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Research Article

Generalized Lefschetz Sets

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We generalize and modify Lefschetz sets defined in 1976 by L. Górniewicz, which leads to more general results in fixed point theory.

1. Introduction

In 1976 L. Górniewicz introduced a notion of a Lefschetz set for multivalued admissible maps. The paper attempts at showing that Lefschetz sets can be defined on a broader class of multivalued maps than admissible maps. This definition can be presented in many ways, and each time it is the generalization of the definition from 1976. These generalizations essentially broaden the class of admissible maps that have a fixed point. Also, they are a homologic tool for examining fixed points for a class of multivalued maps broader than just admissible maps.

2. Preliminaries

Throughout this paper all topological spaces are assumed to be metric. Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$ (see [1, 2]). A space X is acyclic if

(i) X is nonempty,

(ii) $H_q(X) = 0$ for every $q \geq 1$,

(iii) $H_0(X) \approx \mathbb{Q}$.

A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. A proper map $p : X \to Y$ is called Vietoris provided that for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. Let X and Y be two spaces, and assume that for every $x \in X$ a nonempty subset $\varphi(x)$ of Y is given. In such a case we say that $\varphi : X \multimap Y$ is a multivalued mapping. For a multivalued mapping $\varphi : X \multimap Y$ and a subset $U \subset Y$, we let:

$$\varphi^{-1}(U) = \{x \in X;\ \varphi(x) \subset U\}. \tag{2.1}$$

If for every open $U \subset Y$ the set $\varphi^{-1}(U)$ is open, then φ is called an upper semicontinuous mapping; we will write that φ is u.s.c.

Proposition 2.1 (see [1, 2]). *Assume that $\varphi : X \multimap Y$ and $\psi : Y \multimap T$ are u.s.c. mappings with compact values and $p : Z \to X$ is a Vietoris mapping. Then*

(2.1.1) *for any compact $A \subset X$, the image $\varphi(A) = \bigcup_{x\in A} \varphi(x)$ of the set A under φ is a compact set;*

(2.1.2) *the composition $\psi \circ \varphi : X \multimap T$, $(\psi \circ \varphi)(x) = \bigcup_{y\in\varphi(x)} \psi(y)$, is an u.s.c. mapping;*

(2.1.3) *the mapping $\varphi_p : X \multimap Z$, given by the formula $\varphi_p(x) = p^{-1}(x)$, is u.s.c..*

Let $\varphi : X \multimap Y$ be a multivalued map. A pair (p, q) of single-valued, continuous maps is called a selected pair of φ (written $(p, q) \subset \varphi$) if the following two conditions are satisfied:

(i) p is a Vietoris map,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$ for any $x \in X$.

Definition 2.2. A multivalued mapping $\varphi : X \multimap Y$ is called admissible provided that there exists a selected pair (p, q) of φ.

Proposition 2.3 (see [2]). *Let $\varphi : X \multimap Y$ and $\psi : Y \multimap Z$ be two admissible maps. Then the composition $\psi \circ \varphi : X \multimap Z$ is an admissible map.*

Proposition 2.4 (see [2]). *Let $\varphi : X \multimap Y$ and $\psi : Z \multimap T$ be admissible maps. Then the map $\varphi \times \psi : X \times Z \multimap Y \times T$ is admissible.*

Proposition 2.5 (see [2]). *If $\varphi : X \multimap Y$ is an admissible map, $Y_0 \subset Y$, and $X_0 = \varphi^{-1}(Y_0)$, then the contraction $\varphi_0 : X_0 \multimap Y_0$ of φ to the pair (X_0, Y_0) is an admissible map.*

Proposition 2.6 (see [1]). *If $p : X \to Y$ is a Vietoris map, then an induced mapping*

$$p_* : H_*(X) \longrightarrow H_*(Y) \tag{2.2}$$

is a linear isomorphism.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \text{ for some } n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided that $\dim \widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

(i) all u_q are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial.

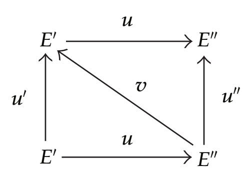
For such a u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q \operatorname{tr}(\widetilde{u_q}), \tag{2.3}$$

where $\operatorname{tr}(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (cf. [1]). The following important property of a Leray endomorphism is a consequence of a well-known formula $\operatorname{tr}(u \circ v) = \operatorname{tr}(v \circ u)$ for the ordinary trace. An endomorphism $u : E \to E$ of a graded vector space E is called weakly nilpotent if for every $q \geq 0$ and for every $x \in E_q$, there exists an integer n such that $u_q^n(x) = 0$. Since for a weakly nilpotent endomorphism $u : E \to E$ we have $N(u) = E$, we get the following.

Proposition 2.7. *If $u : E \to E$ is a weakly nilpotent endomorphism, then $\Lambda(u) = 0$.*

Proposition 2.8. *Assume that in the category of graded vector spaces the following diagram commutes*



(2.4)

If one of u',u'' is a Leray endomorphism, then so is the other and $\Lambda(u') = \Lambda(u'')$.

Let $\varphi : X \multimap X$, be an admissible map. Let $(p, q) \subset \varphi$, where $p : Z \to X$ is a Vietoris mapping and $q : Z \to X$ a continuous map. Assume that $q_* \circ p_*^{-1} : H_*(X) \to H_*(X)$ is a Leray endomorphism for all pairs $(p, q) \subset \varphi$. For such a φ, we define the Lefschetz set $\Lambda(\varphi)$ of φ by putting

$$\Lambda(\varphi) = \left\{\Lambda\left(q_* p_*^{-1}\right); (p, q) \subset \varphi\right\}. \tag{2.5}$$

Let $X_0 \subset X$ and let $\varphi : (X, X_0) \multimap (X, X_0)$ be an admissible map. We define two admissible maps $\varphi_X : X \multimap X$ given by $\varphi_X(x) = \varphi(x)$ for all $x \in X$ and $\varphi_{X_0} : X_0 \multimap X_0$ $\varphi_{X_0}(x) = \varphi(x)$ for all $x \in X_0$. Let $(p, q) \subset \varphi_X$, where $p : Z \to X$ is a Vietoris mapping and $q : Z \to X$ a continuous map. We shall denote by $\widetilde{p} : (Z, p^{-1}(X_0)) \to (X, X_0)$ $\widetilde{p}(z) = p(z)$, $\widetilde{q} : (Z, p^{-1}(X_0)) \to (X, X_0)$ $\widetilde{q}(z) = q(z)$ for all $z \in Z$, $\overline{p} : p^{-1}(X_0) \to X_0$ $\overline{p}(z) = p(z)$, and $\overline{q} : p^{-1}(X_0) \to X_0$ $\overline{q}(z) = q(z)$ for all $z \in p^{-1}(X_0)$. We observe that $(\widetilde{p}, \widetilde{q}) \subset \varphi$ and $(\overline{p}, \overline{q}) \subset \varphi_{X_0}$.

Proposition 2.9 (see [2]). *Let $\varphi : (X, X_0) \multimap (X, X_0)$ be an admissible map of pairs and $(p, q) \subset \varphi_X$. If any two of the endomorphisms $\widetilde{q}_* \widetilde{p}_*^{-1} : H(X, X_0) \to H(X, X_0)$, $q_* p_*^{-1} : H(X) \to H(X)$, $\overline{q}_* \overline{p}_*^{-1} : H(X_0) \to H(X_0)$ are Leray endomorphisms, then so is the third and*

$$\Lambda\left(\widetilde{q}_* \widetilde{p}_*^{-1}\right) = \Lambda\left(q_* p_*^{-1}\right) - \Lambda\left(\overline{q}_* \overline{p}_*^{-1}\right). \tag{2.6}$$

Proposition 2.10 (see [2]). *If $\varphi : X \multimap Y$ and $\psi : Y \multimap T$ are admissible, then the composition $\psi \circ \varphi : X \multimap T$ is admissible, and for every $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$ there exists a pair $(p, q) \subset \psi \circ \varphi$ such that $q_{2*} p_{2*}^{-1} \circ q_{1*} p_{1*}^{-1} = q_* p_*^{-1}$.*

Definition 2.11. An admissible map $\varphi : X \multimap X$ is called a Lefschetz map provided that the Lefschetz set $\Lambda(\varphi)$ of φ is well defined and $\Lambda(\varphi) \neq \{0\}$ implies that the set $\mathrm{Fix}(\varphi) = \{x \in X : x \in \varphi(x)\}$ is nonempty.

Definition 2.12. Let E be a topological vector space. One shall say that E is a Klee admissible space provided that for any compact subset $K \subset E$ and for any open cover $\alpha \in \mathrm{Cov}_E(K)$ there exists a map

$$\pi_\alpha : K \longrightarrow E \tag{2.7}$$

such that the following two conditions are satisfied:

(2.12.1) for each $x \in K$ there exists $V \in \alpha$ such that $x, \pi_\alpha(x) \in V$,

(2.12.2) there exists a natural number $n = n_K$ such that $\pi_\alpha(K) \subset E^n$, where E^n is an n-dimensional subspace of E.

Definition 2.13. One shall say that E is locally convex provided that for each $x \in E$ and for each open set $U \subset E$ such that $x \in U$ there exists an open and convex set $V \subset E$ such that $x \in V \subset U$.

It is clear that if E is a normed space, then E is locally convex.

Proposition 2.14 (see [1, 2]). *Let E be locally convex. Then E is a Klee admissible space.*

Let Y be a metric space, and let $Id_Y : Y \to Y$ be a map given by formula $Id_Y(y) = y$ for each $y \in Y$.

Definition 2.15 (see [3]). A map $r : X \to Y$ of a space X onto a space Y is said to be an mr-map if there is an admissible map $\varphi : Y \multimap X$ such that $r \circ \varphi = Id_Y$.

Definition 2.16 (see [3, 4]). A metric space X is called an absolute multiretract (notation: $X \in$ AMR) provided there exists a locally convex space E and an mr-map $r : E \to X$ from E onto X.

Definition 2.17 (see [3, 4]). A metric space X is called an absolute neighborhood multiretract (notation: $X \in$ ANMR) provided that there exists an open subset U of some locally convex space E and an mr-map $r : U \to X$ from U onto X.

Proposition 2.18 (see [3, 4]). *A space X is an ANMR if and only if there exists a metric space Z and a Vietoris map $p : Z \to X$ which factors through an open subset U of some locally convex E, that is, there are two continuous maps α and β such that the following diagram is commutative.*

$$\begin{array}{ccc} Z & \xrightarrow{p} & X \\ & \searrow^{\alpha} & \uparrow{\scriptstyle \beta} \\ & & U \end{array} \tag{2.8}$$

Proposition 2.19 (see [3]). *Let $X \in$ ANMR, and let $V \subset X$ be an open set. Then $V \in$ ANMR.*

Proposition 2.20 (see [3]). *Assume that X is ANMR. Let U be an open subset in X and $\varphi : U \multimap U$ an admissible and compact map, then φ is a Lefschetz map.*

Let $\varphi_X : X \multimap X$ be a map. Then

$$\varphi_X^n = \begin{cases} Id_X, & \text{for } n = 0, \\ \varphi_X, & \text{for } n = 1, \\ \varphi_X \circ \varphi_X \circ \cdots \circ \varphi_X (n\text{-iterates}) & \text{for } n > 1. \end{cases} \tag{2.9}$$

We denote multivalued maps with $\varphi_{XY} : X \multimap Y$, and $\varphi_Z : Z \multimap Z$. If a nonempty set $A \subset X$, a nonempty set $B \subset Y$ and $\varphi_{XY}(A) \subset B$ then a multivalued map $\varphi_{AB} : A \multimap B$ given by $\varphi_{AB}(x) = \varphi_{XY}(x)$ for each $x \in X$.

Definition 2.21 (see [5]). A multivalued map $\varphi_{XY} : X \multimap Y$ is called locally admissible provided for any compact and nonempty set $K \subset X$ there exists an open set $V \subset X$ such that $K \subset V$ and $\varphi_{VX} : V \multimap X$ is admissible.

Proposition 2.22 (see [5]). *Let $\varphi_{XY} : X \multimap Y$ and $\psi_{YZ} : Y \multimap Z$ be locally admissible maps. Then the map $\Phi_{XZ} = (\psi_{YZ} \circ \varphi_{XY}) : X \multimap Z$ is locally admissible.*

Proposition 2.23 (see [5]). *Let $A \subset X$ be a nonempty set, and let $\varphi_{XY} : X \multimap Y$ be a locally admissible map. Then a map $\varphi_{AY} : A \multimap Y$ is locally admissible.*

Definition 2.24 (see [2, 5]). A multivalued map $\varphi_X : X \multimap X$ is called a compact absorbing contraction (written $\varphi_X \in$ CAC(X)) provided there exists an open set $U \subset X$ such that

(2.24.1) $\varphi_X(U) \subset U$ and the $\varphi_U : U \multimap U$, $\varphi_U(x) = \varphi_X(x)$ for every $x \in X$ is compact $(\overline{\varphi_X(U)} \subset U)$,

(2.24.2) for every $x \in X$ there exists $n = n_x$ such that $\varphi_X^n(x) \subset U$.

Proposition 2.25 (see [3]). *Let $\varphi_X : X \multimap X$ be an admissible map, $X \in$ ANMR, and $\varphi_X \in$ CAC(X) then φ_X is a Lefschetz map.*

Proposition 2.26 (see [5]). *Let $\varphi_X \in \mathrm{CAC(X)}$, and let $U \subset X$ be an open set from Definition 2.24.*

(2.26.1) *Let B be a nonempty set in X and $\varphi_X(B) \subset B$. Then $U \cap B \neq \emptyset$.*

(2.26.2) *For any $n \in \mathbb{N}$ $\varphi_X^n \in \mathrm{CAC(X)}$.*

(2.26.3) *Let $V \subset X$ be a nonempty and open set. Assume that $\overline{\varphi_X(V)} \subset V$. Then $\varphi_V \in \mathrm{CAC(V)}$.*

3. Main result

Let X be a metric space, $\varphi_X : X \multimap X$ a multivalued map, and let

$$\Omega_{\mathrm{AD}}(\varphi) = \left\{ V \subset X : \ V \text{ is open}, \ \varphi_V : V \multimap V \text{ is admissible}, \ \overline{\varphi_V(V)} \subset V \right\}. \tag{3.1}$$

Obviously the above family of sets can be empty. Thus we can define the following class of multivalued maps:

$$\mathrm{ADL} = \{ \varphi_X : X \multimap X, \ \Omega_{\mathrm{AD}}(\varphi) \neq \emptyset \}. \tag{3.2}$$

All the admissible maps $\varphi_X : X \multimap X$ particularly belong to the above class of maps because $X \in \Omega_{\mathrm{AD}}(\varphi)$. We shall remind that the multivalued map $\varphi_X : X \multimap X$ is called acyclic if for every $x \in X$ the set $\varphi_X(x)$ is nonempty, acyclic, and compact. It is known from the mathematical literature that an acyclic map is admissible and the maps

$$r, s : \Gamma \to X \quad \text{given by } r(x,y) = x, \qquad s(x,y) = y \quad \text{for every } (x,y) \in \Gamma, \tag{3.3}$$

where $\Gamma = \{(x,y) \in X \times Y ; y \in \varphi_X(x)\}$, are a selective pair $(r,s) \subset \varphi_X$.

Moreover, for an acyclic map $\varphi_X : X \multimap X$, if the homomorphism $s_* r_*^{-1} : H_*(X) \to H_*(X)$ is a Leray endomorphism, then Lefschetz set $\Lambda(\varphi_X)$ consists of only one element and

$$\Lambda(\varphi_X) = \left\{ \Lambda\left(s_* r_*^{-1} \right) \right\}. \tag{3.4}$$

For a certain class of multivalued maps $\varphi_X \in \mathrm{ADL}$ we define a generalized Lefschetz set $\Lambda^G(\varphi_X)$ of a map φ_X in such a way that the conditions of the following definition are satisfied.

Let $\varphi_V : V \to V$ be an admissible map. One shall say that a set $\Lambda(\varphi_V)$ is well defined if for every $(p,q) \subset \varphi_V$ the map $q_* p_*^{-1} : H_*(V) \to H_*(V)$ is a Leray endomorphism.

Definition 3.1. Assume that there exists a nonempty family of sets $\Upsilon_{\mathrm{AD}}(\varphi) \subset \Omega_{\mathrm{AD}}(\varphi)$ such that if for any $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined, then the following conditions are satisfied:

(3.1.1) if $\varphi_X : X \multimap X$ is acyclic, then $\Lambda^G(\varphi_X) = \{\Lambda(s_* r_*^{-1})\}$ (see (3.3)),

(3.1.2) if $\varphi_X : X \multimap X$ is admissible, then $X \in \Upsilon_{\mathrm{AD}}(\varphi)$ and

$$(\Lambda(\varphi_X) \neq \{0\}) \Longrightarrow \left(\Lambda^G(\varphi_X) \neq \{0\} \right), \tag{3.5}$$

(3.1.3) $(\Lambda^G(\varphi_X) \neq \{0\}) \Rightarrow$ (there exists $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ such that $\Lambda(\varphi_V) \neq \{0\}$).

From the above definition it in particular results that (see (3.1.1)) if $f : X \to X$ is a single-valued map, continuous and $\Lambda(f)$ is well defined, then

$$\Lambda^G(f) = \Lambda(f). \tag{3.6}$$

We shall present a few examples proving that Lefschetz sets can be defined in many ways while retaining the conditions contained in Definition 3.1.

Example 3.2. Let $\varphi_X : X \multimap X$ be an admissible map, and let $\Upsilon_{AD}(\varphi) = \{X\}$. If $\Lambda(\varphi_X)$ is well defined, then we define

$$\Lambda^G(\varphi_X) = \Lambda(\varphi_X). \tag{3.7}$$

The above example consists of Lefschetz set definitions common in mathematical literature and introduced by L. Górniewicz.

Example 3.3. Let $\varphi_X : X \multimap X$ be an admissible map, and let $\Upsilon_{AD}(\varphi)$ be a family of sets $V \in \Omega_{AD}(\varphi)$ such that there exists $(\overline{p},\overline{q}) \subset \varphi_V$ and there exists $(p,q) \subset \varphi_X$ such that the following diagram

$$\begin{array}{ccc} H_*(V) & \xrightarrow{u_*} & H_*(X) \\ {\scriptstyle \overline{q}_*(\overline{p}_*)^{-1}} \uparrow & \nwarrow {\scriptstyle v_*} & \uparrow {\scriptstyle q_* p_*^{-1}} \\ H_*(V) & \xrightarrow{u_*} & H_*(X) \end{array} \tag{3.8}$$

is commutative. It is obvious that $X \in \Upsilon_{AD}(\varphi)$, hence $\Upsilon_{AD}(\varphi) \neq \emptyset$. Assume that for any $V \in \Upsilon_{AD}(\varphi)$ $\Lambda(\varphi_V)$ is well defined. We define

$$\Lambda^G(\varphi_X) = \bigcup_{V \in \Upsilon_{AD}(\varphi)} \Lambda(\varphi_V). \tag{3.9}$$

Justification 1

Let us notice that if φ_X is acyclic, then from the commutativity of the above diagram it results that for every $V \in \Upsilon_{AD}(\varphi)$ $\Lambda(\varphi_V) = \{\Lambda(s_* r_*^{-1}))\}$, hence $\Lambda^G(\varphi_V) = \{\Lambda(s_* r_*^{-1}))\}$. The second and third conditions of Definition 3.1 are obvious.

Let $A \subset X$ be a nonempty set, and let

$$O_\varepsilon(A) = \{x \in X; \text{there exists } y \in A \text{ such that } d(x,y) < \varepsilon\}, \tag{3.10}$$

where d is metric in X.

Example 3.4. Let (X, d) be a metric space, where d is a metric such that, for each $(x, y) \in X \times X$ $d(x, y) \leq 1$, let $\varphi_X : X \multimap X$ be a multivalued map and let $K = \overline{\varphi_X(X)}$. Let

$$\Upsilon_{\mathrm{AD}}(\varphi) = \{V \in \Omega_{\mathrm{AD}}(\varphi) : V = O_{2/n}(K) \text{ for some } n\}. \tag{3.11}$$

Assume that $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$ and for all $V \in \Omega_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined. We define

$$\begin{gathered} \Lambda^G(\varphi_X) = \Lambda(\varphi_U), \quad \text{where} \\ U = O_{2/k}(K), \quad k = \min\{n \in N; O_{2/n}(K) \in \Upsilon_{\mathrm{AD}}(\varphi)\}. \end{gathered} \tag{3.12}$$

Justification 2

The first condition of Definition 3.1 results from the commutativity of the following diagram:

$$\begin{array}{ccc} H_*(U) & \xrightarrow{u_*} & H_*(X) \\ \uparrow {\scriptstyle \overline{q}_*(\overline{p}_*)^{-1}} & \nwarrow^{v_*} & \uparrow {\scriptstyle q_* p_*^{-1}} \\ H_*(U) & \xrightarrow{u_*} & H_*(X), \end{array} \tag{3.13}$$

where $u_* = i_*$ is a homomorphism determined by the inclusions $i : U \to X$, $v_* = \overline{q}_* p_*^{-1}$.

The maps $\overline{p}, \overline{q}$ are the respective contractions of maps p, q, $(p, q) \subset \varphi_X$. Condition (3.1.2) results from the fact that $X = O_2(K) \in \Upsilon_{\mathrm{AD}}(\varphi)$ and

$$\Lambda^G(\varphi_X) = \Lambda(\varphi_{O_2(K)}) = \Lambda(\varphi_X). \tag{3.14}$$

Satisfying Condition (3.1.3) is obvious.

Before the formulation of another example, let us introduce the following definition and provide necessary theorems.

Definition 3.5. Let $\varphi_X : X \multimap X$ be a map. One shall say that a nonempty set $B \subset X$ has an absorbing property (writes $B \in \mathrm{AP}(\varphi)$) if for each $x \in X$ there exists a natural number n such that $\varphi_X^n(x) \subset B$.

Let $\Theta_{\mathrm{AD}}(\varphi) = \Omega_{\mathrm{AD}}(\varphi) \cap \mathrm{AP}(\varphi)$. We observe that if $\varphi_X : X \multimap X$ is admissible then $\Theta_{\mathrm{AD}}(\varphi) \neq \emptyset$ since $X \in \Theta_{\mathrm{AD}}(\varphi)$.

Theorem 3.6 (see [2]). *Let $\varphi_X : X \multimap X$ be an admissible map. Then for any $V \in \Theta_{AD}(\varphi)$ and for all $(p, q) \subset \varphi_X$ the homomorphism*

$$\widetilde{q}_* \widetilde{p}_*^{-1} : H_*(X, V) \longrightarrow H_*(X, V) \tag{3.15}$$

is weakly nilpotent (see Proposition 2.9), where $\widetilde{p}, \widetilde{q}$ denote a respective contraction of p, q.

Theorem 3.7. *Let $\varphi_X : X \multimap X$ be an admissible map. Assume that for each $V \in \Theta_{AD}(\varphi)$ $\Lambda(\varphi_V)$ is well defined. Then*

$$\Lambda(\varphi_X) = \bigcap_{V \in \Theta_{AD}(\varphi)} \Lambda(\varphi_V). \tag{3.16}$$

Proof. Let $V \in \Theta_{\mathrm{AD}}(\varphi)$, $(p,q) \subset \varphi_X$, and let $\Lambda(q_* p_*^{-1}) = c_0$. We observe that a map $\tilde{q}_* \tilde{p}_*^{-1} : H_*(X,V) \multimap H_*(X,V)$ $((\tilde{p},\tilde{q}) \subset \varphi,\ \varphi : (X,V) \multimap (X,V))$ is weakly nilpotent so from Propositions 2.7 and 2.9 $\Lambda(q_* p_*^{-1}) = \Lambda(\overline{q}_* \overline{p}_*^{-1}) = c_0$, where $(\overline{p},\overline{q}) \subset \varphi_V$ and $\overline{p},\overline{q}$ denote a respective contraction of p,q. Hence $c_0 \in \Lambda(\varphi_V)$ and $\Lambda(\varphi_X) \subset \bigcap_{V \in \Theta_{\mathrm{AD}}(\varphi)} \Lambda(\varphi_V)$. It is clear that $\mathrm{X} \in \Theta_{\mathrm{AD}}(\varphi)$ and the proof is complete. □

Example 3.8. Let $\varphi_X : X \multimap X$ be a multivalued map, and let

$$\Upsilon_{\mathrm{AD}}(\varphi) = \Theta_{\mathrm{AD}}(\varphi). \tag{3.17}$$

Assume that the following conditions are satisfied:

(3.8.1) $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$,

(3.8.2) for all $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined,

(3.8.3) $\bigcap_{V \in \Upsilon_{\mathrm{AD}}(\varphi)} \Lambda(\varphi_V) \neq \emptyset$.

We define

$$\Lambda^G(\varphi_X) = \bigcap_{V \in \Upsilon_{\mathrm{AD}}(\varphi)} \Lambda(\varphi_V). \tag{3.18}$$

Justification 3

Condition (3.1.1) results from Proposition 2.7 and Theorem 3.6. Let us notice that if a map $\varphi_X : X \multimap X$ is admissible, then $X \in \Upsilon_{\mathrm{AD}}(\varphi)$ and from Theorem 3.7 we get

$$\Lambda^G(\varphi_X) = \Lambda(\varphi_X), \tag{3.19}$$

and condition (3.1.2) is satisfied. Condition (3.1.3) is obvious.

It is crucial to notice that the definition of Lefschetz set encompassed in this example agrees in the class of admissible maps with the familiar definition of a Lefschetz set introduced by L. Górniewicz. It is possible to create an example (see [5]) of a multivalued map $\varphi_X : X \multimap X$ that is not admissible and satisfies the conditions of Example 3.8.

Example 3.9. Let $\varphi_X : X \multimap X$ be a multivalued map, and let

$$\Upsilon_{\mathrm{AD}}(\varphi) = \Theta_{\mathrm{AD}}(\varphi). \tag{3.20}$$

Assume that the following conditions are satisfied:

(3.9.1) $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$,

(3.9.2) for all $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined.

We define

$$\Lambda^G(\varphi_X) = \bigcup_{V \in \Upsilon_{\mathrm{AD}}(\varphi)} \Lambda(\varphi_V). \tag{3.21}$$

Justification 4

Condition (3.1.1) results from Proposition 2.7 and Theorem 3.6. If a map $\varphi_X : X \multimap X$ is admissible, then $X \in \Upsilon_{\mathrm{AD}}(\varphi)$ and hence condition (3.1.2) is satisfied. Condition (3.1.3) is obvious.

The definition of a Lefschetz set in Example 3.9 is much more general than the definition in Example 3.8, and as consequence it encompasses a broader class of maps. This definition ignores the inconvenient assumption (3.8.3).

Let us define a Lefschetz map by the application of the new Lefschetz set definition.

Definition 3.10. One shall say that a map $\varphi_X \in \mathrm{ADL}$ is a general Lefschetz map provided that the following conditions are satisfied:

(3.10.1) there exists $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$ such that conditions (3.1.1)–(3.1.3) are satisfied,

(3.10.2) for any $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined.

We will formulate, and then prove, a very general fixed point theorem.

Theorem 3.11. *Let* $X \in \mathrm{ANMR}$. *Assume that the following conditions are satisfied:*

(3.11.1) $\varphi_X \in \mathrm{CAC(X)}$ *(see Definiation 2.24),*

(3.11.2) *there exists* $\Upsilon_{AD}(\varphi) \neq \emptyset$ *such that conditions (3.1.1)–(3.1.3) are satisfied.*

Then φ_X *is a general Lefschetz map, and if* $\Lambda^G(\varphi_X) \neq \{0\}$ *then* $\mathrm{Fix}(\varphi_X) \neq \emptyset$.

Proof. From the assumption $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$, thus we show that for all $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ $\Lambda(\varphi_V)$ is well defined. Let $V \in \Upsilon_{\mathrm{AD}}(\varphi)$, then from (2.26.3) $\varphi_V \in \mathrm{CAC(V)}$, so from Propositions 2.19 and 2.25 $\Lambda(\varphi_V)$ is well defined. Assume that $\Lambda^G(\varphi_X) \neq \{0\}$, then from (3.1.3) there exists $V' \in \Upsilon_{\mathrm{AD}}(\varphi)$ such that $\Lambda(\varphi_{V'}) \neq \{0\}$. By the application of (2.26.3), Propositions 2.19, and 2.25, we get $\emptyset \neq \mathrm{Fix}(\varphi_{V'}) \subset \mathrm{Fix}(\varphi_X)$ and the proof is complete. □

The following is a conclusion from Theorem 3.11.

Corollary 3.12. *Let* $X \in \mathrm{ANMR}$, $\varphi_X : X \multimap X$ *be locally admissible (not necessarily admissible), and let* $\varphi_X \in \mathrm{CAC(X)}$. *Then* φ_X *is a general Lefschetz map, and if* $\Lambda^G(\varphi_X) \neq \{0\}$ *then* $\mathrm{Fix}(\varphi_X) \neq \emptyset$.

Proof. Let $U \subset X$ be an open set from Definition 2.24, and let $K = \overline{\varphi_U(U)} \subset U$. We define $\Upsilon_{\mathrm{AD}}(\varphi) = \Theta_{\mathrm{AD}}(\varphi)$ (see Examples 3.8 and 3.9). The map φ_X is locally admissible, so there exists an open set $V \subset X$ such that $K \subset V$ and $\varphi_{VX} : V \multimap X$ is admissible. We observe that $U \cap V \in \Upsilon_{\mathrm{AD}}(\varphi)$ since $\varphi_{U \cap V} : U \cap V \multimap U \cap V$ is admissible, compact and $(U \cap V) \in \mathrm{AP}(\varphi)$, hence $\Upsilon_{\mathrm{AD}}(\varphi) \neq \emptyset$. If we define a generalized Lefschetz set now as in Example 3.9, then from Theorem 3.11 we get a thesis. □

Finally we shall provide an example which shows that the new Lefschetz set definition is more general than the definition of Lefschetz set for admissible maps already familiar in mathematical literature.

Example 3.13 (see [5]). Let $\mathbb{C}$ be a complex number set, and let $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ be single-valued continuous and compact map. Assume that $\operatorname{Fix}(f) = \emptyset$, and choose an open set V such that $\overline{f(\mathbb{C} \setminus \{0\})} \subset V \subset \mathbb{C} \setminus \{0\}$. Let $g : V \to V$ be a compact $(\overline{g(V)} \subset V)$ and continuous map such that $\Lambda(g) \neq 0$. We define a multivalued map $\varphi_{\mathbb{C}\setminus\{0\}} : \mathbb{C} \setminus \{0\} \multimap \mathbb{C} \setminus \{0\}$ given by formula

$$\varphi_{\mathbb{C}\setminus\{0\}}(z) = \begin{cases} f(z), & \text{for z} \notin \text{V}, \\ \{f(z), g(z)\} & \text{for z} \in \text{V}. \end{cases} \tag{3.22}$$

The map $\varphi_{\mathbb{C}\setminus\{0\}}$ is admissible, so $\Upsilon_{\mathrm{AD}}(\varphi) = \Theta_{\mathrm{AD}}(\varphi) \neq \emptyset$ (see Examples 3.8 and 3.9). Let

$$\Lambda^G(\varphi) = \bigcup_{U \in \Upsilon_{\mathrm{AD}}(\varphi)} \Lambda(\varphi_U). \tag{3.23}$$

(see Example 3.9). We observe that

$$\Lambda(\varphi_{\mathbb{C}\setminus\{0\}}) = \{0\} \tag{3.24}$$

since the only selective pair is the pair $(Id_{C\setminus\{0\}}, f) \subset \varphi_{C\setminus\{0\}}$, but

$$\operatorname{Fix}(f) = \emptyset. \tag{3.25}$$

It is clear that $V \in \Upsilon_{\mathrm{AD}}(\varphi)$ and $\Lambda(\varphi_V) \neq \{0\}$, since from the assumption $\Lambda(g) \neq 0$. Hence

$$\Lambda^G(\varphi_X) \neq \{0\}, \qquad \emptyset \neq \operatorname{Fix}(\varphi_V) \subset \operatorname{Fix}(\varphi_{\mathbb{C}\setminus\{0\}}). \tag{3.26}$$

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