Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 186237, 12 pages
doi:10.1155/2011/186237

*Research Article*

# Some Fixed-Point Theorems for Multivalued Monotone Mappings in Ordered Uniform Space

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Received 22 September 2010; Accepted 8 March 2011

Academic Editor: Jong Kim



We use the order relation on uniform spaces defined by Altun and Imdad (2009) to prove some new fixed-point and coupled fixed-point theorems for multivalued monotone mappings in ordered uniform spaces.

## 1. Introduction

There exists considerable literature of fixed-point theory dealing with results on fixed or common fixed-points in uniform space (e.g., between [1–14]). But the majority of these results are proved for contractive or contractive type mapping (notice from the cited references). Also some fixed-point and coupled fixed-point theorems in partially ordered metric spaces are given in [15–20]. Recently, Aamri and El Moutawakil [2] have introduced the concept of $E$-distance function on uniform spaces and utilize it to improve some well-known results of the existing literature involving both $E$-contractive or $E$-expansive mappings. Lately, Altun and Imdad [21] have introduced a partial ordering on uniform spaces utilizing $E$-distance function and have used the same to prove a fixed-point theorem for single-valued nondecreasing mappings on ordered uniform spaces. In this paper, we use the partial ordering on uniform spaces which is defined by [21], so we prove some fixed-point theorems of multivalued monotone mappings and some coupled fixed-point theorems of multivalued mappings which are given for ordered metric spaces in [22] on ordered uniform spaces.

Now, we recall some relevant definitions and properties from the foundation of uniform spaces. We call a pair $(X, \vartheta)$ to be a uniform space which consists of a nonempty set $X$ together with an uniformity $\vartheta$ wherein the latter begins with a special kind of filter on $X \times X$ whose all elements contain the diagonal $\Delta = \{(x, x) : x \in X\}$. If $V \in \vartheta$ and $(x, y) \in V$, $(y, x) \in V$ then $x$ and $y$ are said to be $V$-close. Also a sequence $\{x_n\}$ in $X$, is said to be a Cauchy sequence with regard to uniformity $\vartheta$ if for any $V \in \vartheta$, there exists $N \geq 1$ such that $x_n$ and $x_m$ are $V$-close for $m, n \geq N$. An uniformity $\vartheta$ defines a unique topology $\tau(\vartheta)$ on X for which the neighborhoods of $x \in X$ are the sets $V(x) = \{y \in X : (x, y) \in V\}$ when $V$ runs over $\vartheta$.

A uniform space $(X, \vartheta)$ is said to be Hausdorff if and only if the intersection of all the $V \in \vartheta$ reduces to diagonal $\Delta$ of X, that is, $(x, y) \in V$ for $V \in \vartheta$ implies $x = y$. Notice that Hausdorffness of the topology induced by the uniformity guarantees the uniqueness of limit of a sequence in uniform spaces. An element of uniformity $\vartheta$ is said to be symmetrical if $V = V^{-1} = \{(y, x) : (x, y) \in V\}$. Since each $V \in \vartheta$ contains a symmetrical $W \in \vartheta$ and if $(x, y) \in W$ then $x$ and $y$ are both $W$ and $V$-close and then one may assume that each $V \in \vartheta$ is symmetrical. When topological concepts are mentioned in the context of a uniform space $(X, \vartheta)$, they are naturally interpreted with respect to the topological space $(X, \tau(\vartheta))$.

## 2. Preliminaries

We will require the following definitions and lemmas in the sequel.

*Definition 2.1* (see [2]). Let $(X, \vartheta)$ be a uniform space. A function $p : X \times X \to \mathbb{R}^+$ is said to be an $E$-distance if

($p_1$) for any $V \in \vartheta$, there exists $\delta > 0$, such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$ imply $(x, y) \in V$,

($p_2$) $p(x, y) \leq p(x, z) + p(z, y)$, for all $x, y, z \in X$.

The following lemma embodies some useful properties of $E$-distance.

**Lemma 2.2** (see [1, 2]). *Let $(X, \vartheta)$ be a Hausdorff uniform space and $p$ be an $E$-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be arbitrary sequences in X and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following holds:*

(a) *if $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$,*

(b) *if $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $\{y_n\}$ converges to $z$,*

(c) *if $p(x_n, x_m) \leq \alpha_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in $(X, \vartheta)$.*

*Let $(X, \vartheta)$ be a uniform space equipped with $E$-distance $p$. A sequence in X is $p$-Cauchy if it satisfies the usual metric condition. There are several concepts of completeness in this setting.*

*Definition 2.3* (see [1, 2]). Let $(X, \vartheta)$ be a uniform space and $p$ be an $E$-distance on X. Then

(i) X said to be $S$-complete if for every $p$-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$,

(ii) X is said to be $p$-Cauchy complete if for every $p$-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$,

(iii) $f : X \to X$ is $p$-continuous if $\lim_{n\to\infty} p(x_n, x) = 0$ implies

$$\lim_{n\to\infty} p(fx_n, fx) = 0, \tag{2.1}$$

(iv) $f : X \to X$ is $\tau(\vartheta)$-continuous if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\vartheta)$ implies $\lim_{n\to\infty} f x_n = f x$ with respect to $\tau(\vartheta)$.

*Remark 2.4* (see [2]). Let $(X, \vartheta)$ be a Hausdorff uniform space and let $\{x_n\}$ be a $p$-Cauchy sequence. Suppose that $X$ is $S$-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Then Lemma 2.2(b) gives that $\lim_{n\to\infty} x_n = x$ with respect to the topology $\tau(\vartheta)$ which shows that $S$-completeness implies $p$-Cauchy completeness.

**Lemma 2.5** (see [15]). *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ be E-distance on $X$ and $\varphi : X \to \mathbb{R}$. Define the relation "$\preceq$" on $X$ as follows:*

$$x \preceq y \Longleftrightarrow x = y \quad or \quad p(x, y) \leq \varphi(x) - \varphi(y). \tag{2.2}$$

*Then "$\preceq$" is a (partial) order on $X$ induced by $\varphi$.*

## 3. The Fixed-Point Theorems of Multivalued Mappings

**Theorem 3.1.** *Let $(X, \vartheta)$ a Hausdorff uniform space and $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$ and $M = \{x \in X \mid T(x) \cap [x, +\infty) \neq \emptyset\}$. Suppose that:*

(i) *$T$ is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $T(x) \cap M \cap [x, +\infty) \neq \emptyset$.*

*Then $T$ has a fixed-point $x^*$ and there exists a sequence $\{x_n\}$ with*

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.1}$$

*such that $x_n \to x^*$. Moreover if $\varphi$ is lower semicontinuous, then $x_n \preceq x^*$ for all $n$.*

*Proof.* By the condition (ii), take $x_0 \in M$. From (iii), there exist $x_1 \in T(x_0) \cap M$ and $x_0 \preceq x_1$. Again from (iii), there exist $x_2 \in T(x_1) \cap M$. Thus $x_1 \preceq x_2$.

Continuing this procedure we get a sequence $\{x_n\}$ satisfying

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots. \tag{3.2}$$

So by the definition of "$\preceq$", we have $\cdots \varphi(x_2) \leq \varphi(x_1) \leq \varphi(x_0)$, that is, the sequence $\{\varphi(x_n)\}$ is a nonincreasing sequence in $\mathbb{R}$. Since $\varphi$ is bounded from below, $\{\varphi(x_n)\}$ is convergent and hence it is Cauchy, that is, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $m > n > n_0$ we have $|\varphi(x_m) - \varphi(x_n)| < \varepsilon$. Since $x_n \preceq x_m$, we have $x_n = x_m$ or $p(x_n, x_m) \leq \varphi(x_n) - \varphi(x_m)$. Therefore,

$$\begin{aligned} p(x_n, x_m) &\leq \varphi(x_n) - \varphi(x_m) \\ &= \left|\varphi(x_n) - \varphi(x_m)\right| \\ &< \varepsilon, \end{aligned} \tag{3.3}$$

which shows that (in view of Lemma 2.2(c)) that $\{x_n\}$ is $p$-Cauchy sequence. By the $p$-Cauchy completeness of $X$, $\{x_n\}$ converges to $x^*$. Since $T$ is upper semicontinuous, $x^* \in T(x^*)$.

Moreover, when $\varphi$ is lower semicontinuous, for each $n$

$$\begin{aligned} p(x_n, x^*) &= \lim_{m\to\infty} p(x_n, x_m) \\ &\leq \lim_{m\to\infty} \sup\left(\varphi(x_n) - \varphi(x_m)\right) \\ &= \varphi(x_n) - \lim_{m\to\infty} \inf \varphi(x_m) \\ &\leq \varphi(x_n) - \varphi(x^*). \end{aligned} \tag{3.4}$$

So $x_n \preceq x^*$, for all $n$. □

Similarly, we can prove the following.

**Theorem 3.2.** *Let $(X, \vartheta)$ a Hausdorff uniform space and $p$ an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping, $(-\infty, x] = \{y \in X : y \preceq x\}$ and $M = \{x \in X \mid T(x) \cap (-\infty, x] \neq \emptyset\}$. Suppose that*

(i) *$T$ is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $T(x) \cap M \cap (-\infty, x] \neq \emptyset$.*

*Then $T$ has a fixed-point $x^*$ and there exists a sequence $\{x_n\}$ with*

$$x_{n-1} \succeq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.5}$$

*such that $x_n \to x^*$. Moreover, if $\varphi$ is upper semicontinuous, then $x^* \preceq x_n$ for all $n$.*

**Corollary 3.3.** *Let $(X, \vartheta)$ a Hausdorff uniform space and $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x, +\infty) = \{y \in X : x \preceq y\}$. Suppose that:*

(i) *$T$ is upper semicontinuous, that is, $x_n \in X$ and $y_n \in T(x_n)$ with $x_n \to x_0$ and $y_n \to y_0$, implies $y_0 \in T(x_0)$,*

(ii) *$T$ satisfies the monotonic condition: for any $x,y \in X$ with $x \preceq y$ and any $u \in T(x)$, there exists $v \in T(y)$ such that $u \preceq v$,*

(iii) *there exists an $x_0 \in X$ such that $T(x_0) \cap [x_0, +\infty) \neq \emptyset$.*

*Then $T$ has a fixed-point $x^*$ and there exists a sequence $\{x_n\}$ with*

$$x_{n-1} \preceq x_n \in T(x_{n-1}), \quad n = 1, 2, 3, \ldots, \tag{3.6}$$

*such that $x_n \to x^*$. Moreover if $\varphi$ is lower semicontinuous, then $x_n \preceq x^*$ for all $n$.*

*Proof.* By (iii), $x_0 \in M = \{x \in X : T(x) \cap [x, +\infty) \neq \emptyset\}$. For $x \in M$, take $y \in T(x)$ and $x \preceq y$. By the monotonicity of $T$, there exists $z \in T(y)$ such that $y \preceq z$. So $y \in M$, and $T(x) \cap M \cap [x, +\infty) \neq \emptyset$. The conclusion follows from Theorem 3.1. □

**Corollary 3.4.** *Let $(X, \vartheta)$ a Hausdorff uniform space and $p$ is an $E$-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty, x] = \{y \in X : y \preceq x\}$. Suppose that:*

(i) *$T$ is upper semicontinuous,*

(ii) *$T$ satisfies the monotonic condition; for any $x, y \in X$ with $x \preceq y$ and any $v \in T(y)$, there exists $u \in T(x)$ such that $u \preceq v$,*

(iii) *there exists an $x_0 \in X$ such that $T(x_0) \cap (-\infty, x_0] \neq \emptyset$.*

*Then $T$ has a fixed-point $x^*$ and there exists a sequence $\{x_n\}$ with*

$$x_{n-1} \succeq x_n \in T(x_{n-1}), \quad n = 1, 2, \ldots, \tag{3.7}$$

*such that $x_n \to x^*$. Moreover if $\varphi$ is upper semicontinuous, then $x_n \succeq x^*$ for all $n$.*

**Corollary 3.5.** *Let $(X, \vartheta)$ a Hausdorff uniform space and $p$ is an $E$-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded below and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \preceq f(x)\}$. Suppose that:*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $f(x) \in M$.*

*Then $f$ has a fixed-point $x^*$ and the sequence*

$$x_{n-1} \preceq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.8}$$

*converges to $x^*$. Moreover if $\varphi$ is lower semicontinuous, then $x_n \preceq x^*$ for all $n$.*

**Corollary 3.6.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an $E$-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded above, and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy* *complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $x \in M$, $f(x) \in M$.*

*Then $f$ has a fixed-point $x^*$. And the sequence*

$$x_{n-1} \succeq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.9}$$

*converges to $x^*$. Moreover, if $\varphi$ is upper semicontinuous, then $x_n \succeq x^*$ for all $n$.*

**Corollary 3.7.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an $E$-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded below, and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $f : X \to X$ be a map and $M = \{x \in X : x \succeq f(x)\}$. Suppose that:*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$f$ is monotone increasing, that is, for $x \preceq y$ we have $f(x) \preceq f(y)$,*

(iii) *there exists an $x_0$, with $x_0 \preceq f(x_0)$.*

*Then $f$ has a fixed-point $x^*$ and the sequence*

$$x_{n-1} \preceq x_n = f(x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{3.10}$$

*converges to $x^*$. Moreover if $\varphi$ is lower semicontinuous, then $x_n \preceq x^*$ for all $n$.*

*Example 3.8.* Let $X = \{k, l, m\}$ and $\vartheta = \{V \subset X \times X : \Delta \subset V\}$. Define $p : X \times X \to \mathbb{R}^+$ as $p(x, x) = 0$ for all $x \in X$, $p(k, l) = p(l, k) = 2$, $p(k, m) = p(m, k) = 1$ ve $p(l, m) = p(m, l) = 3$. Since definition of $\vartheta$, $\bigcap_{V \in \vartheta} V = \Delta$ and this show that the uniform space $(X, \vartheta)$ is a Hausdorff uniform space. On the other hand, $p(k, l) \leq p(k, m) + p(m, l)$, $p(k, m) \leq p(k, l) + p(l, m)$ and $p(l, m) \leq p(l, k) + p(k, m)$ for $k, l, m \in X$ and thus $p$ is an $E$-distance as it is a metric on $X$. Next define $\varphi : X \to \mathbb{R}$ $\varphi(k) = 3$, $\varphi(l) = 2$, $\varphi(m) = 1$. Since $p(k, m) = p(m, k) = 1 \leq \varphi(k) - \varphi(m)$, therefore $k \preceq m$. But as $p(l, k) = p(k, l) = 2 \nleq |\varphi(k) - \varphi(l)|$ therefore $k \npreceq l$ and $l \npreceq k$. Again similarly $l \npreceq m$ and $m \npreceq l$ which show that this ordering is partial and hence $X$ is a partially ordered uniform space. Define $f : X \to X$ as $f(k) = k$, $f(l) = l$ and $f(m) = m$, then by a routine calculation one can verify that all the conditions of Corollary 3.7 are satisfied and $f$ has a fixed-point. Notice that $p(f(k), f(l)) = p(k, l)$ which shows that $f$ is neither $E$-contractive nor $E$ expansive, therefore the results of [2] are not applicable in the context of this example. Thus, this example demonstrates the utility of our result.

**Corollary 3.9.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an $E$-distance on $X$, $\varphi : X \to \mathbb{R}$ be a function which is bounded above and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space and $f : X \to X$ be a map. Suppose that*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$f$ is monotone increasing, that is, for $x \preceq y$ we have $f(x) \preceq f(y)$,*

(iii) *there exists an $x_0$ with $x_0 \succeq f(x_0)$.*

*Then $f$ has a fixed-point $x^*$. And the sequence*

$$x_{n-1} \succeq x_n = f(x_{n-1}), \quad n = 1,2,3,\ldots \tag{3.11}$$

*converges to $x^*$. Moreover if $\varphi$ is upper semicontinuous, then $x_n \succeq x^*$ for all $n$.*

**Theorem 3.10.** *Let $(X,\vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a continuous function bounded below and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $[x,+\infty) = \{y \in X : x \preceq y\}$. Suppose that*

(i) *$T$ satisfies the monotonic condition: for each $x \preceq y$ and each $u \in T(x)$ there exists $v \in T(y)$ such that $u \preceq v$,*

(ii) *$T(x)$ is compact for each $x \in X$,*

(iii) *$M = \{x \in X : T(x) \cap [x,+\infty) \neq \emptyset\} \neq \emptyset$.*

*Then $T$ has a fixed-point $x_0$.*

*Proof.* We will prove that $M$ has a maximum element. Let $\{x_v\}_{v\in\Lambda}$ be a totally ordered subset in $M$, where $\Lambda$ is a directed set. For $v,\mu \in \Lambda$ and $v \le \mu$, one has $x_v \preceq x_\mu$, which implies that $\varphi(x_v) \ge \varphi(x_\mu)$ for $v \le \mu$. Since $\varphi$ is bounded below, $\{\varphi(x_v)\}$ is a convergence net in $\mathbb{R}$. From $p(x_v,x_\mu) \le \varphi(x_v) - \varphi(x_\mu)$, we get that $\{x_v\}$ is a $p$-cauchy net in $X$. By the $p$-Cauchy completeness of $X$, let $x_v$ converge to $z$ in $X$.

For given $\mu \in \Lambda$
$p(x_\mu,z) = \lim_v p(x_\mu,x_v) \le \lim_v(\varphi(x_\mu) - \varphi(x_v)) = \varphi(x_\mu) - \varphi(x_z)$. So $x_\mu \preceq z$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu)$, there exists a $v_\mu \in T(z)$ such that $u_\mu \preceq v_\mu$. By the compactness of $T(z)$, there exists a convergence subnet $\{v_{\mu^|}\}$ of $\{v_\mu\}$. Suppose that $\{v_{\mu^|}\}$ converges to $w \in T(z)$. Take $\Lambda^|$ such that $\mu^| \ge \Lambda^|$ implies $u_\mu \preceq v_\mu \preceq v_{\mu^|}$.

We have

$$p(u_\mu,w) = \lim_{\mu^|} p\left(u_\mu, v_{\mu^|}\right) \le \lim_{\mu^|}\left(\varphi(u_\mu) - \varphi\left(v_{\mu^|}\right)\right) = \varphi(u_\mu) - \varphi(w). \tag{3.12}$$

So $u_\mu \preceq w$ for all $\mu$ and

$$p(z,w) = \lim_{\mu} p(u_\mu,w) \le \lim_{\mu}(\varphi(u_\mu) - \varphi(w)) = \varphi(z) - \varphi(w). \tag{3.13}$$

So $z \preceq w$ and this gives that $z \in M$. Hence we have proven that $\{x_\mu\}$ has an upper bound in $M$.

By Zorn's Lemma, there exists a maximum element $x_0$ in $M$. By the definition of $M$, there exists a $y_0 \in T(x_0)$ such that $x_0 \preceq y_0$. By the condition (i), there exists a $z_0 \in T(y_0)$ such that $y_0 \preceq z_0$. Hence $y_0 \in M$. Since $x_0$ is the maximum element in $M$, it follows that $y_0 = x_0$ and $x_0 \in T(x_0)$. So $x_0$ is a fixed-point of $T$. □

**Theorem 3.11.** *Let $(X,\vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a continuous function bounded above and "$\preceq$" the order introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \to 2^X$ be a multivalued mapping and $(-\infty,x] = \{y \in X : y \preceq x\}$. Suppose* *that*

(i) $T$ *satisfies the following condition; for each* $x \preceq y$ *and* $v \in T(x)$, *there exists* $u \in T(y)$ *such that* $u \preceq v$,

(ii) $T(x)$ *is compact for each* $x \in X$,

(iii) $M = \{x \in X : T(x) \cap (-\infty, x] \neq \emptyset\} \neq \emptyset$.

*Then* $T$ *has a fixed-point.*

**Corollary 3.12.** *Let* $(X, \vartheta)$ *be a Hausdorff uniform space,* $p$ *is an E-distance on* $X$, $\varphi : X \to \mathbb{R}$ *be a continuous function bounded below and "*$\preceq$*" the order introduced by* $\varphi$. *Let* $X$ *be also a* $p$*-Cauchy complete space and* $f : X \to X$ *be a map. Suppose that;*

(i) $f$ *is monotone increasing, that is for* $x \preceq y$, $f(x) \preceq f(y)$,

(ii) *there is an* $x_0 \in X$ *such that* $x_0 \preceq f(x_0)$.

*Then* $f$ *has a fixed-point.*

**Corollary 3.13.** *Let* $(X, \vartheta)$ *be a Hausdorff uniform space,* $p$ *is an E-distance on* $X$, $\varphi : X \to \mathbb{R}$ *be a continuous function bounded above and "*$\preceq$*" the order introduced by* $\varphi$. *Let* $X$ *be also a* $p$*-Cauchy complete space and* $f : X \to X$ *be a map. Suppose that;*

(i) $f$ *is monotone increasing, that is, for* $x \preceq y$, $f(x) \preceq f(y)$;

(ii) *there is an* $x_0 \in X$ *such that* $x_0 \succeq f(x_0)$.

*Then* $f$ *has a fixed-point.*

## 4. The Coupled Fixed-Point Theorems of Multivalued Mappings

*Definition 4.1.* An element $(x, y) \in X \times X$ is called a coupled fixed-point of the multivalued mapping $T : X \times X \to 2^X$ if $x \in T(x, y)$, $y \in T(y, x)$.

**Theorem 4.2.** *Let* $(X, \vartheta)$ *be a Hausdorff uniform space,* $p$ *is an E-distance on* $X$, $\varphi : X \to \mathbb{R}$ *be a function bounded below and "*$\preceq$*" be the order in* $X$ *introduced by* $\varphi$. *Let* $X$ *be also a* $p$*-Cauchy complete space,* $T : X \times X \to 2^X$ *be a multivalued mapping,* $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, *and* $M = \{(x, y) \in X \times X : x \preceq y,\ T(x, y) \cap [x, +\infty) \neq \emptyset$ *and* $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. *Suppose that:*

(i) $T$ *is upper semicontinuous, that is,* $x_n \in X$, $y_n \in X$ *and* $z_n \in T(x_n, y_n)$, *with* $x_n \to x_0$, $y_n \to y_0$ *and* $z_n \to z_0$ *implies* $z_0 \in T(x_0, y_0)$,

(ii) $M \neq \emptyset$,

(iii) *for each* $(x, y) \in M$, *there is* $(u, v) \in M$ *such that* $u \in T(x, y) \cap [x, +\infty)$ *and* $v \in T(y, x) \cap (-\infty, y]$.

*Then* $T$ *has a coupled fixed-point* $(x^*, y^*)$, *that is,* $x^* \in T(x^*, y^*)$ *and* $y^* \in T(y^*, x^*)$. *And there exist two sequences* $\{x_n\}$ *and* $\{y_n\}$ *with*

$$x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}), \quad y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}), \quad n = 1, 2, 3, \ldots \tag{4.1}$$

*such that* $x_n \to x^*$ *and* $y_n \to y^*$.

*Proof.* By the condition (ii), take $(x_0, y_0) \in M$. From (iii), there exist $(x_1, y_1) \in M$ such that $x_1 \in T(x_0, y_0)$, $x_0 \preceq x_1$ and $y_1 \in T(y_0, x_0)$, $y_1 \preceq y_0$. Again from (iii), there exist $(x_2, y_2) \in M$ such that $x_2 \in T(x_1, y_1)$, $x_1 \preceq x_2$ and $y_2 \in T(y_1, x_1)$, $y_2 \preceq y_1$.

Continuing this procedure we get two sequences $\{x_n\}$ and $\{y_n\}$ satisfying $(x_n, y_n) \in M$ and

$$x_{n-1} \preceq x_n \in T(x_{n-1}, y_{n-1}), \quad n = 1, 2, \ldots,$$
$$y_{n-1} \succeq y_n \in T(y_{n-1}, x_{n-1}), \quad n = 1, 2, \ldots. \tag{4.2}$$

So

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \preceq y_n \preceq \cdots \preceq y_2 \preceq y_1. \tag{4.3}$$

Hence,

$$\varphi(x_0) \geq \varphi(x_1) \geq \cdots \geq \varphi(x_n) \geq \cdots \geq \varphi(y_n) \geq \cdots \geq \varphi(y_1) \geq \varphi(y_0). \tag{4.4}$$

From this we get that $\varphi(x_n)$ and $\varphi(y_n)$ are convergent sequences. By the definition of "$\preceq$" as in the proof of Theorem 3.1, it is easy to prove that $\{x_n\}$ and $\{y_n\}$ are $p$-Cauchy sequences. Since X is $p$-Cauchy complete, let $\{x_n\}$ converge to $x^*$ and $\{y_n\}$ converge to $y^*$. Since $T$ is upper semicontinuous, $x^* \in T(x^*, y^*)$ and $y^* \in T(y^*, x^*)$. Hence $(x^*, y^*)$ is a coupled fixed-point of $T$. □

**Corollary 4.3.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on X, $\varphi : X \to \mathbb{R}$ be a function bounded below, and "$\preceq$" be the order in X introduced by $\varphi$. Let X be also a $p$-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *for each $(x, y) \in M$, $x \preceq f(x, y)$ and $f(y, x) \preceq y$.*

*Then $f$ has a coupled fixed-point $(x^*, y^*)$, that is, $x^* = f(x^*, y^*)$ and $y^* = f(y^*, x^*)$. And there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$ such that $x_n \to x^*$ and $y_n \to y^*$.*

**Corollary 4.4.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on X,$\varphi : X \to \mathbb{R}$ be a function bounded below, and "$\preceq$" be the order in X introduced by $\varphi$. Let X be also a $p$-Cauchy complete space, $f : X \times X \to X$ be a mapping and $M = \{(x, y) \in X \times X : x \preceq y$ and $x \preceq f(x, y)$ and $f(x, y) \preceq y\}$. Suppose that;*

(i) *$f$ is $\tau(\vartheta)$-continuous,*

(ii) *$M \neq \emptyset$,*

(iii) *$f$ is mixed monotone, that is for each $x_1 \preceq x_2$ and $y_1 \succeq y_2$, $f(x_1, y_1) \preceq f(x_2, y_2)$.*

*Then $f$ has a coupled fixed-point $(x^*, y^*)$. And there exist two sequences $\{x_n\}$ and $\{y_n\}$ with $x_{n-1} \preceq x_n = f(x_{n-1}, y_{n-1})$, $y_{n-1} \succeq y_n = f(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$ such that $x_n \to x^*$ and $y_n \to y^*$.*

**Theorem 4.5.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a continuous function, and "$\preceq$" be the order in $X$ introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space, $T : X \times X \to 2^X$ be a multivalued mapping, $[x, +\infty) = \{y \in X : x \preceq y\}$, $(-\infty, y] = \{x \in X : x \preceq y\}$, and $M = \{(x, y) \in X \times X : x \preceq y, T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Suppose that;*

(i) *$T$ is mixed monotone, that is, for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $u \in T(x_1, y_1)$, $v \in T(y_1, x_1)$, there exist $w \in T(x_2, y_2)$, $z \in T(y_2, x_2)$ such that $u \preceq w$, $v \succeq z$,*

(ii) *$M \neq \emptyset$,*

(iii) *$T(x, y)$ is compact for each $(x, y) \in X \times X$.*

*Then $T$ has a coupled fixed-point.*

*Proof.* By (ii), there exists $(x_0, y_0) \in M$ with $x_0 \preceq y_0$, $T(x_0, y_0) \cap [x_0, +\infty) \neq \emptyset$ and $T(y_0, x_0) \cap (-\infty, y_0] \neq \emptyset$. Let $C = \{(x, y) : x_0 \preceq x, y \preceq y_0, T(x, y) \cap [x, +\infty) \neq \emptyset$ and $T(y, x) \cap (-\infty, y] \neq \emptyset\}$. Then $(x_0, y_0) \in C$. Define the order relation "$\preceq$" in $C$ by

$$(x_1, y_1) \preceq (x_2, y_2) \Longleftrightarrow x_1 \preceq x_2, y_2 \preceq y_1. \tag{4.5}$$

It is easy to prove that $(C, \preceq)$ becomes an ordered space.

We will prove that $C$ has a maximum element. Let $\{x_v, y_v\}_{v \in \Lambda}$ be a totally ordered subset in $C$, where $\Lambda$ is a directed set. For $v, \mu \in \Lambda$ and $v \leq \mu$, one has $(x_v, y_v) \preceq (x_\mu, y_\mu)$. So $x_v \preceq x_\mu$ and $y_\mu \preceq y_v$, which implies that

$$\begin{aligned} &\varphi(x_0) \geq \varphi(x_v) \geq \varphi(x_\mu) \geq \varphi(y_0), \\ &\varphi(y_0) \leq \varphi(y_\mu) \leq \varphi(y_v) \leq \varphi(x_0) \end{aligned} \tag{4.6}$$

for $v \leq \mu$.

Since $\{\varphi(x_v)\}$ and $\{\varphi(y_v)\}$ are convergence nets in $\mathbb{R}$. From

$$p(x_v, x_\mu) \leq \varphi(x_v) - \varphi(x_\mu), \qquad p(y_\mu, y_v) \leq \varphi(y_\mu) - \varphi(y_v), \tag{4.7}$$

we get that $\{x_v\}$ and $\{y_v\}$ are $p$-Cauchy nets in $X$. By the $p$-Cauchy completeness of $X$, let $x_v$ convergence to $x^*$ and $y_v$ convergence to $y^*$ in $X$. For given $\mu \in \Lambda$,

$$\begin{aligned} p(x_\mu, x^*) &= \lim_v p(x_\mu, x_v) \leq \lim_v (\varphi(x_\mu) - \varphi(x_v)) = \varphi(x_\mu) - \varphi(x^*), \\ p(y_\mu, y^*) &= \lim_v p(y_\mu, y_v) \leq \lim_v (\varphi(y_v) - \varphi(y_\mu)) = \varphi(y_v) - \varphi(y^*). \end{aligned} \tag{4.8}$$

So $x_0 \preceq x_\mu \preceq x^*$ and $y_\mu \succeq y^* \succeq y_0$ for all $\mu \in \Lambda$.

For $\mu \in \Lambda$, by the condition (i), for each $u_\mu \in T(x_\mu, y_\mu)$ with $x_\mu \preceq u_\mu$ and $v_\mu \in T(y_\mu, x_\mu)$ with $v_\mu \preceq y_\mu$, there exist $w_\mu \in T(x^*, y^*)$ and $z_\mu \in T(y^*, x^*)$ such that $u_\mu \preceq w_\mu$ and $v_\mu \succeq z_\mu$. By the compactness of $T(x^*, y^*)$ and $T(y^*, x^*)$, there exist convergence subnets $\{w_{\mu^l}\}$ of $\{w_\mu\}$

and $\{z_{\mu^|}\}$ of $\{z_\mu\}$. Suppose that $\{w_{\mu^|}\}$ converges to $w \in T(x^*, y^*)$ and $\{z_{\mu^|}\}$ converges to $z \in T(y^*, x^*)$. Take $\Lambda^|$, such that $\mu^| \geq \Lambda^|$ implies $u_\mu \preceq v_\mu \preceq v_{\mu^|}$. We have

$$\begin{aligned} p(u_\mu, w) &= \lim_{\mu^|} p\left(u_\mu, u_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi(u_\mu) - \varphi\left(u_{\mu^|}\right)\right) = \varphi(u_\mu) - \varphi(w), \\ p(z, v_\mu) &= \lim_{\mu^|} p\left(v_{\mu^|}, v_\mu\right) \leq \lim_{\mu^|}\left(\varphi\left(v_{\mu^|}\right) - \varphi(v_\mu)\right) = \varphi(z) - \varphi(v_\mu). \end{aligned} \tag{4.9}$$

So $x_\mu \preceq u_\mu \preceq w$ and $z \preceq v_\mu \preceq y_\mu$ for all $\mu$. And

$$\begin{aligned} p(x^*, w) &= \lim_{\mu^|} p\left(x_{\mu^|}, u_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi\left(x_{\mu^|}\right) - \varphi\left(u_{\mu^|}\right)\right) = \varphi(x^*) - \varphi(w), \\ p(z, y^*) &= \lim_{\mu^|} p\left(v_{\mu^|}, y_{\mu^|}\right) \leq \lim_{\mu^|}\left(\varphi\left(v_{\mu^|}\right) - \varphi\left(y_{\mu^|}\right)\right) = \varphi(z) - \varphi(y^*). \end{aligned} \tag{4.10}$$

So $x^* \preceq w$ and $z \preceq y^*$, this gives that $(x^*, y^*) \in C$. Hence we have proven that $\{x_\mu, y_\mu\}_{\mu\in\Lambda}$ has an upper bound in $C$.

By Zorn's lemma, there exists a maximum element $(\overline{x}, \overline{y})$ in $C$. By the definition of $C$, there exist $\overline{u} \in T(\overline{x}, \overline{y})$, $\overline{v} \in T(\overline{y}, \overline{x})$, such that $x_0 \preceq \overline{u}, \overline{v} \preceq y_0$ and $\overline{x} \preceq \overline{u}, \overline{v} \preceq \overline{y}$. By the condition (i) there exist $\overline{w} \in T(\overline{u}, \overline{v})$, $\overline{z} \in T(\overline{v}, \overline{u})$ such that $x_0 \preceq \overline{u} \preceq \overline{w}$ and $\overline{z} \preceq \overline{v} \preceq y_0$. Hence $(\overline{u}, \overline{v}) \in C$ and $(\overline{x}, \overline{y}) \preceq (\overline{u}, \overline{v})$. Since $(\overline{x}, \overline{y})$ is maximum element in $C$, it follows that $(\overline{x}, \overline{y}) = (\overline{u}, \overline{v})$, and it follows that $\overline{x} = \overline{u} \in T(\overline{x}, \overline{u})$ and $\overline{y} = \overline{v} \in T(\overline{y}, \overline{x})$. So $(\overline{x}, \overline{y})$ is a coupled fixed-point of $T$. □

**Corollary 4.6.** *Let $(X, \vartheta)$ be a Hausdorff uniform space, $p$ is an E-distance on $X$, $\varphi : X \to \mathbb{R}$ be a continuous function, and "$\preceq$" be the order in $X$ introduced by $\varphi$. Let $X$ be also a $p$-Cauchy complete space and $f : X \times X \to X$ be a mapping. Suppose that;*

(i) *$f$ is mixed monotone, that is for $x_1 \preceq y_1$, $x_2 \succeq y_2$ and $f(x_1, y_1) \preceq f(y_2, x_2)$,*

(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$ and $f(y_0, x_0) \preceq y_0$.*

*Then $f$ has a coupled fixed-point.*

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