Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 232163, 15 pages
doi:10.1155/2011/232163

Research Article

A Hybrid-Extragradient Scheme for System of Equilibrium Problems, Nonexpansive Mappings, and Monotone Mappings

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Received 21 October 2010; Accepted 24 November 2010

Academic Editor: Jen Chih Yao

We introduce a new iterative scheme based on both hybrid method and extragradient method for finding a common element of the solutions set of a system of equilibrium problems, the fixed points set of a nonexpansive mapping, and the solutions set of a variational inequality problems for a monotone and k-Lipschitz continuous mapping in a Hilbert space. Some convergence results for the iterative sequences generated by these processes are obtained. The results in this paper extend and improve some known results in the literature.

1. Introduction

In this paper, we always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and C is a nonempty closed convex subset of H, $S : C \to C$ is a nonexpansive mapping; that is, $\|Sx - Sy\| \le \|x - y\|$ for all $x, y \in C$, P_C denotes the metric projection of H onto C, and $\mathrm{Fix}(S)$ denotes the fixed points set of S.

Let $\{F_k\}_{k\in\Gamma}$ be a countable family of bifunctions from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Combettes and Hirstoaga [1] introduced the following system of equilibrium problems:

$$\text{finding } x \in C, \quad \text{such that } \forall k \in \Gamma, \quad \forall y \in C, \quad F_k(x, y) \ge 0, \tag{1.1}$$

where Γ is an arbitrary index set. If Γ is a singleton, the problem (1.1) becomes the following equilibrium problem:

$$\text{finding } x \in C, \quad \text{such that } F(x, y) \ge 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\mathrm{EP}(F)$. And it is easy to see that the set of solutions of (1.1) can be written as $\bigcap_{k\in\Gamma}\mathrm{EP}(F_k)$.

Given a mapping $T : C \to H$, let $F(x,y) = \langle Tx, y-x\rangle$ for all $x,y \in C$. Then, the problem (1.2) becomes the following variational inequality:

$$\text{finding } x \in C, \quad \text{such that } \langle Tx, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\mathrm{VI}(C,A)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see, for instance, [1–4].

In 1953, Mann [5] introduced the following iteration algorithm: let $x_0 \in C$ be an arbitrary point, let $\{\alpha_n\}$ be a real sequence in $[0,1]$, and let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n. \tag{1.4}$$

Mann iteration algorithm has been extensively investigated for nonexpansive mappings, for example, please see [6, 7]. Takahashi et al. [8] modified the Mann iteration method (1.4) and introduced the following hybrid projection algorithm:

$$\begin{gathered} x_0 \in H, \qquad C_1 = C, \qquad x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)Sx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \qquad \forall n \in \mathbb{N}, \end{gathered} \tag{1.5}$$

where $0 \leq \alpha_n < a < 1$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{\mathrm{Fix}(S)}x_0$.

In 1976, Korpelevič [9] introduced the following so-called extragradient algorithm:

$$\begin{gathered} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n) \end{gathered} \tag{1.6}$$

for all $n \geq 0$, where $\lambda \in (0, 1/k)$, A is monotone and k-Lipschitz continuous mapping of C into $\mathbb{R}^n$. She proved that, if $\mathrm{VI}(C,A)$ is nonempty, the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.6), converge to the same point $z \in \mathrm{VI}(C,A)$.

Some methods have been proposed to solve the problem (1.2); see, for instance, [10, 11] and the references therein. S. Takahashi and W. Takahashi [10] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of the solution (1.2) and the set of fixed points of a nonexpansive mapping in a real Hilbert space: starting with an arbitrary initial $x_1 \in C$, define sequences $\{x_n\}$ and $\{u_n\}$ recursively by

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Su_n, \quad n \geq 1. \tag{1.7}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$ and $\{r_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to $z \in \mathrm{Fix}(S) \cap \mathrm{EP}(F)$, where $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)} f(z)$.

Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty closed convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, and let S be a relatively nonexpansive mapping from C into itself such that $\mathrm{Fix}(S) \cap \mathrm{EP}(f) \neq \emptyset$. Takahashi and Zembayashi [11] introduced the following hybrid method in Banach space: let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, and

$$y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n),$$
$$u_n \in C, \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall y \in C,$$
$$C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\},$$
$$x_{n+1} = \prod\nolimits_{C_{n+1}} x \tag{1.8}$$

for every $n \in \mathbb{N} \cup \{0\}$, where J is the duality napping on E, $\phi(x, y) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$ for all $x, y \in E$, and $\prod_C x = \arg\min_{y \in C} \phi(y, x)$ for all $x \in E$. They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $\prod_{\mathrm{Fix}(S)\cap \mathrm{EP}(f)} x$ if $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$.

On the other hand, Combettes and Hirstoaga [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) in a Hilbert space and obtained a weak convergence theorem. Peng and Yao [4] introduced a new viscosity approximation scheme based on the extragradient method for finding a common element of the set of solutions of problem (1.1), the set of fixed points of an infinite family of nonexpansive mappings, and the set of solutions to the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space and obtained two strong convergence theorems. Colao et al. [3] introduced an implicit method for finding common solutions of variational inequalities and systems of equilibrium problems and fixed points of infinite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Peng et al. [12] introduced a new iterative scheme based on extragradient method and viscosity approximation method for finding a common element of the solutions set of a system of equilibrium problems, fixed points set of a family of infinitely nonexpansive mappings, and the solution set of a variational inequality for a relaxed coercive mapping in a Hilbert space and obtained a strong convergence theorem.

In this paper, motivated by the above results, we introduce a new hybrid extragradient method to find a common element of the set of solutions to a system of equilibrium problems, the set of fixed points of a nonexpansive mapping, and the set of solutions of the variational inequality for monotone and k-Lipschitz continuous mappings in a Hilbert space and obtain some strong convergence theorems. Our results unify, extend, and improve those corresponding results in [8, 11] and the references therein.

2. Preliminaries

Let symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively. It is well known that

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$.

For any $x \in H$, there exists a unique nearest point in C denoted by $P_C(x)$ such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C(x) \in C$ and

$$\langle x - P_C(x), P_C(x) - y\rangle \ge 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.2) is equivalent to

$$\|x-y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

A mapping A of C into H is called monotone if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in C$. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a positive real number L such that $\|Ax - Ay\| \le L\|x - y\|$ for all $x, y \in C$.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem, the characterization of projection (2.2) implies the following:

$$\begin{aligned} u \in \mathrm{VI}(C,A) &\Longrightarrow u = P_C(u - \lambda Au), \quad \forall\lambda > 0, \\ u = P_C(u - \lambda Au), &\quad \text{for some } \lambda > 0 \Longrightarrow u \in \mathrm{VI}(C,A). \end{aligned} \tag{2.4}$$

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions which were imposed in [2]:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone; that is, $F(x,y) + F(y,x) \le 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y); \tag{2.5}$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 2.1 (See [2]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.6}$$

Lemma 2.2 (See [1]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r^F : H \to 2^C$ as follows:*

$$T_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\} \tag{2.7}$$

for all $x \in H$. Then, the following statements hold:

(1) T_r^F *is single-valued;*

(2) T_r^F *is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\left\| T_r^F(x) - T_r^F(y) \right\|^2 \leq \left\langle T_r^F(x) - T_r^F(y), x - y \right\rangle; \tag{2.8}$$

(3) $\mathrm{Fix}(T_r^F) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

3. Main Results

In this section, we will introduce a new algorithm based on hybrid and extragradient method to find a common element of the set of solutions to a system of equilibrium problems, the set of fixed points of a nonexpansive mapping, and the set of solutions of the variational inequality for monotone and k-Lipschitz continuous mappings in a Hilbert space and show that the sequences generated by the processes converge strongly to a same point.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F_k, $k \in \{1, 2, \ldots, M\}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let A be a monotone and k-Lipschitz continuous mapping of C into H, and let S be a nonexpansive mapping from C into itself such that $\Omega = \mathrm{Fix}(S) \cap \mathrm{VI}(\mathrm{C}, \mathrm{A}) \cap (\bigcap_{\mathrm{k}=1}^{\mathrm{M}} \mathrm{EP}(\mathrm{F_k})) \neq \emptyset$. Pick any $x_0 \in H$, and set $C_1 = C$. Let $\{x_n\}, \{y_n\}, \{w_n\}$, and $\{u_n\}$ be sequences generated by $x_1 = P_{C_1} x_0$ and*

$$\begin{aligned} u_n &= T_{r_{M\cdot n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ w_n &= \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n A y_n), \\ C_{n+1} &= \{ z \in C_n : \|w_n - z\| \leq \|x_n - z\| \}, \\ x_{n+1} &= P_{C_{n+1}} x_0 \end{aligned} \tag{3.1}$$

for each $n \in \mathbb{N}$. *If* $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$, $\{\alpha_n\} \subset [c,d]$ *for some* $c,d \in (0,1)$, *and* $\{r_{k,n}\} \subset (0,\infty)$ *satisfies* $\liminf_{n\to\infty} r_{k,n} > 0$ *for each* $k \in \{1,2,\ldots,M\}$, *then* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, *and* $\{w_n\}$ *generated by* (3.1) *converge strongly to* $P_\Omega x_0$.

Proof. It is obvious that C_n is closed for each $n \in \mathbb{N}$. Since

$$C_{n+1} = \left\{ z \in C_n : \|w_n - x_n\|^2 + 2\langle w_n - x_n, x_n - z\rangle \leq 0 \right\}, \tag{3.2}$$

we also have that C_n is convex for each $n \in \mathbb{N}$. Thus, $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, and $\{w_n\}$ are welldefined. By taking $\Theta_n^k = T_{r_{k,n}}^{F_k} T_{r_{k-1,n}}^{F_{k-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1}$ for $k \in \{1,2,\ldots,M\}$ and $n \in \mathbb{N}$, $\Theta_n^0 = I$ for each $n \in \mathbb{N}$, where I is the identity mapping on H. Then, it is easy to see that $u_n = \Theta_n^M x_n$. We divide the proof into several steps.

Step 1. We show by induction that $\Omega \subset C_n$ for $n \in \mathbb{N}$. It is obvious that $\Omega \subset C = C_1$. Suppose that $\Omega \subset C_n$ for some $n \in \mathbb{N}$. Let $v \in \Omega$. Then, by Lemma 2.2 and $v = P_C(v - \lambda_n A v) = \Theta_n^M v$, we have

$$\|u_n - v\| = \left\| \Theta_n^M x_n - \Theta_n^M v \right\| \leq \|x_n - v\|, \quad \forall n \in \mathbb{N}. \tag{3.3}$$

Putting $v_n = P_C(u_n - \lambda_n A y_n)$ for each $n \in \mathbb{N}$, from (2.3) and the monotonicity of A, we have

$$\begin{aligned}
\|v_n - v\|^2 &\leq \left\|u_n - \lambda_n A y_n - v\right\|^2 - \left\|u_n - \lambda_n A y_n - v_n\right\|^2 \\
&= \|u_n - v\|^2 - \|u_n - v_n\|^2 + 2\lambda_n \langle A y_n, v - v_n\rangle \\
&= \|u_n - v\|^2 - \|u_n - v_n\|^2 \\
&\quad + 2\lambda_n \left(\langle A y_n - A v, v - y_n\rangle + \langle A v, v - y_n\rangle + \langle A y_n, y_n - v_n\rangle \right) \\
&\leq \|u_n - v\|^2 - \|u_n - v_n\|^2 + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\
&= \|u_n - v\|^2 - \left\|u_n - y_n\right\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \left\|y_n - v_n\right\|^2 \\
&\quad + 2\lambda_n \langle A y_n, y_n - v_n\rangle \\
&= \|u_n - v\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - v_n\right\|^2 \\
&\quad + 2\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle.
\end{aligned} \tag{3.4}$$

Moreover, from $y_n = P_C(u_n - \lambda_n A u_n)$ and (2.2), we have

$$\langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle \leq 0. \tag{3.5}$$

Since A is k-Lipschitz continuous, it follows that

$$\begin{aligned}\left\langle u_n-\lambda_n A y_n-y_n, v_n-y_n\right\rangle &= \left\langle u_n-\lambda_n A u_n-y_n, v_n-y_n\right\rangle+\left\langle \lambda_n A u_n-\lambda_n A y_n, v_n-y_n\right\rangle \\ &\leq \left\langle \lambda_n A u_n-\lambda_n A y_n, v_n-y_n\right\rangle \\ &\leq \lambda_n k\left\|u_n-y_n\right\|\left\|v_n-y_n\right\|.\end{aligned} \tag{3.6}$$

So, we have

$$\begin{aligned}\left\|v_n-v\right\|^2 &\leq \left\|u_n-v\right\|^2-\left\|u_n-y_n\right\|^2-\left\|y_n-v_n\right\|^2+2\lambda_n k\left\|u_n-y_n\right\|\left\|v_n-y_n\right\| \\ &\leq \left\|u_n-v\right\|^2-\left\|u_n-y_n\right\|^2-\left\|y_n-v_n\right\|^2+\lambda_n^2 k^2\left\|u_n-y_n\right\|^2+\left\|v_n-y_n\right\|^2 \\ &= \left\|u_n-v\right\|^2+\left(\lambda_n^2 k^2-1\right)\left\|u_n-y_n\right\|^2 \\ &\leq \left\|u_n-v\right\|^2.\end{aligned} \tag{3.7}$$

From (3.7) and the definition of w_n, we have

$$\begin{aligned}\left\|w_n-v\right\|^2 &\leq \alpha_n\left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left\|S v_n-v\right\|^2 \\ &\leq \alpha_n\left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left\|v_n-v\right\|^2 \\ &\leq \alpha_n\left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left\{\left\|u_n-v\right\|^2+\left(\lambda_n^2 k^2-1\right)\left\|u_n-y_n\right\|^2\right\}\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\leq \alpha_n\left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left(\lambda_n^2 k^2-1\right)\left\|u_n-y_n\right\|^2 \\ &= \left\|x_n-v\right\|^2+\left(1-\alpha_n\right)\left(\lambda_n^2 k^2-1\right)\left\|u_n-y_n\right\|^2 \\ &\leq \left\|x_n-v\right\|^2,\end{aligned} \tag{3.9}$$

and hence $v \in C_{n+1}$. This implies that $\Omega \subset C_n$ for all $n \in \mathbb{N}$.

Step 2. We show that $\lim_{n\to\infty}\|x_n-w_n\| \to 0$ and $\lim_{n\to\infty}\|u_n-y_n\| = 0$.

Let $l_0 = P_\Omega x_0$. From $x_n = P_{C_n}x_0$ and $l_0 \in \Omega \subset C_n$, we have

$$\left\|x_n-x_0\right\| \leq \left\|l_0-x_0\right\|, \quad \forall n \in \mathbb{N}. \tag{3.10}$$

Therefore, $\{x_n\}$ is bounded. From (3.3)–(3.9), we also obtain that $\{w_n\}$, $\{v_n\}$, and $\{u_n\}$ are bounded. Since $x_{n+1} \in C_{n+1} \subseteq C_n$ and $x_n = P_{C_n}x_0$, we have

$$\left\|x_n-x_0\right\| \leq \left\|x_{n+1}-x_0\right\|, \quad \forall n \in \mathbb{N}. \tag{3.11}$$

Therefore, $\lim_{n\to\infty}\|x_n-x_0\|$ exists.

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we have

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0, \quad \forall n \in \mathbb{N}. \tag{3.12}$$

So

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_n + x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2,
\end{aligned} \tag{3.13}$$

which implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Since $x_{n+1} \in C_{n+1}$, we have $\|w_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$, and hence

$$\|x_n - w_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \leq 2\|x_n - x_{n+1}\|, \quad \forall n \in \mathbb{N}. \tag{3.15}$$

It follows from (3.14) that $\|x_n - w_n\| \to 0$.

For $v \in \Omega$, it follows from (3.9) that

$$\begin{aligned}
\left\|u_n - y_n\right\|^2 &\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}\left(\|x_n - v\|^2 - \|w_n - v\|^2\right) \\
&= \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - v\| - \|w_n - v\|)(\|x_n - v\| + \|w_n - v\|) \\
&\leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}\|x_n - w_n\|(\|x_n - v\| + \|w_n - v\|),
\end{aligned} \tag{3.16}$$

which implies that $\lim_{n\to\infty}\|u_n - y_n\| = 0$.

Step 3. We now show that

$$\lim_{n\to\infty}\left\|\Theta_n^k x_n - \Theta_n^{k-1} x_n\right\| = 0, \quad k = 1, 2, \ldots, M. \tag{3.17}$$

Indeed, let $v \in \Omega$. It follows form the firmly nonexpansiveness of $T^{F_k}_{r_{k,n}}$ that we have, for each $k \in \{1, 2, \ldots, M\}$,

$$\begin{aligned}
\left\|\Theta^k_n x_n - v\right\|^2 &= \left\|T^{F_k}_{r_{k,n}}\Theta^{k-1}_n x_n - T^{F_k}_{r_{k,n}} v\right\|^2 \\
&\le \left\langle \Theta^k_n x_n - v, \Theta^{k-1}_n x_n - v\right\rangle \\
&= \frac{1}{2}\left(\left\|\Theta^k_n x_n - v\right\|^2 + \left\|\Theta^{k-1}_n x_n - v\right\|^2 - \left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2\right).
\end{aligned} \tag{3.18}$$

Thus, we get

$$\left\|\Theta^k_n x_n - v\right\|^2 \le \left\|\Theta^{k-1}_n x_n - v\right\|^2 - \left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2, \quad k = 1, 2, \ldots, M, \tag{3.19}$$

which implies that, for each $k \in \{1, 2, \ldots, M\}$,

$$\begin{aligned}
\left\|\Theta^k_n x_n - v\right\|^2 &\le \left\|\Theta^0_n x_n - v\right\|^2 - \left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2 - \left\|\Theta^{k-1}_n x_n - \Theta^{k-2}_n x_n\right\|^2 \\
&\quad - \cdots - \left\|\Theta^2_n x_n - \Theta^1_n x_n\right\|^2 - \left\|\Theta^1_n x_n - \Theta^0_n x_n\right\|^2 \\
&\le \|x_n - v\|^2 - \left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2.
\end{aligned} \tag{3.20}$$

By (3.8), $u_n = \Theta^M_n x_n$, and (3.20), we have, for each $k \in \{1, 2, \ldots, M\}$,

$$\begin{aligned}
\|w_n - v\|^2 &\le \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\|u_n - v\|^2 \\
&\le \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\left\|\Theta^k_n x_n - v\right\|^2, \quad \forall k \in \{1, 2, \ldots, M\} \\
&\le \alpha_n\|x_n - v\|^2 + (1 - \alpha_n)\left(\|x_n - v\|^2 - \left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2\right) \\
&\le \|x_n - v\|^2 - (1 - \alpha_n)\left\|\Theta^k_n x_n - \Theta^{k-1}_n x_n\right\|^2,
\end{aligned} \tag{3.21}$$

which implies that

$$\begin{aligned}
(1 - \alpha_n)\left\|\Theta^k_n x_n - \Theta^{k-1} x_n\right\| &\le \|x_n - v\|^2 - \|w_n - v\|^2 \\
&= (\|x_n - v\| + \|w_n - v\|)(\|x_n - v\| - \|w_n - v\|) \\
&\le (\|x_n - v\| + \|w_n - v\|)\|x_n - w_n\|.
\end{aligned} \tag{3.22}$$

It follows from $\|x_n - w_n\| \to 0$ and $0 < c \le \alpha_n \le d < 1$ that (3.17) holds.

Step 4. We now show that $\lim_{n\to\infty}\|Sv_n - v_n\| = 0$.

It follows from (3.17) that $\|x_n - u_n\| \to 0$. Since $\|x_n - y_n\| \leq \|x_n - u_n\| + \|u_n - y_n\|$, we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.23}$$

We observe that

$$\begin{aligned} \|v_n - y_n\| &= \|P_C(u_n - \lambda_n A y_n) - P_C(u_n - \lambda_n A u_n)\| \\ &\leq \|\lambda_n A u_n - \lambda_n A y_n\| \leq \lambda_n k \|u_n - y_n\|, \end{aligned} \tag{3.24}$$

which implies that

$$\lim_{n\to\infty} \|v_n - y_n\| = 0. \tag{3.25}$$

Since $\|x_n - w_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n) S v_n\| = \|(1-\alpha_n)(x_n - S v_n)\|$, we obtain

$$\lim_{n\to\infty} \|x_n - S v_n\| = 0. \tag{3.26}$$

Since $\|S v_n - v_n\| \leq \|S v_n - x_n\| + \|x_n - y_n\| + \|y_n - v_n\|$, we get

$$\lim_{n\to\infty} \|S v_n - v_n\| = 0. \tag{3.27}$$

Step 5. We show that $x_n \to w$, where $w = P_\Omega x_0$.

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to w. From $\|\Theta_n^k x_n - \Theta_n^{k-1} x_n\| \to 0$ for each $k = 1, 2, \ldots, M$, we obtain that $\Theta_{n_i}^k x_{n_i} \rightharpoonup w$ for $k = 1, 2, \ldots, M$. It follows from $\|x_n - w_n\| \to 0$, $\|v_n - y_n\| \to 0$, and $\|u_n - y_n\| \to 0$ that $w_{n_i} \rightharpoonup w$, $y_{n_i} \rightharpoonup w$, and $v_{n_i} \rightharpoonup w$.

In order to show that $w \in \Omega$, we first show that $w \in \bigcap_{k=1}^M \mathrm{EP}(F_k)$. Indeed, by definition of $T_{r_{k,n}}^{F_k}$, we have that, for each $k \in \{1, 2, \ldots, M\}$,

$$F_k\left(\Theta_n^k x_n, y\right) + \frac{1}{r_{k,n}} \left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

From (A2), we also have

$$\frac{1}{r_{k,n}} \left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\rangle \geq F_k\left(y, \Theta_n^k x_n\right), \quad \forall y \in C. \tag{3.29}$$

And hence

$$\left\langle y - \Theta_{n_i}^{k} x_{n_i}, \frac{\Theta_{n_i}^{k} x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \right\rangle \geq F_k\left(y, \Theta_{n_i}^{k} x_{n_i}\right), \quad \forall y \in C. \tag{3.30}$$

From (A4), $(\Theta_{n_i}^{k} x_{n_i} - \Theta_{n_i}^{k-1} x_{n_i})/r_{k,n_i} \to 0$ and $\Theta_{n_i}^{k} x_{n_i} \rightharpoonup w$ imply that, for each $k \in \{1, 2, \ldots, M\}$,

$$F_k(y, w) \leq 0, \quad \forall y \in C. \tag{3.31}$$

Since $x_{n_i} \subset C$, $x_{n_i} \rightharpoonup w$ and C is closed and convex, C is weakly closed, and hence $w \in C$. Thus, for t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1 - t)w$. Since $y \in C$ and $w \in C$, we have $y_t \in C$, and hence $F_k(y_t, w) \leq 0$. So, from (A1) and (A4), we have, for each $k \in \{1, 2, \ldots, M\}$,

$$0 = F_k(y_t, y_t) \leq tF_k(y_t, y) + (1 - t)F_k(y_t, w) \leq tF_k(y_t, y), \tag{3.32}$$

and hence, for each $k \in \{1, 2, \ldots, M\}$, $0 \leq F_k(y_t, y)$. From (A3), we have, for each $k \in \{1, 2, \ldots, M\}$, $0 \leq F_k(w, y)$, for all $y \in C$. Thus, $w \in \bigcap_{k=1}^{M} \mathrm{EP}(F_k)$.

We now show that $w \in \mathrm{Fix}(S)$. Assume that $w \notin \mathrm{Fix}(S)$. Since $v_{n_i} \rightharpoonup w$ and $w \neq Sw$, from Opial's condition [13], we have

$$\begin{aligned}
\liminf_{i \to \infty} \|v_{n_i} - w\| &< \liminf_{i \to \infty} \|v_{n_i} - Sw\| \\
&\leq \limsup_{i \to \infty} \|v_{n_i} - Sv_{n_i}\| + \liminf_{i \to \infty} \|Sv_{n_i} - Sw\| \\
&= \liminf_{i \to \infty} \|Sv_{n_i} - Sw\| \\
&\leq \liminf_{i \to \infty} \|v_{n_i} - w\|,
\end{aligned} \tag{3.33}$$

which is a contradiction. Thus, we obtain $w \in \mathrm{Fix}(S)$.

We next show that $w \in \mathrm{VI}(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.34}$$

It is worth to note that in this case the mapping T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$ (see [14]). Let $(v, u) \in G(T)$. Since $u - Av \in N_C v$ and $v_n \in C$, we have

$\langle v - v_n, u - Av\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n A y_n)$ and $v \in C$, we have $\langle v - v_n, v_n - (u_n - \lambda_n A y_n)\rangle \geq 0$, and hence $\langle v - v_n, (v_n - u_n)/\lambda_n + Ay_n\rangle \geq 0$. Therefore, we have

$$\begin{aligned}
\langle v - v_{n_i}, u\rangle &\geq \langle v - v_{n_i}, Av\rangle \\
&\geq \langle v - v_{n_i}, Av\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\
&= \left\langle v - v_{n_i}, Av - Ay_{n_i} - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.35}$$

Since $\lim_{n\to\infty}\|v_n - y_n\| = 0$ and A is k-Lipschitz continuous, we obtain that $\lim_{n\to\infty}\|Av_n - Ay_n\| = 0$. From $v_{n_i} \rightharpoonup w$, $\liminf_{n\to\infty}\lambda_n > 0$, and $\lim_{n\to\infty}\|v_n - u_n\| = 0$, we obtain

$$\langle v - w, u\rangle \geq 0. \tag{3.36}$$

Since T is maximal monotone, we have $w \in T^{-1}0$, and hence $w \in \mathrm{VI}(C, A)$, which implies that $w \in \Omega$. Finally, we show that $x_n \to w$, where

$$w = P_\Omega x_0. \tag{3.37}$$

Since $x_n = P_{C_n}x_0$ and $w \in \Omega \subset C_n$, we have $\|x_n - x_0\| \leq \|w - x_0\|$. It follows from $l_0 = P_\Omega x_0$ and the lower semicontinuousness of the norm that

$$\|l_0 - x_0\| \leq \|w - x_0\| \leq \liminf_{i\to\infty}\|x_{n_i} - x_0\| \leq \limsup_{i\to\infty}\|x_{n_i} - x_0\| \leq \|l_0 - x_0\|. \tag{3.38}$$

Thus, we obtain $w = l_0$ and

$$\lim_{i\to\infty}\|x_{n_i} - x_0\| = \|w - x_0\|. \tag{3.39}$$

From $x_{n_i} - x_0 \rightharpoonup w - x_0$ and the Kadec-Klee property of H, we have $x_{n_i} - x_0 \to w - x_0$, and hence $x_{n_i} \to w$. This implies that $x_n \to w$. It is easy to see that $u_n \to w$, $y_n \to w$, and $w_n \to w$. The proof is now complete. □

By Theorem 3.1, we can easily obtain some new results as follows.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), let A be a monotone and k-Lipschitz continuous mapping of C into H, and let S be a nonexpansive mapping from C into itself such that $\Omega =$*

$\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \cap \mathrm{EP}(F) \neq \emptyset$. *Pick any* $x_0 \in H$, *and set* $C_1 = C$. *Let* $\{x_n\}, \{y_n\}, \{w_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_1 = P_{C_1}(x_0)$ *and*

$$\begin{aligned} u_n \in C, \quad &\text{such that } F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \lambda_n A u_n), \\ w_n &= \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A y_n), \\ C_{n+1} &= \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0) \end{aligned} \tag{3.40}$$

for each $n \in \mathbb{N}$. *If* $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, 1/k)$, $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0,1)$, *and* $\{r_n\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_n > 0$, *then* $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, *and* $\{w_n\}$ *converge strongly to* $P_\Omega(x_0)$.

Proof. Putting $F_M = F_{M-1} = \cdots = F_1 = F$ in Theorem 3.1, we obtain Corollary 3.2. □

Corollary 3.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F_k, $k \in \{1, 2, \ldots, M\}$ *be a family of bifunctions from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)–(A4), and let* S *be a nonexpansive mapping from* C *into itself such that* $\Omega = \mathrm{Fix}(S) \cap (\bigcap_{k=1}^{M} \mathrm{EP}(F_k)) \neq \emptyset$. *Pick any* $x_0 \in H$, *and set* $C_1 = C$. *Let* $\{x_n\}, \{w_n\}$, *and* $\{u_n\}$ *be sequences generated by* $x_1 = P_{C_1}(x_0)$ *and*

$$\begin{aligned} u_n &= T_{r_{M\cdot n}}^{F_M} T_{r_{M-1,n}}^{F_{M-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} x_n, \\ w_n &= \alpha_n x_n + (1-\alpha_n) S u_n, \\ C_{n+1} &= \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0) \end{aligned} \tag{3.41}$$

for each $n \in \mathbb{N}$. *If* $\{\alpha_n\} \subset [c, d]$ *for some* $c, d \in (0,1)$ *and* $\{r_{k,n}\} \subset (0, \infty)$ *satisfies* $\liminf_{n\to\infty} r_{k,n} > 0$ *for each* $k \in \{1, 2, \ldots, M\}$, *then* $\{x_n\}$, $\{u_n\}$, *and* $\{w_n\}$ *converge strongly to* $P_\Omega(x_0)$.

Proof. Let $A = 0$ in Theorem 3.1, then complete the proof. □

Corollary 3.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H, *and let* S *be a nonexpansive mapping from* C *into itself such that* $\Omega = \mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. *Pick any* $x_0 \in H$, *and set* $C_1 = C$. *Let* $\{x_n\}, \{y_n\}$, *and* $\{w_n\}$ *be sequences generated by* $x_1 = P_{C_1}(x_0)$ *and*

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n A x_n), \\ w_n &= \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A y_n), \\ C_{n+1} &= \{z \in C_n : \|w_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} &= P_{C_{n+1}}(x_0) \end{aligned} \tag{3.42}$$

for each $n \in \mathbb{N}$*. If* $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$, $\{\alpha_n\} \subset [c,d]$ *for some* $c,d \in (0,1)$*, then* $\{x_n\}$, $\{y_n\}$*, and* $\{w_n\}$ *converge strongly to* $P_\Omega(x_0)$.

Proof. Putting $F_M = F_{M-1} = \cdots = F_1 = 0$ in Theorem 3.1, we obtain Corollary 3.4. □

Remark 3.5. Letting $F_M = F_{M-1} = \cdots = F_1 = F$ in Corollary 3.3, we obtain the Hilbert space version of Theorem 3.1 in [11]. Letting $A = 0$ in Corollary 3.4, we recover Theorem 4.1 in [8]. Hence, Theorem 3.1 unifies, generalizes, and extends the corresponding results in [8, 11] and the references therein.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (Grants 10771228 and 10831009), the Natural Science Foundation of Chongqing (Grant no. CSTC, 2009BB8240), and the Research Project of Chongqing Normal University (Grant 08XLZ05). The authors are grateful to the referees for the detailed comments and helpful suggestions, which have improved the presentation of this paper.

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