

Research Article

A General Iterative Approach to Variational Inequality Problems and Optimization Problems

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We introduce a new general iterative scheme for finding a common element of the set of solutions of variational inequality problem for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space and then establish strong convergence of the sequence generated by the proposed iterative scheme to a common element of the above two sets under suitable control conditions, which is a solution of a certain optimization problem. Applications of the main result are also given.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \rightarrow C$ be self-mapping on C . We denote by $F(S)$ the set of fixed points of S and by P_C the metric projection of H onto C .

Let A be a nonlinear mapping of C into H . The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0, \quad \forall v \in C. \quad (1.1)$$

We denote the set of solutions of the variational inequality problem (1.1) by $VI(C, A)$. The variational inequality problem has been extensively studied in the literature; see [1–5] and the references therein.

Recently, in order to study the problem (1.1) coupled with the fixed point problem, many authors have introduced some iterative schemes for finding a common element of the set of the solutions of the problem (1.1) and the set of fixed points of nonexpansive mappings; see [6–9] and the references therein. In particular, in 2005, Iiduka and Takahashi [8]

introduced an iterative scheme for finding a common point of the set of fixed points of a nonexpansive mapping S and the set of solutions of the problem (1.1) for an inverse-strong monotone mapping A : $x_1 \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \quad n \geq 1, \quad (1.2)$$

where $\{\alpha_n\} \subset [0, 1)$ and $\{\lambda_n\} \subset [0, 2\alpha]$. They proved that the sequence generated by (1.2) strongly converges strongly to $P_{F(S) \cap VI(C, A)} x$. In 2010, Jung [10] provided the following new composite iterative scheme for the fixed point problem and the problem (1.1): $x_1 = x \in C$ and

$$\begin{aligned} y_n &= \alpha_n f(x_n) + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1 - \beta_n) y_n + \beta_n SP_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned} \quad (1.3)$$

where f is a contraction with constant $k \in (0, 1)$, $\{\alpha_n\}, \{\beta_n\} \in [0, 1]$, and $\{\lambda_n\} \subset [0, 2\alpha]$. He proved that the sequence $\{x_n\}$ generated by (1.3) strongly converges strongly to a point in $F(S) \cap VI(C, A)$, which is the unique solution of a certain variational inequality.

On the other hand, the following optimization problem has been studied extensively by many authors:

$$\min_{x \in \Omega} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (1.4)$$

where $\Omega = \bigcap_{n=1}^{\infty} C_n$, C_1, C_2, \dots are infinitely many closed convex subsets of H such that $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, $u \in H$, $\mu \geq 0$ is a real number, B is a strongly positive bounded linear operator on H (i.e., there is a constant $\bar{\gamma} > 0$ such that $\langle Bx, x \rangle \geq \bar{\gamma} \|x\|^2$, for all $x \in H$), and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$). For this kind of optimization problems, see, for example, Deutsch and Yamada [11], Jung [10], and Xu [12, 13] when $\Omega = \bigcap_{i=1}^N C_i$ and $h(x) = \langle x, b \rangle$ for a given point b in H .

In 2007, related to a certain optimization problem, Marino and Xu [14] introduced the following general iterative scheme for the fixed point problem of a nonexpansive mapping:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) S x_n, \quad n \geq 0, \quad (1.5)$$

where $\{\alpha_n\} \in (0, 1)$ and $\gamma > 0$. They proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in F(S), \quad (1.6)$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2} \langle Bx, x \rangle - h(x), \quad (1.7)$$

where h is a potential function for γf . The result improved the corresponding results of Moudafi [15] and Xu [16].

In this paper, motivated by the above-mentioned results, we introduce a new general composite iterative scheme for finding a common point of the set of solutions of the variational inequality problem (1.1) for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping and then prove that the sequence generated by the proposed iterative scheme converges strongly to a common point of the above two sets, which is a solution of a certain optimization problem. Applications of the main result are also discussed. Our results improve and complement the corresponding results of Chen et al. [6], Iiduka and Takahashi [8], Jung [10], and others.

2. Preliminaries and Lemmas

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H . We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x . $x_n \rightarrow x$ implies that $\{x_n\}$ converges strongly to x .

First we recall that a mapping $f : C \rightarrow C$ is a *contraction* on C if there exists a constant $k \in (0, 1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|$, $x, y \in C$. A mapping $T : C \rightarrow C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$. We denote by $F(T)$ the set of fixed points of T .

For every point $x \in H$, there exists a unique nearest point in C , denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \leq \|x - y\| \quad (2.1)$$

for all $y \in C$. P_C is called the *metric projection* of H onto C . It is well known that P_C is nonexpansive and P_C satisfies

$$\langle x - y, P_C(x) - P_C(y) \rangle \geq \|P_C(x) - P_C(y)\|^2 \quad (2.2)$$

for every $x, y \in H$. Moreover, $P_C(x)$ is characterized by the properties:

$$\begin{aligned} \|x - y\|^2 &\geq \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \\ u = P_C(x) &\iff \langle x - u, u - y \rangle \geq 0, \quad \forall x \in H, y \in C. \end{aligned} \quad (2.3)$$

In the context of the variational inequality problem for a nonlinear mapping A , this implies that

$$u \in \text{VI}(C, A) \iff u = P_C(u - \lambda Au), \quad \text{for any } \lambda > 0. \quad (2.4)$$

It is also well known that H satisfies the *Opial condition*, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n \rightarrow \infty} \|x_n - x\| < \liminf_{n \rightarrow \infty} \|x_n - y\| \quad (2.5)$$

holds for every $y \in H$ with $y \neq x$.

A mapping A of C into H is called *inverse-strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 \quad (2.6)$$

for all $x, y \in C$; see [4, 7, 17]. For such a case, A is called α -inverse-strongly monotone. We know that if $A = I - T$, where T is a nonexpansive mapping of C into itself and I is the identity mapping of H , then A is $1/2$ -inverse-strongly monotone and $\text{VI}(C, A) = F(T)$. A mapping A of C into H is called *strongly monotone* if there exists a positive real number η such that

$$\langle x - y, Ax - Ay \rangle \geq \eta \|x - y\|^2 \quad (2.7)$$

for all $x, y \in C$. In such a case, we say A is η -strongly monotone. If A is η -strongly monotone and κ -Lipschitz continuous, that is, $\|Ax - Ay\| \leq \kappa \|x - y\|$ for all $x, y \in C$, then A is η/κ^2 -inverse-strongly monotone. If A is an α -inverse-strongly monotone mapping of C into H , then it is obvious that A is $1/\alpha$ -Lipschitz continuous. We also have that for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda \langle x - y, Ax - Ay \rangle + \lambda^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 + \lambda(\lambda - 2\alpha) \|Ax - Ay\|^2. \end{aligned} \quad (2.8)$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping of C into H . The following result for the existence of solutions of the variational inequality problem for inverse strongly-monotone mappings was given in Takahashi and Toyoda [9].

Proposition 2.1. *Let C be a bounded closed convex subset of a real Hilbert space and let A be an α -inverse-strongly monotone mapping of C into H . Then, $\text{VI}(C, A)$ is nonempty.*

A set-valued mapping $T : H \rightarrow 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \rightarrow 2^H$ is *maximal* if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the *normal cone* to C at v , that is, $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \text{ for all } u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \quad (2.9)$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \text{VI}(C, A)$; see [18, 19].

We need the following lemmas for the proof of our main results.

Lemma 2.2. *In a real Hilbert space H , there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad (2.10)$$

for all $x, y \in H$.

Lemma 2.3 (Xu [12]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \beta_n + \gamma_n, \quad n \geq 1, \quad (2.11)$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the following conditions:

- (i) $\{\lambda_n\} \subset [0, 1]$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ or, equivalently, $\prod_{n=1}^{\infty} (1 - \lambda_n) = 0$;
- (ii) $\limsup_{n \rightarrow \infty} (\beta_n / \lambda_n) \leq 0$ or $\sum_{n=1}^{\infty} |\beta_n| < \infty$;
- (iii) $\gamma_n \geq 0$ ($n \geq 1$), $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n \rightarrow \infty} s_n = 0$.

Lemma 2.4 (Marino and Xu [14]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with constant $\bar{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

The following lemma can be found in [20, 21] (see also Lemma 2.2 in [22]).

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $g : C \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper lower semicontinuous differentiable convex function. If x^* is a solution to the minimization problem*

$$g(x^*) = \inf_{x \in C} g(x), \quad (2.12)$$

then

$$\langle g'(x), x - x^* \rangle \geq 0, \quad x \in C. \quad (2.13)$$

In particular, if x^* solves the optimization problem

$$\min_{x \in C} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (2.14)$$

then

$$\langle u + (\gamma f - (I + \mu B))x^*, x - x^* \rangle \leq 0, \quad x \in C, \quad (2.15)$$

where h is a potential function for γf .

3. Main Results

In this section, we present a new general composite iterative scheme for inverse-strongly monotone mappings and a nonexpansive mapping.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H such that $C \pm C \subset C$. Let A be an α -inverse-strongly monotone mapping of C into H and S a nonexpansive mapping of C into itself such that $F(S) \cap \text{VI}(C, A) \neq \emptyset$. Let $u \in C$ and let B be a strongly positive bounded linear operator on C with constant $\bar{\gamma} \in (0, 1)$ and f a contraction of C into itself with constant $k \in (0, 1)$. Assume that $\mu > 0$ and $0 < \gamma < (1 + \mu)\bar{\gamma}/k$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))SP_C(x_n - \lambda_n Ax_n), \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n SP_C(y_n - \lambda_n Ay_n), \quad n \geq 1, \end{aligned} \quad (\text{IS})$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$, and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the following conditions:

- (i) $\alpha_n \rightarrow 0$ ($n \rightarrow \infty$); $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\beta_n \subset [0, a)$ for all $n \geq 0$ and for some $a \in (0, 1)$;
- (iii) $\lambda_n \in [c, d]$ for some c, d with $0 < c < d < 2\alpha$;
- (iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.

Then $\{x_n\}$ converges strongly to $q \in F(S) \cap \text{VI}(C, A)$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap \text{VI}(C, A)} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (\text{OP1})$$

where h is a potential function for γf .

Proof. We note that from the control condition (i), we may assume, without loss of generality, that $\alpha_n \leq (1 + \mu\|B\|)^{-1}$. Recall that if B is bounded linear self-adjoint operator on H , then

$$\|B\| = \sup\{|\langle Bu, u \rangle| : u \in H, \|u\| = 1\}. \quad (3.1)$$

Observe that

$$\begin{aligned} \langle (I - \alpha_n(I + \mu B))u, u \rangle &= 1 - \alpha_n - \alpha_n \mu \langle Bu, u \rangle \\ &\geq 1 - \alpha_n - \alpha_n \mu \|B\| \\ &\geq 0, \end{aligned} \quad (3.2)$$

which is to say that $I - \alpha_n(I + \mu B)$ is positive. It follows that

$$\begin{aligned}
\|I - \alpha_n(I + \mu B)\| &= \sup\{\langle (I - \alpha_n(I + \mu B))u, u \rangle : u \in H, \|u\| = 1\} \\
&= \sup\{1 - \alpha_n - \alpha_n\mu\langle Bu, u \rangle : u \in H, \|u\| = 1\} \\
&\leq 1 - \alpha_n(1 + \mu\bar{\gamma}) \\
&< 1 - \alpha_n(1 + \mu)\bar{\gamma}.
\end{aligned} \tag{3.3}$$

Now we divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $z_n = P_C(x_n - \lambda_n Ax_n)$ and $w_n = P_C(y_n - \lambda_n Ay_n)$ for every $n \geq 1$. Let $p \in F(S) \cap \text{VI}(C, A)$. Since $I - \lambda_n A$ is nonexpansive and $p = P_C(p - \lambda_n Ap)$ from (2.4), we have

$$\begin{aligned}
\|z_n - p\| &\leq \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\| \\
&\leq \|x_n - p\|.
\end{aligned} \tag{3.4}$$

Similarly, we have

$$\|w_n - p\| \leq \|y_n - p\|. \tag{3.5}$$

Now, set $\bar{B} = (I + \mu B)$. Let $p \in F(S) \cap \text{VI}(C, A)$. Then, from (IS) and (3.4), we obtain

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n u + \alpha_n(\gamma f(x_n) - \bar{B}p) + (I - \alpha_n \bar{B})(Sz_n - p)\| \\
&\leq (1 - (1 + \mu)\bar{\gamma}\alpha_n)\|z_n - p\| + \alpha_n\|u\| \\
&\quad + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - \bar{B}p\| \\
&\leq (1 - (1 + \mu)\bar{\gamma}\alpha_n)\|z_n - p\| + \alpha_n\|u\| \\
&\quad + \alpha_n\gamma k\|x_n - p\| + \alpha_n\|\gamma f(p) - \bar{B}p\| \\
&= (1 - ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n)\|x_{n+1} - p\| \\
&\quad + ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n \frac{\|\gamma f(p) - \bar{B}p\| + \|u\|}{(1 + \mu)\bar{\gamma} - \gamma k}.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \beta_n)(y_n - p) + \beta_n(Sw_n - p)\| \\
&\leq (1 - \beta_n)\|y_n - p\| + \beta_n\|w_n - p\| \\
&\leq (1 - \beta_n)\|y_n - p\| + \beta_n\|y_n - p\| \\
&= \|y_n - p\| \\
&\leq \max \left\{ \|x_n - p\|, \frac{\|\gamma f(p) - \bar{B}p\| + \|u\|}{(1 + \mu)\bar{\gamma} - \gamma k} \right\}.
\end{aligned} \tag{3.7}$$

By induction, it follows from (3.7) that

$$\|x_n - p\| \leq \max \left\{ \|x_1 - p\|, \frac{\|\gamma f(p) - \bar{B}p\| + \|u\|}{(1 + \mu)\bar{\gamma} - \gamma k} \right\} \quad n \geq 1. \tag{3.8}$$

Therefore, $\{x_n\}$ is bounded. So $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{f(x_n)\}$, $\{Ax_n\}$, $\{Ay_n\}$, and $\{\bar{B}Sz_n\}$ are bounded. Moreover, since $\|Sz_n - p\| \leq \|x_n - p\|$ and $\|Sw_n - p\| \leq \|y_n - p\|$, $\{Sz_n\}$ and $\{Sw_n\}$ are also bounded. And by the condition (i), we have

$$\begin{aligned}
\|y_n - Sz_n\| &= \alpha_n \|(u + \gamma f(x_n)) - (I + \mu B)Sz_n\| \\
&= \alpha_n \|(u + \gamma f(x_n)) - \bar{B}Sz_n\| \longrightarrow 0 \quad (\text{as } n \longrightarrow \infty).
\end{aligned} \tag{3.9}$$

Step 2. We show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0$. Indeed, since $I - \lambda_n A$ and P_C are nonexpansive and $z_n = P_C(x_n - \lambda_n Ax_n)$, we have

$$\begin{aligned}
\|z_n - z_{n-1}\| &\leq \|(x_n - \lambda_n Ax_n) - (x_{n-1} - \lambda_{n-1} Ax_{n-1})\| \\
&\leq \|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Ax_{n-1}\|.
\end{aligned} \tag{3.10}$$

Similarly, we get

$$\|w_n - w_{n-1}\| \leq \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Ay_{n-1}\|. \tag{3.11}$$

Simple calculations show that

$$\begin{aligned}
y_n - y_{n-1} &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n \bar{B})Sz_n - \alpha_{n-1}(u + \gamma f(x_{n-1})) - (I - \alpha_{n-1} \bar{B})Sz_{n-1} \\
&= (\alpha_n - \alpha_{n-1})(u + \gamma f(x_{n-1}) - \bar{B}Sz_{n-1}) + \alpha_n \gamma (f(x_n) - f(x_{n-1})) \\
&\quad + (I - \alpha_n \bar{B})(Sz_n - Sz_{n-1}).
\end{aligned} \tag{3.12}$$

So, we obtain

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq |\alpha_n - \alpha_{n-1}| \left(\|u\| + \gamma \|f(x_{n-1})\| + \|\bar{B}\| \|Sz_{n-1}\| \right) \\
&\quad + \alpha_n \gamma k \|x_n - x_{n-1}\| + (1 - (1 + \mu)\bar{\gamma}\alpha_n) \|z_n - z_{n-1}\| \\
&\leq |\alpha_n - \alpha_{n-1}| \left(\|u\| + \gamma \|f(x_{n-1})\| + \|\bar{B}\| \|Sz_{n-1}\| \right) \\
&\quad + \alpha_n \gamma k \|x_n - x_{n-1}\| + (1 - (1 + \mu)\bar{\gamma}\alpha_n) \|x_n - x_{n-1}\| \\
&\quad + |\lambda_n - \lambda_{n-1}| \|Ax_{n-1}\|.
\end{aligned} \tag{3.13}$$

Also observe that

$$\begin{aligned}
x_{n+1} - x_n &= (1 - \beta_n)(y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(Sw_{n-1} - y_{n-1}) \\
&\quad + \beta_n(Sw_n - Sw_{n-1}).
\end{aligned} \tag{3.14}$$

By (3.11), (3.13), and (3.14), we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq (1 - \beta_n) \|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}| (\|Sx_{n-1}\| + \|y_{n-1}\|) \\
&\quad + \beta_n \|w_n - w_{n-1}\| \\
&\leq (1 - \beta_n) \|y_n - y_{n-1}\| + \beta_n \|y_n - y_{n-1}\| + \beta_n |\lambda_n - \lambda_{n-1}| \|Ay_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| (\|Sw_{n-1}\| + \|y_{n-1}\|) \\
&\leq \|y_n - y_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Ay_{n-1}\| + |\beta_n - \beta_{n-1}| (\|Sw_{n-1}\| + \|y_{n-1}\|) \\
&\leq (1 - ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n) \|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left(\|u\| + \gamma \|f(x_{n-1})\| + \|\bar{B}\| \|Sz_{n-1}\| \right) \\
&\quad + |\lambda_n - \lambda_{n-1}| (\|Ay_{n-1}\| + \|Ax_{n-1}\|) + |\beta_n - \beta_{n-1}| (\|Sw_{n-1}\| + \|y_{n-1}\|) \\
&\leq (1 - ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n) \|x_n - x_{n-1}\| \\
&\quad + M_1 |\alpha_n - \alpha_{n-1}| + M_2 |\lambda_n - \lambda_{n-1}| + M_3 |\beta_n - \beta_{n-1}|,
\end{aligned} \tag{3.15}$$

where $M_1 = \sup\{\|u\| + \gamma \|f(x_n)\| + \|\bar{B}\| \|T_n z_n\| : n \geq 1\}$, $M_2 = \sup\{\|Ay_n\| + \|Ax_n\| : n \geq 1\}$, and $M_3 = \sup\{\|Sw_n\| + \|y_n\| : n \geq 1\}$. From the conditions (i) and (iv), it is easy to see that

$$\begin{aligned}
\lim_{n \rightarrow \infty} ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n &= 0, \quad \sum_{n=1}^{\infty} ((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n < \infty, \\
\sum_{n=2}^{\infty} (M_1 |\alpha_n - \alpha_{n-1}| + M_2 |\lambda_n - \lambda_{n-1}| + M_3 |\beta_n - \beta_{n-1}|) &< \infty.
\end{aligned} \tag{3.16}$$

Applying Lemma 2.3 to (3.15), we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \quad (3.17)$$

Moreover, by (3.10) and (3.13), we also have

$$\lim_{n \rightarrow \infty} \|z_{n+1} - z_n\| = 0, \quad \lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0. \quad (3.18)$$

Step 3. We show that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ and $\lim_{n \rightarrow \infty} \|x_n - Sz_n\| = 0$. Indeed,

$$\begin{aligned} \|x_{n+1} - y_n\| &= \beta_n \|S\omega_n - y_n\| \\ &\leq \beta_n (\|S\omega_n - Sz_n\| + \|Sz_n - y_n\|) \\ &\leq a (\|\omega_n - z_n\| + \|Sz_n - y_n\|) \\ &\leq a (\|y_n - x_n\| + \|Sz_n - y_n\|) \\ &\leq a (\|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| + \|Sz_n - y_n\|) \end{aligned} \quad (3.19)$$

which implies that

$$\|x_{n+1} - y_n\| \leq \frac{a}{1-a} (\|x_{n+1} - x_n\| + \|Sz_n - y_n\|). \quad (3.20)$$

Obviously, by (3.9) and Step 2, we have $\|x_{n+1} - y_n\| \rightarrow 0$ as $n \rightarrow \infty$. This implies that

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.21)$$

By (3.9) and (3.21), we also have

$$\|x_n - Sz_n\| \leq \|x_n - y_n\| + \|y_n - Sz_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.22)$$

Step 4. We show that $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$ and $\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0$. To this end, let $p \in F(S) \cap VI(C, A)$. Since $z_n = P_C(x_n - \lambda_n Ax_n)$ and $p = P_C(p - \lambda_n p)$, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \left\| \alpha_n (u + \gamma f(x_n) - \bar{B}p) + (I - \alpha_n \bar{B})(Sz_n - p) \right\|^2 \\
&\leq \left(\alpha_n \|u + \gamma f(x_n) - \bar{B}p\| + \|I - \alpha_n \bar{B}\| \|Sz_n - p\| \right)^2 \\
&\leq \alpha_n \|u + \gamma f(x_n) - \bar{B}p\|^2 + (1 - \alpha_n(1 + \mu)\bar{\gamma}) \|z_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n(1 + \mu)\bar{\gamma}) \|u + \gamma f(x_n) - \bar{B}p\| \|z_n - p\| \\
&\leq \alpha_n \|u + \gamma f(x_n) - \bar{B}p\|^2 \\
&\quad + (1 - \alpha_n(1 + \mu)\bar{\gamma}) \left[\|x_n - p\|^2 + \lambda_n(\lambda_n - 2\alpha) \|Ax_n - Ap\|^2 \right] \\
&\quad + 2\alpha_n(1 - \alpha_n(1 + \mu)\bar{\gamma}) \left\| \gamma u + f(x_n) - \bar{B}p \right\| \|z_n - p\| \\
&\leq \alpha_n \|u + \gamma f(x_n) - \bar{B}p\|^2 + \|x_n - p\|^2 \\
&\quad + (1 - \alpha_n(1 + \mu)\bar{\gamma}) c(d - 2\alpha) \|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n \left\| \gamma u + f(x_n) - \bar{B}p \right\| \|z_n - p\|.
\end{aligned} \tag{3.23}$$

So we obtain

$$\begin{aligned}
&- (1 - \alpha_n(1 + \mu)\bar{\gamma}) c(d - 2\alpha) \|Ax_n - Ap\|^2 \\
&\leq \alpha_n \left\| \gamma u + f(x_n) - \bar{B}p \right\|^2 + (\|x_n - p\| + \|y_n - p\|) (\|x_n - p\| - \|y_n - p\|) \\
&\quad + 2\alpha_n \left\| \gamma u + f(x_n) - \bar{B}p \right\| \|z_n - p\| \\
&\leq \alpha_n \left\| \gamma u + f(x_n) - \bar{B}p \right\|^2 + (\|x_n - p\| + \|y_n - p\|) \|x_n - y_n\| \\
&\quad + 2\alpha_n \left\| \gamma u + f(x_n) - \bar{B}p \right\| \|z_n - p\|.
\end{aligned} \tag{3.24}$$

Since $\alpha_n \rightarrow 0$ from the condition (i) and $\|x_n - y_n\| \rightarrow 0$ from Step 3, we have $\|Ax_n - Ap\| \rightarrow 0$ ($n \rightarrow \infty$). Moreover, from (2.4) we obtain

$$\begin{aligned}
\|z_n - p\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(p - \lambda_n Ap)\|^2 \\
&\leq \langle x_n - \lambda_n Ax_n - (p - \lambda_n Ap), z_n - p \rangle \\
&= \frac{1}{2} \left\{ \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap)\|^2 + \|z_n - p\|^2 \right. \\
&\quad \left. - \|(x_n - \lambda_n Ax_n) - (p - \lambda_n Ap) - (z_n - p)\|^2 \right\} \\
&\leq \frac{1}{2} \left\{ \|x_n - p\|^2 + \|z_n - p\|^2 - \|x_n - z_n\|^2 \right. \\
&\quad \left. + 2\lambda_n \langle x_n - z_n, Ax_n - Ap \rangle - \lambda_n^2 \|Ax_n - Ap\|^2 \right\},
\end{aligned} \tag{3.25}$$

and so

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - z_n\|^2 + 2\lambda_n \langle x_n - z_n, Ax_n - Ap \rangle - \lambda_n^2 \|Ax_n - Ap\|^2. \tag{3.26}$$

Thus

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n \|u + \gamma f(x_n) - \bar{B}p\|^2 + (1 - \alpha_n(1 + \mu)\bar{\gamma}) \|z_n - p\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n(1 + \mu)\bar{\gamma}) \|\gamma u + f(x_n) - \bar{B}p\| \|z_n - p\| \\
&\leq \alpha_n \|u + \gamma f(x_n) - \bar{B}p\|^2 + \|x_n - p\|^2 - (1 - \alpha_n(1 + \mu)\bar{\gamma}) \|x_n - z_n\|^2 \\
&\quad + 2(1 - \alpha_n(1 + \mu)\bar{\gamma}) \lambda_n \langle x_n - z_n, Ax_n - Ap \rangle \\
&\quad - (1 - \alpha_n(1 + \mu)\bar{\gamma}) \lambda_n^2 \|Ax_n - Ap\|^2 \\
&\quad + 2\alpha_n \|u + \gamma f(x_n) - \bar{B}p\| \|z_n - p\|.
\end{aligned} \tag{3.27}$$

Then, we have

$$\begin{aligned}
& (1 - \alpha_n(1 + \mu)\bar{\gamma})\|x_n - z_n\|^2 \\
& \leq \alpha_n \left\| u + \gamma f(x_n) - \bar{B}p \right\|^2 + (\|x_n - p\| + \|y_n - p\|)(\|x_n - p\| - \|y_n - p\|) \\
& \quad + 2(1 - \alpha_n(1 + \mu)\bar{\gamma})\lambda_n \langle x_n - z_n, Ax_n - Ap \rangle - (1 - \alpha_n(1 + \mu)\bar{\gamma})\lambda_n^2 \|Ax_n - Ap\|^2 \\
& \quad + 2\alpha_n \left\| u + \gamma f(x_n) - \bar{B}p \right\| \|z_n - p\| \\
& \leq \alpha_n \left\| u + \gamma f(x_n) - \bar{B}p \right\|^2 + (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\
& \quad + 2(1 - \alpha_n(1 + \mu)\bar{\gamma})\lambda_n \langle x_n - z_n, Ax_n - Ap \rangle - (1 - \alpha_n(1 + \mu)\bar{\gamma})\lambda_n^2 \|Ax_n - Ap\|^2 \\
& \quad + 2\alpha_n \left\| u + \gamma f(x_n) - \bar{B}p \right\| \|z_n - p\|.
\end{aligned} \tag{3.28}$$

Since $\alpha_n \rightarrow 0$, $\|x_n - y_n\| \rightarrow 0$ and $\|Ax_n - Au\| \rightarrow 0$, we get $\|x_n - z_n\| \rightarrow 0$. Also by (3.21)

$$\|y_n - z_n\| \leq \|y_n - x_n\| + \|x_n - z_n\| \rightarrow 0 \quad (n \rightarrow \infty). \tag{3.29}$$

Step 5. We show that $\lim_{n \rightarrow \infty} \|Sz_n - z_n\| = 0$. In fact, since

$$\begin{aligned}
\|Sz_n - z_n\| & \leq \|Sz_n - y_n\| + \|y_n - z_n\| \\
& = \alpha_n \left\| u + \gamma f(x_n) - \bar{B}Sz_n \right\| + \|y_n - z_n\|,
\end{aligned} \tag{3.30}$$

from (3.9) and (3.29), we have $\lim_{n \rightarrow \infty} \|Sz_n - z_n\| = 0$.

Step 6. We show that

$$\limsup_{n \rightarrow \infty} \langle u + (\gamma f - (I + \mu B))q, y_n - q \rangle = \limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \leq 0, \tag{3.31}$$

where q is a solution of the optimization problem (OP1). First we prove that

$$\limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_n - q \rangle \leq 0. \tag{3.32}$$

Since $\{z_n\}$ is bounded, we can choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_n - q \rangle = \lim_{i \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_{n_i} - q \rangle. \quad (3.33)$$

Without loss of generality, we may assume that $\{z_{n_i}\}$ converges weakly to $z \in C$.

Now we will show that $z \in F(S) \cap VI(C, A)$. First we show that $z \in F(S)$. Assume that $z \notin F(S)$. Since $z_{n_i} \rightharpoonup z$ and $Sz \neq z$, by the Opial condition and Step 5, we obtain

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|z_{n_i} - z\| &< \liminf_{i \rightarrow \infty} \|z_{n_i} - Sz\| \\ &\leq \liminf_{i \rightarrow \infty} (\|z_{n_i} - Sz_{n_i}\| + \|Sz_{n_i} - Sz\|) \\ &= \liminf_{i \rightarrow \infty} \|Sz_{n_i} - Sz\| \\ &\leq \liminf_{i \rightarrow \infty} \|z_{n_i} - z\|, \end{aligned} \quad (3.34)$$

which is a contradiction. Thus we have $z \in F(S)$.

Next, let us show that $z \in VI(C, A)$. Let

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \quad (3.35)$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, w - Av \rangle \geq 0. \quad (3.36)$$

On the other hand, from $z_n = P_C(x_n - \lambda_n Ax_n)$, we have $\langle v - z_n, z_n - (x_n - \lambda_n Ax_n) \rangle \geq 0$ and hence

$$\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + Ax_n \rangle \geq 0. \quad (3.37)$$

Therefore, we have

$$\begin{aligned}
\langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Av \rangle \\
&\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i} \right\rangle \\
&= \left\langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&= \langle v - z_{n_i}, Av - Az_{n_i} \rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle.
\end{aligned} \tag{3.38}$$

Since $\|z_n - x_n\| \rightarrow 0$ in Step 4 and A is α -inverse-strongly monotone, we have $\langle v - z, w \rangle \geq 0$ as $i \rightarrow \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \text{VI}(C, A)$.

Therefore, $z \in F(S) \cap \text{VI}(C, A)$. Now from Lemma 2.5 and Step 5, we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_n - q \rangle &= \lim_{i \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_{n_i} - q \rangle \\
&= \lim_{i \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, z_{n_i} - q \rangle \\
&= \langle u + (\gamma f - \bar{B})q, z - q \rangle \\
&\leq 0.
\end{aligned} \tag{3.39}$$

By (3.9) and (3.39), we conclude that

$$\begin{aligned}
&\limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \\
&\leq \limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, y_n - Sz_n \rangle + \limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_n - q \rangle \\
&\leq \limsup_{n \rightarrow \infty} \|u + (\gamma f - \bar{B})q\| \|y_n - Sz_n\| + \limsup_{n \rightarrow \infty} \langle u + (\gamma f - \bar{B})q, Sz_n - q \rangle \\
&\leq 0.
\end{aligned} \tag{3.40}$$

Step 7. We show that $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$ and $\lim_{n \rightarrow \infty} \|u_n - q\| = 0$, where q is a solution of the optimization problem (OP1). Indeed from (IS) and Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \|y_n - q\|^2 \\
&= \left\| \alpha_n (u + \gamma f(x_n) - \bar{B}q) + (I - \alpha_n \bar{B})(Sz_n - q) \right\|^2 \\
&\leq \left\| (I - \alpha_n \bar{B})(Sz_n - q) \right\|^2 + 2\alpha_n \langle u + \gamma f(x_n) - \bar{B}q, y_n - q \rangle \\
&\leq (1 - (1 + \mu)\bar{\gamma}\alpha_n)^2 \|z_n - q\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(q), y_n - q \rangle \\
&\quad + 2\alpha_n \langle u + \gamma f(q) - \bar{B}q, y_n - q \rangle \\
&\leq (1 - (1 + \mu)\bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|x_n - q\| \|y_n - q\| \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \\
&\leq (1 - (1 + \mu)\bar{\gamma}\alpha_n)^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|x_n - q\| (\|y_n - x_n\| + \|x_n - q\|) \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \\
&= (1 - 2((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n) \|x_n - q\|^2 \\
&\quad + \alpha_n^2 ((1 + \mu)\bar{\gamma})^2 \|x_n - q\|^2 + 2\alpha_n \gamma k \|x_n - q\| \|y_n - x_n\| \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{B})q, y_n - q \rangle,
\end{aligned} \tag{3.41}$$

that is,

$$\begin{aligned}
\|x_{n+1} - q\|^2 &\leq (1 - 2((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n) \|x_n - q\|^2 \\
&\quad + \alpha_n^2 ((1 + \mu)\bar{\gamma})^2 M_4^2 + 2\alpha_n \gamma k \|y_n - x_n\| M_4 \\
&\quad + 2\alpha_n \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \\
&= (1 - \bar{\alpha}_n) \|x_n - q\|^2 + \bar{\beta}_n,
\end{aligned} \tag{3.42}$$

where $M_4 = \sup\{\|x_n - q\| : n \geq 1\}$, $\bar{\alpha}_n = 2((1 + \mu)\bar{\gamma} - \gamma k)\alpha_n$, and

$$\bar{\beta}_n = \alpha_n \left[\alpha_n (1 + \mu\bar{\gamma})^2 M_4^2 + 2\gamma k \|y_n - x_n\| M_4 + 2 \langle u + (\gamma f - \bar{B})q, y_n - q \rangle \right]. \tag{3.43}$$

From (i), $\|y_n - x_n\| \rightarrow 0$ in Steps 3, and 6, it is easily seen that $\bar{\alpha}_n \rightarrow 0$, $\sum_{n=1}^{\infty} \bar{\alpha}_n = \infty$, and $\limsup_{n \rightarrow \infty} (\bar{\beta}_n / \bar{\alpha}_n) \leq 0$. Hence, by Lemma 2.3, we conclude $x_n \rightarrow q$ as $n \rightarrow \infty$. This completes the proof. \square

As a direct consequence of Theorem 3.1, we have the following results.

Corollary 3.2. Let $H, C, S, B, f, u, \gamma, \bar{\gamma}, k$, and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))Sx_n, \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n S y_n, \quad n \geq 1, \end{aligned} \quad (3.44)$$

where $\{\alpha_n\}$ and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i), (ii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S)$, which is a solution of the optimization problem

$$\min_{x \in F(S)} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (\text{OP2})$$

where h is a potential function for γf .

Corollary 3.3. Let $H, C, A, B, f, u, \gamma, \bar{\gamma}, k$, and μ be as in Theorem 3.1. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))P_C(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n P_C(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned} \quad (3.45)$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$, and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in \text{VI}(C, A)$, which is a solution of the optimization problem

$$\min_{x \in \text{VI}(C, A)} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (\text{OP3})$$

where h is a potential function for γf .

Remark 3.4. (1) Theorem 3.1 and Corollary 3.3 improve and develop the corresponding results in Chen et al. [6], Iiduka and Takahashi [8], and Jung [10].

(2) Even though $\beta_n = 0$ for $n \geq 1$, the iterative scheme (3.44) in Corollary 3.2 is a new one for fixed point problem of a nonexpansive mapping.

4. Applications

In this section, as in [6, 8, 10], we prove two theorems by using Theorem 3.1. First of all, we recall the following definition.

A mapping $T : C \rightarrow C$ is called *strictly pseudocontractive* if there exists α with $0 \leq \alpha < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \alpha\|(I - T)x - (I - T)y\|^2 \quad (4.1)$$

for every $x, y \in C$. If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \rightarrow C$ is a strictly pseudo-contractive mapping with constant α . Then A is $(1 - \alpha)/2$ -inverse-strongly monotone; see [2]. Actually, we have, for all $x, y \in C$,

$$\|(I - A)x - (I - A)y\|^2 \leq \|x - y\|^2 + \alpha\|Ax - Ay\|^2. \quad (4.2)$$

On the other hand, since H is a real Hilbert space, we have

$$\|(I - A)x - (I - A)y\|^2 = \|x - y\|^2 + \|Ax - Ay\|^2 - 2\langle x - y, Ax - Ay \rangle. \quad (4.3)$$

Hence we have

$$\langle x - y, Ax - Ay \rangle \geq \frac{1 - \alpha}{2} \|Ax - Ay\|^2. \quad (4.4)$$

Using Theorem 3.1, we found a strong convergence theorem for finding a common fixed point of a nonexpansive mapping and a strictly pseudo-contractive mapping.

Theorem 4.1. *Let $H, C, S, B, f, u, \gamma, \bar{\gamma}, k$, and μ be as in Theorem 3.1. Let T be an α -strictly pseudo-contractive mapping of C into itself such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))S((1 - \lambda_n)x_n + \lambda_n T x_n), \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n S((1 - \lambda_n)y_n + \lambda_n T y_n), \quad n \geq 1, \end{aligned} \quad (4.5)$$

where $\{\lambda_n\} \subset [0, 1 - \alpha]$, $\{\alpha_n\} \subset [0, 1)$, and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap F(T)$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap F(T)} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (\text{OP4})$$

where h is a potential function for γf .

Proof. Put $A = I - T$. Then A is $(1 - \alpha)/2$ -inverse-strongly monotone. We have $F(T) = \text{VI}(C, A)$ and $P_C(x_n - \lambda_n A x_n) = (1 - \lambda_n)x_n + \lambda_n T x_n$. Thus, the desired result follows from Theorem 3.1. \square

Using Theorem 3.1, we also obtain the following result.

Theorem 4.2. Let H be a real Hilbert space. Let A be an α -inverse-strongly monotone mapping of H into H and S a nonexpansive mapping of H into itself such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $u \in H$, and let B be a strongly positive bounded linear operator on H with constant $\bar{\gamma} > 0$ and $f : H \rightarrow H$ a contraction with constant $k \in (0, 1)$. Assume that $\mu > 0$ and $0 < \gamma < (1 + \mu)\bar{\gamma}/k$. Let $\{x_n\}$ be a sequence generated by

$$\begin{aligned} x_1 &= x \in H, \\ y_n &= \alpha_n(u + \gamma f(x_n)) + (I - \alpha_n(I + \mu B))S(x_n - \lambda_n A x_n), \\ x_{n+1} &= (1 - \beta_n)y_n + \beta_n S(y_n - \lambda_n A y_n), \quad n \geq 1, \end{aligned} \quad (4.6)$$

where $\{\lambda_n\} \subset [0, 2\alpha]$, $\{\alpha_n\} \subset [0, 1)$, and $\{\beta_n\} \subset [0, 1]$. Let $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\beta_n\}$ satisfy the conditions (i), (ii), (iii), and (iv) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $q \in F(S) \cap A^{-1}0$, which is a solution of the optimization problem

$$\min_{x \in F(S) \cap A^{-1}0} \frac{\mu}{2} \langle Bx, x \rangle + \frac{1}{2} \|x - u\|^2 - h(x), \quad (\text{OP5})$$

where h is a potential function for γf .

Proof. We have $A^{-1}0 = VI(H, A)$. So, putting $P_H = I$, by Theorem 3.1, we obtain the desired result. \square

Remark 4.3. (1) Theorems 4.1 and 4.2 complement and develop the corresponding results in Chen et al. [6] and Jung [10].

(2) In all our results, we can replace the condition $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ on the control parameter $\{\alpha_n\}$ by the condition $\alpha_n \in (0, 1]$ for $n \geq 1$, $\lim_{n \rightarrow \infty} \alpha_n / \alpha_{n+1} = 1$ [12, 13] or by the perturbed control condition $|\alpha_{n+1} - \alpha_n| < o(\alpha_{n+1}) + \sigma_n, \sum_{n=1}^{\infty} \sigma_n < \infty$ [23].

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