)\langle Sz-z, x_{s,t}-z\rangle \\
&\quad + (1-s)\langle Tx_{s,t}-Tz, x_{s,t}-z\rangle \\
&\leq st\rho\|x_{s,t}-z\|^2 + st\|f(z)-z\|\|x_{s,t}-z\| + s(1-t)\|x_{s,t}-z\|^2 \\
&\quad + s(1-t)\|Sz-z\|\|x_{s,t}-z\| + (1-s)\|x_{s,t}-z\|^2 \\
&= \left[1-(1-\rho)st\right]\|x_{s,t}-z\|^2 + s\left[t\|f(z)-z\| + (1-t)\|Sz-z\|\right]\|x_{s,t}-z\|.
\end{aligned} \tag{3.4}$$

It follows that

$$\|x_{s,t}-z\| \leq \frac{1}{(1-\rho)t}\max\{\|f(z)-z\|, \|Sz-z\|\}. \tag{3.5}$$

It follows that for each fixed $t \in (0,1)$, $\{x_{s,t}\}$ is bounded, so are the nets $\{f(x_{s,t})\}$, $\{Sx_{s,t}\}$, and $\{Tx_{s,t}\}$. □

We will use $M_t > 0$ to denote possible constant appearing in the following.

Lemma 3.3. $x_{s,t} \to x_t \in \mathrm{Fix}(T)$ *as* $s \to 0$.

Proof. From (3.2), we have

$$x_{s,t} - Tx_{s,t} = s\left[tf(x_{s,t}) + (1-t)Sx_{s,t} - Tx_{s,t}\right] \longrightarrow 0 \quad \text{as } s \longrightarrow 0 \text{ for each fixed } t \in (0,1). \tag{3.6}$$

Next, we show that, for each fixed $t \in (0,1)$, the net $\{x_{s,t}\}$ is relatively norm compact as $s \to 0$. It follows from (3.2) that

$$\begin{aligned}\|x_{s,t}-z\|^2 &= st\langle f(x_{s,t})-f(z), x_{s,t}-z\rangle + st\langle f(z)-z, x_{s,t}-z\rangle + s(1-t)\langle Sx_{s,t}-Sz, x_{s,t}-z\rangle \\ &\quad + s(1-t)\langle Sz-z, x_{s,t}-z\rangle + (1-s)\langle Tx_{s,t}-z, x_{s,t}-z\rangle \\ &\le \left[1-(1-\rho)st\right]\|x_{s,t}-z\|^2 + st\langle f(z)-z, x_{s,t}-z\rangle + s(1-t)\langle Sz-z, x_{s,t}-z\rangle.\end{aligned} \tag{3.7}$$

It turns out that

$$\|x_{s,t}-z\|^2 \le \frac{1}{(1-\rho)t}\langle tf(z)+(1-t)Sz-z, x_{s,t}-z\rangle, \quad \forall z \in \mathrm{Fix}(T). \tag{3.8}$$

Assume that $\{s_n\} \subset (0,1)$ is such that $s_n \to 0$ as $n \to \infty$. By (3.8), we obtain immediately that

$$\|x_{s_n,t}-z\|^2 \le \frac{1}{(1-\rho)t}\langle tf(z)+(1-t)Sz-z, x_{s_n,t}-z\rangle, \quad \forall z \in \mathrm{Fix}(T). \tag{3.9}$$

Since $\{x_{s_n,t}\}$ is bounded, without loss of generality, we may assume that as $s_n \to 0$, $\{x_{s_n,t}\}$ converges weakly to a point x_t. From (3.6), we get $\|x_{s_n,t}-Tx_{s_n,t}\| \to 0$. So, Lemma 2.1 implies that $x_t \in \mathrm{Fix}(T)$. We can then substitute x_t for z in (3.9) to get

$$\|x_{s_n,t}-x_t\|^2 \le \frac{1}{(1-\rho)t}\langle tf(x_t)+(1-t)Sx_t-x_t, x_{s_n,t}-x_t\rangle. \tag{3.10}$$

Consequently, the weak convergence of $\{x_{s_n,t}\}$ to x_t actually implies that $x_{s_n,t} \to x_t$ strongly. This has proved the relative norm compactness of the net $\{x_{s,t}\}$ as $s \to 0$.

Now, we return to (3.9) and take the limit as $n \to \infty$ to get

$$\|x_t-z\|^2 \le \frac{1}{(1-\rho)t}\langle tf(z)+(1-t)Sz-z, x_t-z\rangle, \quad \forall z \in \mathrm{Fix}(T). \tag{3.11}$$

In particular, x_t solves the following variational inequality

$$x_t \in \mathrm{Fix}(T), \quad \langle tf(z)+(1-t)Sz-z, x_t-z\rangle \ge 0, \quad \forall z \in \mathrm{Fix}(T), \tag{3.12}$$

or the equivalent dual variational inequality (see Lemma 2.2)

$$x_t \in \mathrm{Fix}(T), \quad \langle tf(x_t)+(1-t)Sx_t-x_t, x_t-z\rangle \ge 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.13}$$

Next, we show that as $s \to 0$, the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in \mathrm{Fix}(T)$. We assume $x_{s'_n,t} \to x'_t$ where $s'_n \to 0$. Similarly, by the above proof, we deduce $x'_t \in \mathrm{Fix}(T)$ which solves the following variational inequality

$$x'_t \in \mathrm{Fix}(T), \quad \langle tf(x'_t) + (1-t)Sx'_t - x'_t, x'_t - z \rangle \geq 0, \quad \forall z \in \mathrm{Fix}(T). \tag{3.14}$$

In (3.13), we take $z = x'_t$ to get

$$t\langle (I-f)x_t, x_t - x'_t \rangle + (1-t)\langle (I-S)x_t, x_t - x'_t \rangle \leq 0. \tag{3.15}$$

In (3.14), we take $z = x_t$ to get

$$t\langle (I-f)x'_t, x'_t - x_t \rangle + (1-t)\langle (I-S)x'_t, x'_t - x_t \rangle \leq 0. \tag{3.16}$$

Adding up (3.15) and (3.16) yields

$$t\langle (I-f)x_t - (I-f)x'_t, x_t - x'_t \rangle + (1-t)\langle (I-S)x_t - (I-S)x'_t, x_t - x'_t \rangle \leq 0. \tag{3.17}$$

At the same time, we note that

$$\begin{gathered} \langle (I-f)x_t - (I-f)x'_t, x_t - x'_t \rangle \geq (1-\rho)\|x_t - x'_t\|^2, \\ \langle (I-S)x_t - (I-S)x'_t, x_t - x'_t \rangle \geq 0. \end{gathered} \tag{3.18}$$

Therefore,

$$\begin{aligned} 0 &\geq t\langle (I-f)x_t - (I-f)x'_t, x_t - x'_t \rangle + (1-t)\langle (I-S)x_t - (I-S)x'_t, x_t - x'_t \rangle \\ &\geq (1-\rho)t\|x_t - x'_t\|^2. \end{aligned} \tag{3.19}$$

It follows that

$$x'_t = x_t. \tag{3.20}$$

Hence, we conclude that the entire net $\{x_{s,t}\}$ converges in norm to $x_t \in \mathrm{Fix}(T)$ as $s \to 0$. □

Lemma 3.4. *The net $\{x_t\}$ is bounded.*

Proof. In (3.13), we take any $y \in \Omega$ to deduce

$$\langle tf(x_t) + (1-t)Sx_t - x_t, x_t - y \rangle \geq 0. \tag{3.21}$$

By virtue of the monotonicity of $I - S$ and the fact that $y \in \Omega$, we have

$$\langle Sx_t - x_t, x_t - y \rangle \leq \langle Sy - y, x_t - y \rangle \leq 0. \tag{3.22}$$

It follows from (3.21) and (3.22) that

$$\langle f(x_t) - x_t, x_t - y \rangle \geq 0, \quad \forall y \in \Omega. \tag{3.23}$$

Hence

$$\|x_t - y\|^2 \leq \langle f(x_t) - f(y), x_t - y \rangle + \langle f(y) - y, x_t - y \rangle \leq \rho \|x_t - y\|^2 + \langle f(y) - y, x_t - y \rangle. \tag{3.24}$$

Therefore,

$$\|x_t - y\|^2 \leq \frac{1}{1-\rho} \langle f(y) - y, x_t - y \rangle, \quad \forall y \in \Omega. \tag{3.25}$$

In particular,

$$\|x_t - y\| \leq \frac{1}{1-\rho} \|f(y) - y\|, \quad \forall t \in (0,1). \tag{3.26}$$

□

Lemma 3.5. *The net $x_t \to x^* \in \Omega$ which solves the variational inequality* (3.3).

Proof. First, we note that the solution of the variational inequality VI (3.3) is unique.

We next prove that $\omega_w(x_t) \subset \Omega$; namely, if (t_n) is a null sequence in $(0,1)$ such that $x_{t_n} \to x'$ weakly as $n \to \infty$, then $x' \in \Omega$. To see this, we use (3.13) to get

$$\langle (I-S)x_t, z - x_t \rangle \geq \frac{t}{1-t} \langle (I-f)x_t, z - x_t \rangle, \quad z \in \mathrm{Fix}(T). \tag{3.27}$$

However, since $I - S$ is monotone,

$$\langle (I-S)z, z - x_t \rangle \geq \langle (I-S)x_t, z - x_t \rangle. \tag{3.28}$$

Combining the last two relations yields

$$\langle (I-S)z, z - x_t \rangle \geq \frac{t}{1-t} \langle (I-f)x_t, z - x_t \rangle, \quad z \in \mathrm{Fix}(T). \tag{3.29}$$

Letting $t = t_n \to 0$ as $n \to \infty$ in (3.29), we get

$$\langle (I-S)z, z - x' \rangle \geq 0, \quad z \in \mathrm{Fix}(T), \tag{3.30}$$

which is equivalent to its dual variational inequality

$$\langle (I-S)x', z - x' \rangle \geq 0, \quad z \in \mathrm{Fix}(T). \tag{3.31}$$

Namely, x' is a solution of VI (1.2); hence, $x' \in \Omega$. We further prove that $x' = x^*$, the unique solution of VI (3.3). As a matter of fact, we have by (3.25),

$$\left\| x_{t_n} - x' \right\|^2 \leq \frac{1}{1-\rho} \left\langle f\left(x'\right) - x', x_{t_n} - x' \right\rangle, \quad x' \in \Omega. \tag{3.32}$$

Therefore, the weak convergence to x' of $\{x_{t_n}\}$ right implies that that $x_{t_n} \to x'$ in norm. Now, we can let $t = t_n \to 0$ in (3.23) to get

$$\left\langle f\left(x'\right) - x', y - x' \right\rangle \leq 0, \quad \forall y \in \Omega. \tag{3.33}$$

It turns out that $x' \in \Omega$ solves VI (3.3). By uniqueness, we have $x' = x^*$. This is sufficient to guarantee that $x_t \to x^*$ in norm, as $t \to 0$. The proof is complete. □

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