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*Research Article*

# An Implicit Iteration Method for Variational Inequalities over the Set of Common Fixed Points for a Finite Family of Nonexpansive Mappings in Hilbert Spaces

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We introduce a new implicit iteration method for finding a solution for a variational inequality involving Lipschitz continuous and strongly monotone mapping over the set of common fixed points for a finite family of nonexpansive mappings on Hilbert spaces.

## 1. Introduction

Let $C$ be a nonempty closed and convex subset of a real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let $F : H \to H$ be a nonlinear mapping. The variational inequality problem is formulated as finding a point $p^* \in C$ such that

$$\langle F(p^*), p - p^* \rangle \geq 0, \quad \forall p \in C. \tag{1.1}$$

Variational inequalities were initially studied by Kinderlehrer and Stampacchia in [1] and ever since have been widely investigated, since they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics, and finance (see [1–3]).

It is well known that if $F$ is an $L$-Lipschitz continuous and $\eta$-strongly monotone, that is, $F$ satisfies the following conditions:

$$\|F(x) - F(y)\| \leq L\|x - y\|,$$
$$\langle F(x) - F(y), x - y\rangle \geq \eta\|x - y\|^2, \tag{1.2}$$

where $L$ and $\eta$ are fixed positive numbers, then (1.1) has a unique solution. It is also known that (1.1) is equivalent to the fixed-point equation

$$p = P_C(p - \mu F(p)), \tag{1.3}$$

where $P_C$ denotes the metric projection from $x \in H$ onto $C$ and $\mu$ is an arbitrarily fixed positive constant.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of $C$. For finding an element $p \in \cap_{i=1}^N \operatorname{Fix}(T_i)$, Xu and Ori introduced in [4] the following implicit iteration process. For $x_0 \in C$ and $\{\beta_k\}_{k=1}^\infty \subset (0,1)$, the sequence $\{x_k\}$ is generated as follows:

$$\begin{aligned}
x_1 &= \beta_1 x_0 + (1-\beta_1)T_1 x_1,\\
x_2 &= \beta_2 x_1 + (1-\beta_2)T_2 x_2,\\
&\vdots\\
x_N &= \beta_N x_{N-1} + (1-\beta_N)T_N x_N,\\
x_{N+1} &= \beta_{N+1} x_N + (1-\beta_{N+1})T_1 x_{N+1},\\
&\vdots
\end{aligned} \tag{1.4}$$

The compact expression of the method is the form

$$x_k = \beta_k x_{k-1} + (1-\beta_k)T_{[k]}x_k, \quad k \geq 1, \tag{1.5}$$

where $T_{[n]} = T_{n \bmod N}$, for integer $n \geq 1$, with the mod function taking values in the set $\{1, 2, \ldots, N\}$. They proved the following result.

**Theorem 1.1.** *Let $H$ be a real Hilbert space and $C$ a nonempty closed convex subset of $H$. Let $\{T_i\}_{i=1}^N$ be $N$ nonexpansive self-maps of $C$ such that $\cap_{i=1}^N \operatorname{Fix}(T_i) \neq \emptyset$, where $\operatorname{Fix}(T_i) = \{x \in C : T_i x = x\}$. Let $x_0 \in C$ and $\{\beta_k\}_{k=1}^\infty$ be a sequence in $(0,1)$ such that $\lim_{k\to\infty}\beta_k = 0$. Then, the sequence $\{x_k\}$ defined implicitly by (1.5) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^N$.*

Further, Zeng and Yao introduced in [5] the following implicit method. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_k\}_{k=1}^{\infty}$ is generated as follows:

$$\begin{aligned} x_1 &= \beta_1 x_0 + (1-\beta_1)\left[T_1 x_1 - \lambda_1 \mu F(T_1 x_1)\right], \\ x_2 &= \beta_2 x_1 + (1-\beta_2)\left[T_2 x_2 - \lambda_2 \mu F(T_2 x_2)\right], \\ &\vdots \\ x_N &= \beta_N x_{N-1} + (1-\beta_N)\left[T_N x_N - \lambda_N \mu F(T_N x_N)\right], \\ x_{N+1} &= \beta_{N+1} x_N + (1-\beta_{N+1})\left[T_1 x_{N+1} - \lambda_{N+1} \mu F(T_1 x_{N+1})\right], \\ &\vdots \end{aligned} \tag{1.6}$$

The scheme is written in a compact form as

$$x_k = \beta_k x_{k-1} + (1-\beta_k)\left[T_{[k]} x_k - \lambda_k \mu F\left(T_{[k]} x_k\right)\right], \quad k \geq 1. \tag{1.7}$$

They proved the following result.

**Theorem 1.2.** *Let $H$ be a real Hilbert space and $F : H \to H$ a mapping such that for some constants $L, \eta > 0$, $F$ is $L$-Lipschitz continuous and $\eta$-strongly monotone. Let $\{T_i\}_{i=1}^{N}$ be $N$ nonexpansive self-maps of $H$ such that $C = \cap_{i=1}^{N} \operatorname{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, and let $x_0 \in H$, with $\{\lambda_k\}_{k=1}^{\infty} \subset [0,1)$ and $\{\beta_k\}_{k=1}^{\infty} \subset (0,1)$ satisfying the conditions: $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $\alpha \leq \beta_k \leq \beta$, $k \geq 1$, for some $\alpha, \beta \in (0,1)$. Then, the sequence $\{x_k\}$ defined by (1.7) converges weakly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$. The convergence is strong if and only if $\liminf_{k\to\infty} d(x_k, C) = 0$.*

Recently, Ceng et al. [6] extended the above result to a finite family of asymptotically self-maps.

Clearly, from $\sum_{k=1}^{\infty} \lambda_k < \infty$ we have that $\lambda_k \to 0$ as $k \to \infty$. To obtain strong convergence without the condition $\sum_{k=1}^{\infty} \lambda_k < \infty$, in this paper we propose the following implicit algorithm:

$$x_t = T^t x_t, \quad T^t := T_0^t T_N^t \cdots T_1^t, \quad t \in (0,1), \tag{1.8}$$

where $T_i^t$ are defined by

$$T_i^t x = \left(1-\beta_t^i\right)x + \beta_t^i T_i x, \quad i = 1, \ldots, N, \qquad T_0^t y = (I - \lambda_t \mu F) y, \quad x, y \in H, \tag{1.9}$$

$I$ denotes the identity mapping of $H$, and the parameters $\{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$ for all $t \in (0,1)$ satisfy the following conditions: $\lambda_t \to 0$ as $t \to 0$ and $0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1$, $i = 1, \ldots, N$.

## 2. Main Result

We formulate the following facts for the proof of our results.

**Lemma 2.1** (see [7]). (i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *and for any fixed* $t \in [0,1]$, (ii) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - (1-t)t\|x-y\|^2$, *for all* $x, y \in H$.

Put $T^{\lambda}x = Tx - \lambda\mu F(Tx)$, $x \in H$, $\lambda \in [0,1]$; for any nonexpansive mapping $T$ of $H$, we have the following lemma.

**Lemma 2.2** (see [8]). $\|T^{\lambda}x - T^{\lambda}y\| \leq (1-\lambda\tau)\|x-y\|$, *for all* $x, y \in H$ *and for a fixed number* $\mu \in (0, 2\eta/L^2)$, *where* $\tau = 1 - \sqrt{1-\mu(2\eta-\mu L^2)} \in (0,1)$.

**Lemma 2.3** (Demiclosedness Principle [9]). *Assume that $T$ is a nonexpansive self-mapping of a closed convex subset $K$ of a Hibert space $H$. If $T$ has a fixed point, then $I-T$ is demiclosed; that is, whenever $\{x_k\}$ is a sequence in $K$ weakly converging to some $x \in K$ and the sequence $\{(I-T)x_k\}$ strongly converges to some $y$, it follows that $(I-T)x = y$.*

Now, we are in a position to prove the following result.

**Theorem 2.4.** *Let $H$ be a real Hilbert space and $F : H \to H$ a mapping such that for some constants $L, \eta > 0$, $F$ is L-Lipschitz continuous and $\eta$-strongly monotone. Let $\{T_i\}_{i=1}^N$ be $N$ nonexpansive self-maps of $H$ such that $C = \cap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$ and let $t \in (0,1)$, $\{\lambda_t\}, \{\beta_t^i\} \subset (0,1)$, such that*

$$\lambda_t \longrightarrow 0, \quad \text{as } t \longrightarrow 0, \quad 0 < \liminf_{t\to 0} \beta_t^i \leq \limsup_{t\to 0} \beta_t^i < 1, \quad i = 1, \ldots, N. \tag{2.1}$$

*Then, the net $\{x_t\}$ defined by* (1.8)-(1.9) *converges strongly to the unique element $p^*$ in* (1.1).

*Proof.* By using Lemma 2.2 with $T^{\lambda} = T_0^t$, that is, $T = I$, we have that

$$\begin{aligned}
\left\|T^t x - T^t y\right\| &\leq (1-\lambda_t\tau)\left\|T_N^t \cdots T_1^t x - T_N^t \cdots T_1^t y\right\| \\
&\vdots \\
&\leq (1-\lambda_t\tau)\left\|T_i^t \cdots T_1^t x - T_i^t \cdots T_1^t y\right\| \\
&\vdots \\
&\leq (1-\lambda_t\tau)\left\|T_1^t x - T_1^t y\right\| \leq (1-\lambda_t\tau)\left\|x-y\right\| \quad \forall x, y \in H.
\end{aligned} \tag{2.2}$$

So, $T^t$ is a contraction in $H$. By Banach's Contraction Principle, there exists a unique element $x_t \in H$ such that $x_t = T^t x_t$ for all $t \in (0,1)$.

Next, we show that $\{x_t\}$ is bounded. Indeed, for a fixed point $p \in C$, we have that $T_i^t p = p$ for $i = 1, \ldots, N$, and hence

$$\begin{aligned}
\|x_t - p\| &= \|T^t x_t - p\| = \|T^t x_t - T_N^t \cdots T_1^t p\| \\
&= \left\|(I - \lambda_t \mu F) T_N^t \cdots T_1^t x_t - (I - \lambda_t \mu F) T_N^t \cdots T_1^t p - \lambda_t \mu F(p)\right\| \\
&\le (1 - \lambda_t \tau)\|T_N^t \cdots T_1^t x_t - T_N^t \cdots T_1^t p\| + \lambda_t \mu \|F(p)\| \\
&\le (1 - \lambda_t \tau)\|T_{N-1}^t \cdots T_1^t x_t - T_{N-1}^t \cdots T_1^t p\| + \lambda_t \mu \|F(p)\| \\
&\vdots \\
&\le (1 - \lambda_t \tau)\|T_i^t \cdots T_1^t x_t - T_i^t \cdots T_1^t p\| + \lambda_t \mu \|F(p)\| \\
&\vdots \\
&\le (1 - \lambda_t \tau)\|T_1^t x_t - T_1^t p\| + \lambda_t \mu \|F(p)\| \\
&\le (1 - \lambda_t \tau)\|x_t - p\| + \lambda_t \mu \|F(p)\|.
\end{aligned} \tag{2.3}$$

Therefore,

$$\|x_t - p\| \le \frac{\mu}{\tau}\|F(p)\| \tag{2.4}$$

that implies the boundedness of $\{x_t\}$. So, are the nets $\{F(y_t^N)\}, \{y_t^i\},\ i = 1, \ldots, N$.

Put

$$\begin{aligned}
y_t^1 &= \left(1 - \beta_t^1\right) x_t + \beta_t^1 T_1 x_t, \\
y_t^2 &= \left(1 - \beta_t^2\right) y_t^1 + \beta_t^2 T_2 y_t^1, \\
&\vdots \\
y_t^i &= \left(1 - \beta_t^i\right) y_t^{i-1} + \beta_t^i T_i y_t^{i-1}, \\
&\vdots \\
y_t^N &= \left(1 - \beta_t^N\right) y_t^{N-1} + \beta_t^N T_N y_t^{N-1}.
\end{aligned} \tag{2.5}$$

Then,

$$x_t = (I - \lambda_t \mu F) y_t^N. \tag{2.6}$$

Moreover,

$$\begin{aligned}\|x_t - p\|^2 &= \left\|(I - \lambda_t\mu F)y_t^N - p\right\|^2 \\ &= \left\|y_t^N - p\right\|^2 - 2\lambda_t\mu\left\langle F\left(y_t^N\right), y_t^N - p\right\rangle + \lambda_t^2\mu^2\left\|F\left(y_t^N\right)\right\|^2 \\ &\le \left\|y_t^{N-1} - p\right\|^2 - 2\lambda_t\mu\left\langle F\left(y_t^N\right), y_t^N - p\right\rangle + \lambda_t^2\mu^2\left\|F\left(y_t^N\right)\right\|^2 \\ &\vdots \\ &\le \left\|y_t^1 - p\right\|^2 - 2\lambda_t\mu\left\langle F\left(y_t^N\right), y_t^N - p\right\rangle + \lambda_t^2\mu^2\left\|F\left(y_t^N\right)\right\|^2 \\ &\le \|x_t - p\|^2 - 2\lambda_t\mu\left\langle F\left(y_t^N\right), y_t^N - p\right\rangle + \lambda_t^2\mu^2\left\|F\left(y_t^N\right)\right\|^2.\end{aligned} \tag{2.7}$$

Thus,

$$\eta\left\|y_t^N - p\right\|^2 + \left\langle F(p), y_t^N - p\right\rangle \le \frac{\lambda_t\mu}{2}\left\|F\left(y_t^N\right)\right\|^2. \tag{2.8}$$

Further, for the sake of simplicity, we put $y_t^0 = x_t$ and prove that

$$\left\|y_t^{i-1} - T_i y_t^{i-1}\right\| \longrightarrow 0, \tag{2.9}$$

as $t \to 0$ for $i = 1, \ldots, N$.

Let $\{t_k\} \subset (0,1)$ be an arbitrary sequence converging to zero as $k \to \infty$ and $x_k := x_{t_k}$. We have to prove that $\|y_k^{i-1} - T_i y_k^{i-1}\| \to 0$, where $y_k^i$ are defined by (2.5) with $t = t_k$ and $y_k^i = y_{t_k}^i$. Let $\{x_l\}$ be a subsequence of $\{x_k\}$ such that

$$\limsup_{k\to\infty}\left\|y_k^{i-1} - T_i y_k^{i-1}\right\| = \lim_{l\to\infty}\left\|y_l^{i-1} - T_i y_l^{i-1}\right\|. \tag{2.10}$$

Let $\{x_{k_j}\}$ be a subsequence of $\{x_l\}$ such that

$$\limsup_{k\to\infty}\|x_k - p\| = \lim_{j\to\infty}\left\|x_{k_j} - p\right\|. \tag{2.11}$$

From (2.6) and Lemma 2.1, it implies that

$$\begin{aligned}
\left\|x_{k_j}-p\right\|^2 &= \left\|\left(I-\lambda_{k_j}\mu F\right)y_{k_j}^N-p\right\|^2 \\
&\leq \left\|y_{k_j}^N-p\right\|^2-2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle \\
&= \left\|\left(1-\beta_{k_j}^N\right)\left(y_{k_j}^{N-1}-p\right)+\beta_{k_j}^N\left(T_N y_{k_j}^{N-1}-T_N p\right)\right\|^2 \\
&\quad -2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle \\
&\leq \left(1-\beta_{k_j}^N\right)\left\|y_{k_j}^{N-1}-p\right\|^2+\beta_{k_j}^N\left\|T_N y_{k_j}^{N-1}-T_N p\right\|^2 \\
&\quad -2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle \\
&\leq \left\|y_{k_j}^{N-1}-p\right\|^2-2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle \\
&\leq \cdots \leq \left\|y_{k_j}^1-p\right\|^2-2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle \\
&\leq \left\|x_{k_j}-p\right\|^2-2\lambda_{k_j}\mu\left\langle F\left(y_{k_j}^N\right),x_{k_j}-p\right\rangle.
\end{aligned} \tag{2.12}$$

Hence,

$$\lim_{j\to\infty}\left\|x_{k_j}-p\right\| = \lim_{j\to\infty}\left\|y_{k_j}^i-p\right\|, \quad i=1,\ldots,N. \tag{2.13}$$

By Lemma 2.1,

$$\begin{aligned}
\left\|y_{k_j}^i-p\right\|^2 &= \left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-p\right\|^2+\beta_{k_j}^i\left\|T_i y_{k_j}^{i-1}-p\right\|^2 \\
&\quad -\beta_{k_j}^i\left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-T_i y_{k_j}^{i-1}\right\|^2 \\
&\leq \left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-p\right\|^2+\beta_{k_j}^i\left\|y_{k_j}^{i-1}-p\right\|^2 \\
&\quad -\beta_{k_j}^i\left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-T_i y_{k_j}^{i-1}\right\|^2 \\
&= \left\|y_{k_j}^{i-1}-p\right\|^2-\beta_{k_j}^i\left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-T_i y_{k_j}^{i-1}\right\|^2 \\
&\leq \cdots = \left\|y_{k_j}^0-p\right\|^2-\beta_{k_j}^i\left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-T_i y_{k_j}^{i-1}\right\|^2 \\
&= \left\|x_{k_j}-p\right\|^2-\beta_{k_j}^i\left(1-\beta_{k_j}^i\right)\left\|y_{k_j}^{i-1}-T_i y_{k_j}^{i-1}\right\|^2, \quad i=1,\ldots,N.
\end{aligned} \tag{2.14}$$

Without loss of generality, we can assume that $\alpha \leq \beta_t^i \leq \beta$ for some $\alpha, \beta \in (0,1)$. Then, we have

$$\alpha(1-\beta)\left\|y_{k_j}^{i-1} - T_i y_{k_j}^{i-1}\right\|^2 \leq \left\|x_{k_j} - p\right\|^2 - \left\|y_{k_j}^i - p\right\|^2. \tag{2.15}$$

This together with (2.13) implies that

$$\lim_{j\to\infty}\left\|y_{k_j}^{i-1} - T_i y_{k_j}^{i-1}\right\|^2 = 0, \quad i = 1,\ldots,N. \tag{2.16}$$

It means that $\|y_t^{i-1} - T_i y_t^{i-1}\| \to 0$ as $t \to 0$ for $i = 1,\ldots,N$.

Next, we show that $\|x_t - T_i x_t\| \to 0$ as $t \to 0$. In fact, in the case that $i = 1$ we have $y_t^0 = x_t$. So, $\|x_t - T_1 x_t\| \to 0$ as $t \to 0$. Further, since

$$\left\|y_t^1 - T_1 x_t\right\| = \left(1 - \beta_t^1\right)\|x_t - T_1 x_t\|, \tag{2.17}$$

and $\|x_t - T_1 x_t\| \to 0$, we have that $\|y_t^1 - T_1 x_t\| \to 0$. Therefore, from

$$\left\|x_t - y_t^1\right\| \leq \|x_t - T_1 x_t\| + \left\|T_1 x_t - y_t^1\right\|, \tag{2.18}$$

it follows that $\|x_t - y_t^1\| \to 0$ as $t \to 0$. On the other hand, since

$$\begin{aligned}
\left\|y_t^2 - T_2 y_t^1\right\| &= \left(1 - \beta_t^2\right)\left\|y_t^1 - T_2 y_t^1\right\| \longrightarrow 0,\\
\left\|y_t^2 - x_t\right\| &\leq \left(1 - \beta_t^2\right)\left\|y_t^1 - x_t\right\| + \beta_t^2\left\|T_2 y_t^1 - x_t\right\|\\
&\leq \left(1 - \beta_t^2\right)\left\|y_t^1 - x_t\right\| + \beta_t^2\left\|T_2 y_t^1 - y_t^1\right\| + \left\|y_t^1 - x_t\right\|,
\end{aligned} \tag{2.19}$$

we obtain that $\|y_t^2 - x_t\| \to 0$ as $t \to 0$. Now, from

$$\begin{aligned}
\|x_t - T_2 x_t\| &\leq \left\|x_t - y_t^2\right\| + \left\|y_t^2 - T_2 y_t^1\right\| + \left\|T_2 y_t^1 - T_2 x_t\right\|\\
&\leq \left\|x_t - y_t^2\right\| + \left\|y_t^2 - T_2 y_t^1\right\| + \left\|y_t^1 - x_t\right\|,
\end{aligned} \tag{2.20}$$

and $\|x_t - y_t^2\|, \|y_t^2 - T_2 y_t^1\|, \|y_t^1 - x_t\| \to 0$, it follows that $\|x_t - T_2 x_t\| \to 0$. Similarly, we obtain that $\|x_t - T_i x_t\| \to 0$, for $i = 1,\ldots,N$ and $\|y_t^N - x_t\| \to 0$ as $t \to 0$.

Let $\{x_k\}$ be any sequence of $\{x_t\}$ converging weakly to $\tilde{p}$ as $k \to \infty$. Then, $\|x_k - T_i x_k\| \to 0$, for $i = 1,\ldots,N$ and $\{y_k^N\}$ also converges weakly to $\tilde{p}$. By Lemma 2.3, we have $\tilde{p} \in C = \cap_{i=1}^N \mathrm{Fix}(T_i)$ and from (2.8), it follows that

$$\langle F(p), p - \tilde{p}\rangle \geq 0 \quad \forall p \in C. \tag{2.21}$$

Since $p, \widetilde{p} \in C$, by replacing $p$ by $tp + (1-t)\widetilde{p}$ in the last inequality, dividing by $t$ and taking $t \to 0$ in the just obtained inequality, we obtain

$$\langle F(\widetilde{p}), p - \widetilde{p} \rangle \geq 0 \quad \forall p \in C. \tag{2.22}$$

The uniqueness of $p^*$ in (1.1) guarantees that $\widetilde{p} = p^*$. Again, replacing $p$ in (2.8) by $p^*$, we obtain the strong convergence for $\{x_t\}$. This completes the proof. □

## 3. Application

Recall that a mapping $S : H \to H$ is called a $\gamma$-strictly pseudocontractive if there exists a constant $\gamma \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \gamma \|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in H. \tag{3.1}$$

It is well known [10] that a mapping $T : H \to H$ by $Tx = \alpha x + (1-\alpha)Sx$ with a fixed $\alpha \in [\gamma, 1)$ for all $x \in H$ is a nonexpansive mapping and $\mathrm{Fix}(T) = \mathrm{Fix}(S)$. Using this fact, we can extend our result to the case $C = \cap_{i=1}^N \mathrm{Fix}(S_i)$, where $S_i$ is $\gamma_i$-strictly pseudocontractive as follows.

Let $\alpha_i \in [\gamma_i, 1)$ be fixed numbers. Then, $C = \cap_{i=1}^N \mathrm{Fix}(\widetilde{T}_i)$ with $\widetilde{T}_i y = \alpha_i y + (1-\alpha_i)S_i y$, a nonexpansive mapping, for $i = 1, \ldots, N$, and hence

$$\begin{aligned} \widetilde{T}_i^t y &= \left(1 - \beta_t^i\right) y + \beta_t^i \widetilde{T}_i y \\ &= \left(1 - \beta_t^i (1-\alpha_i)\right) y + \beta_t^i (1-\alpha_i) S_i y, \quad i = 1, \ldots, N. \end{aligned} \tag{3.2}$$

So, we have the following result.

**Theorem 3.1.** *Let $H$ be a real Hilbert space and $F : H \to H$ a mapping such that for some constants $L, \eta > 0$, $F$ is L-Lipschitz continuous and $\eta$-strongly monotone. Let $\{S_i\}_{i=1}^N$ be $N$ $\gamma_i$-strictly pseudocontractive self-maps of $H$ such that $C = \cap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$. Let $\alpha_i \in [\gamma_i, 1)$, $\mu \in (0, 2\eta/L^2)$ and let $t \in (0,1)$, $\{\lambda_t\}$, $\{\beta_t^i\} \subset (0,1)$, such that*

$$\lambda_t \longrightarrow 0, \qquad \text{as } t \longrightarrow 0, \quad 0 < \liminf_{t \to 0} \beta_t^i \leq \limsup_{t \to 0} \beta_t^i < 1, \quad i = 1, \ldots, N. \tag{3.3}$$

*Then, the net $\{x_t\}$ defined by*

$$x_t = \widetilde{T}^t x_t, \quad \widetilde{T}^t := T_0^t \widetilde{T}_N^t \cdots \widetilde{T}_1^t, \quad t \in (0,1), \tag{3.4}$$

*where $\widetilde{T}_i^t$, for $i = 1, \ldots, N$, are defined by (3.2) and $T_0^t x = (I - \lambda_t \mu F)x$, converges strongly to the unique element $p^*$ in (1.1).*

It is known in [11] that $\mathrm{Fix}(\widetilde{S}) = C$ where $\widetilde{S} = \sum_{i=1}^N \xi_i S_i$ with $\xi_i > 0$ and $\sum_{i=1}^N \xi_i = 1$ for $N$ $\gamma_i$-strictly pseudocontractions $\{S_i\}_{i=1}^N$. Moreover, $\widetilde{S}$ is $\gamma$-strictly pseudocontractive with $\gamma = \max\{\gamma_i : 1 \leq i \leq N\}$. So, we also have the following result.

**Theorem 3.2.** *Let $H$ be a real Hilbert space and $F : H \to H$ a mapping such that for some constants $L, \eta > 0$, $F$ is $L$-Lipschitz continuous and $\eta$-strongly monotone. Let $\{S_i\}_{i=1}^N$ be $N$ $\gamma_i$-strictly pseudocontractive self-maps of $H$ such that $C = \cap_{i=1}^N \operatorname{Fix}(S_i) \neq \emptyset$. Let $\alpha \in [\gamma, 1)$, where $\gamma = \max\{\gamma_i : 1 \leq i \leq N\}$, $\mu \in (0, 2\eta/L^2)$, and let $t \in (0,1)$, $\{\lambda_t\}, \{\beta_t\} \subset (0,1)$, such that*

$$\lambda_t \longrightarrow 0, \quad \text{as } t \longrightarrow 0, \qquad 0 < \liminf_{t \to 0} \beta_t \leq \limsup_{t \to 0} \beta_t < 1. \tag{3.5}$$

*Then, the net $\{x_t\}$, defined by*

$$x_t = \widetilde{T}^t x_t, \quad \widetilde{T}^t := T_0^t \left( (1 - \beta_t(1-\alpha)) I + \beta_t(1-\alpha) \sum_{i=1}^N \xi_i S_i \right), \quad t \in (0,1), \tag{3.6}$$

*where $T_0^t = (I - \lambda_t \mu F)$, $\xi_i > 0$, and $\sum_{i=1}^N \xi_i = 1$, converges strongly to the unique element $p^*$ in (1.1).*

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