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Research Article

Existence and Stability of Solutions for Implicit Multivalued Vector Equilibrium Problems

Sanhua Wang[1] and Qiuying Li[2]

[1] *Department of Mathematics, Nanchang University, Nanchang, Jiangxi 330031, China*
[2] *College of Science and Technology, Nanchang University, Nanchang, Jiangxi 330029, China*

Correspondence should be addressed to Sanhua Wang, wsh_315@163.com

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A class of implicit multivalued vector equilibrium problems is studied. By using the generalized Fan-Browder fixed point theorem, some existence results of solutions for the implicit multivalued vector equilibrium problems are obtained under some suitable assumptions. Moreover, a stability result of solutions for the implicit multivalued vector equilibrium problems is derived. These results extend and unify some recent results for implicit vector equilibrium problems, multivalued vector variational inequality problems, and vector variational inequality problems.

1. Introduction

Let X be a Hausdorff topological vector space, K a nonempty subset of X, and $f : K \times K \to R$ a function. Then, the scalar equilibrium problem consists in finding $\overline{x} \in K$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in K. \tag{1.1}$$

This problem provides a unifying framework for many important problems, such as, optimization problems, variational inequality problems, complementary problems, minimal inequality problems, and fixed point problems, and has been widely applied to study the problems arising in economics, mechanics, and engineering science (see [1]). In recent years, lots of existence results concerning equilibrium problems and variational inequality problems have been established by many authors in different ways. For details, we refer the reader to [1–26] and the references therein.

Now, let Y be another Hausdorff topological vector space and $C \subseteq Y$ a closed convex cone with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the topological interior of C. Let $f : K \times K \to Y$ be a given map. Recently, Ansari et al. [2] studied the following vector equilibrium problems to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \notin -\operatorname{int} C, \quad \forall y \in K, \tag{1.2}$$

or to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \in C, \quad \forall y \in K. \tag{1.3}$$

In case that the map f is multivalued, Ansari et al. [2] also studied the following multivalued vector equilibrium problem (for short, MVEP) to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \nsubseteq -\operatorname{int} C, \quad \forall y \in K, \tag{1.4}$$

or to find $\overline{x} \in K$ such that

$$f(\overline{x}, y) \subseteq C, \quad \forall y \in K. \tag{1.5}$$

By using an abstract monotonicity condition, they gave some existence theorems of solutions for MVEP.

Very recently, in [3, 4], the authors studied the following implicit vector equilibrium problems (for short, IVEP): to find $\overline{x} \in K$ such that

$$f(g(\overline{x}), y) \notin -\operatorname{int} C(\overline{x}), \quad \forall y \in K, \tag{1.6}$$

where $g : K \to K$ is a vector-valued map and $C : K \to 2^Y$ is a multivalued map such that, for all $x \in K$, $C(x)$ is a closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. By using the famous FKKM theorem and section theorem, they gave some existence results of solutions for IVEP.

Inspired and motivated by the research work mentioned above, in this paper, we consider a class of implicit multivalued vector equilibrium problems and introduce the concepts of $C_x - h$-pseudomonotonicity and $V - h$-hemicontinuity for multivalued maps. By using the fixed point theorem of Chowdhury and Tan [27], we obtain some existence results of solutions for the implicit multivalued vector equilibrium problems in the setting of topological vector spaces. Furthermore, we derive a stability result of solutions for the implicit multivalued vector equilibrium problems. These results extend and unify some recent results for implicit vector equilibrium problems, multivalued vector variational inequality problems, and vector variational inequality problems.

2. Preliminaries

Throughout this paper, unless otherwise specified, we suppose that X, Y, and Z are topological vector spaces, and $K \subseteq X$ and $D \subseteq Z$ are nonempty subsets. We also suppose that $C : X \to 2^Y$ is a multivalued map such that, for any $x \in X$, $C(x)$ is a proper, closed, and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$, $g : K \times D \to X$, $h : K \times K \to X$ are vector-valued maps, and $F : X \times X \to 2^Y$, $T : K \to 2^D$ are multivalued maps.

In this paper, we consider the following implicit multivalued vector equilibrium problem (for short, IMVEP) to find $\overline{x} \in K$ such that

$$\forall y \in K,\ \exists\, u \in T(\overline{x}) : F\big(g(\overline{x}, u), h(\overline{x}, y)\big) \nsubseteq -\operatorname{int} C(\overline{x}). \tag{2.1}$$

We call this $\overline{x}$ a solution for IMVEP.

Some special cases of IMVEP.

(1) If F is a single-valued map, then IMVEP reduces to the problem of finding $\overline{x} \in K$ such that

$$\forall y \in K,\ \exists u \in T(\overline{x}) : F\big(g(\overline{x}, u), h(\overline{x}, y)\big) \notin -\operatorname{int} C(\overline{x}), \tag{2.2}$$

which has been studied in [5].

(2) If F is a single-valued map, $g(x, v) = g(x)$, $h(x, y) = y$, then IMVEP reduces to IVEP.

(3) If F is a single-valued map, $g(x, v) = x$, $h(x, y) = y$, then IMVEP reduces to the problem of finding $\overline{x} \in K$ such that

$$F(\overline{x}, y) \notin -\operatorname{int} C(\overline{x}), \quad \forall y \in K, \tag{2.3}$$

which has been studied in [6].

(4) If $g(x, v) = x$, $h(x, y) = y$, $C(x) = C$ (C is a closed convex cone in Y with $\operatorname{int} C \neq \emptyset$), then IMVEP reduces to MVEP.

Definition 2.1 (see[28]). Let X and Y be two topological spaces. A multivalued map $T : X \to 2^Y$ is said to be

(i) upper semicontinuous (for short, u.s.c.) at $x_0 \in X$ if, for each open set V in Y with $T(x_0) \subseteq V$, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(x) \subseteq V$ for all $x \in U(x_0)$;

(ii) lower semicontinuous (for short, l.s.c.) at $x_0 \in X$ if, for each open set V in Y with $T(x_0) \cap V \neq \emptyset$, there exists an open neighborhood $U(x_0)$ of x_0 such that $T(x) \cap V \neq \emptyset$ for all $x \in U(x_0)$;

(iii) closed if the graph $\operatorname{Gr}(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is a closed subset of $X \times Y$.

(iv) compact-valued if, for each $x \in X$, $T(x)$ is a nonempty compact subset of Y.

Definition 2.2. Let X and Y be topological vector spaces, K a nonempty convex subset of X and C a nonempty convex cone of Y. A multivalued map $F : X \to 2^Y$ is said to be C-convex if, for any $x, y \in X$ and $t \in [0, 1]$, one has

$$F\big(tx + (1 - t)y\big) \subseteq tF(x) + (1 - t)F(y) - C. \tag{2.4}$$

Definition 2.3. Let X, Y, and Z be topological vector spaces, K a nonempty convex subset of X, and D a nonempty subset of Z. Let $C : X \to 2^Y$ be a multivalued map such that, for any

$x \in X$, $C(x)$ is a proper, closed, and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. Given two vector-valued maps $g : K \times D \to X$, $h : K \times K \to X$, and two multivalued maps $F : X \times X \to 2^Y$, $T : K \to 2^D$. Then, T is said to be

(i) $C_x - h$-pseudomonotone with respect to F and g on K if, for any $x, y \in K$ and any $u \in T(x)$, $v \in T(y)$, one has

$$F(g(x,u), h(x,y)) \not\subseteq -\operatorname{int} C(x) \quad \text{implies } F(g(y,v), h(x,y)) \not\subseteq -\operatorname{int} C(x); \tag{2.5}$$

(ii) weakly $C_x - h$-pseudomonotone with respect to F and g on K if, for any $x, y \in K$ and any $u \in T(x)$, one has

$$F(g(x,u), h(x,y)) \not\subseteq -\operatorname{int} C(x) \quad \text{implies } F(g(y,v), h(x,y)) \not\subseteq -\operatorname{int} C(x) \text{ for some } v \in T(y); \tag{2.6}$$

(iii) $V - h$-hemicontinuous with respect to F and g on K if, for any $x, y \in K$, $t \in (0,1)$, $y_t = x + t(y - x)$ and any $v_t \in T(y_t)$, there exists $u \in T(x)$ such that, for any open set V with $F(g(x,u), h(x,y)) \subseteq V$, there exists $t_0 \in (0,1)$ such that

$$\forall t \in (0, t_0], \quad F(g(y_t, v_t), h(x,y)) \subseteq V. \tag{2.7}$$

Remark 2.4. The above $V - h$-hemicontinuity for multivalued map is a generalization of V-hemicontinuity for continuous linear operator.

Example 2.5. Let $X = Y = Z = R$, $K = D = [0,1]$, and $C(x) = R_+$ for all $x \in X$. Let $g : K \times D \to X$, $h : K \times K \to X$, $T : K \to 2^D$ and $F : X \times X \to 2^Y$ be defined as follows:

$$\begin{gathered} g(x,u) = -(1 + x + u), \quad \forall (x,u) \in K \times D, \\ h(x,y) = y - x, \quad \forall (x,y) \in K \times K, \\ T(x) = [0,x], \quad \forall x \in K, \\ F(x,y) = \begin{cases} [x,y], & \text{if } x \geq y, \\ [y,x], & \text{otherwise}. \end{cases} \quad \forall (x,y) \in X \times X. \end{gathered} \tag{2.8}$$

Then, T is $C_x - h$-pseudomonotone with respect to F and g on K. Moreover, T is $V - h$-hemicontinuous with respect to F and g on K.

Proof. Firstly, we show that T is $C_x - h$-pseudomonotone with respect to F and g on K.

Indeed, for any $x, y \in [0,1]$, $u \in T(x) = [0,x]$, and $v \in T(y) = [0,y]$, it is obvious that $-(1 + x + u) \leq -x \leq y - x$. If

$$F(g(x,u), h(x,y)) = F(-(1 + x + u), y - x) = [-(1 + x + u), y - x] \not\subseteq (-\infty, 0), \tag{2.9}$$

then, $y - x \geq 0$, that is, $y \geq x$. It follows that

$$F(g(y,v),h(x,y)) = F(-(1+y+v), y-x) = [-(1+y+v), y-x] \not\subseteq (-\infty, 0). \tag{2.10}$$

Hence T is $C_x - h$-pseudomonotone with respect to F and g on K.

Secondly, we show that T is $V - h$-hemicontinuous with respect to F and g on K.

Indeed, let $x, y \in [0,1]$. Taking $u = x \in T(x)$, for any open set V with $F(g(x,u),h(x,y)) \subseteq V$, that is,

$$F(g(x,u),h(x,y)) = F(-(1+x+u), y-x) = F(-(1+2x), y-x) = [-(1+2x), y-x] \subseteq V, \tag{2.11}$$

there exists $r > 0$ such that

$$[-(1+2x) - 2r, y-x] \subseteq V. \tag{2.12}$$

Let $y_t = x + t(y-x)$, for all $t \in (0,1)$. Clearly, $y_t \geq 0$.

If $y \leq x$, then for any $t_0 \in (0,1)$ and any $v_t \in T(y_t) = [0, y_t]$, we have

$$0 \leq y_t \leq x. \tag{2.13}$$

And so

$$\begin{aligned} F(g(y_t,v_t),h(x,y)) &= F(-(1+y_t+v_t), y-x) \\ &= [-(1+y_t+v_t), y-x] \\ &\subseteq [-(1+2y_t), y-x] \\ &\subseteq [-(1+2x), y-x] \\ &\subseteq V. \end{aligned} \tag{2.14}$$

If $y > x$, taking $t_0 = \min\{r/(y-x), 1/2\} \in (0,1)$, then for any $t \in (0, t_0)$ and $v_t \in T(y_t)$, we have

$$\begin{aligned} y_t &\leq y_{t_0}, \\ -(1+y_t+v_t) &\leq -(1+y_t) \\ &= -1 - x - t(y-x) \\ &= (y-x) + tx - (1+t)y - 1 \\ &\leq y - x + 1 - 0 - 1 \\ &\leq y - x. \end{aligned} \tag{2.15}$$

It follows that

$$\begin{aligned}
F(g(y_t, v_t), h(x, y)) &= F(-(1 + y_t + v_t), y - x) \\
&= [-(1 + y_t + v_t), y - x] \\
&\subseteq [-(1 + 2y_t), y - x] \\
&\subseteq [-(1 + 2y_{t_0}), y - x] \\
&= [-(1 + 2(x + t_0(y - x))), y - x] \\
&\subseteq [-(1 + 2(x + r)), y - x] \\
&\subseteq V.
\end{aligned} \tag{2.16}$$

Thus, T is $V - h$-hemicontinuous with respect to F and g on K. □

Lemma 2.6 (see [29]). *Let X and Y be two topological spaces and $F : X \to 2^Y$ a multivalued map.*

(i) *F is closed if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and any net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ with $y_\alpha \to y$, one has $y \in F(x)$.*

(ii) *If F is compact valued, then F is u.s.c. at $x \in X$ if and only if for any net $\{x_\alpha\} \subseteq X$ with $x_\alpha \to x$ and any net $\{y_\alpha\}$ with $y_\alpha \in F(x_\alpha)$, there exists $y \in F(x)$ and a subnet $\{y_\beta\} \subseteq \{y_\alpha\}$ such that $y_\beta \to y$.*

The following lemma, which is a generalized form of Fan-Browder fixed piont theorem [30, 31], is very important to establish our existence results of solutions for IMVEP.

Lemma 2.7 (see [27]). *Let K be a nonempty convex subset of a topological vector space X and F, $G : K \to 2^K$ be two multivalued maps such that*

(i) *for any $x \in K$, $F(x) \subseteq G(x)$;*

(ii) *for any $x \in K$, $G(x)$ is convex;*

(iii) *for any $y \in K$, $F^{-1}(y)$ is compactly open (i.e., $F^{-1}(y) \cap L$ is open in L for each nonempty compact subset L of K);*

(iv) *there exists a nonempty, closed, and compact subset $A \subseteq K$ and $\overline{y} \in A$ such that $K \setminus A \subseteq G^{-1}(\overline{y})$;*

(v) *for any $x \in K$, $F(x) \neq \emptyset$.*

Then, there exists $x_0 \in K$ such that $x_0 \in G(x_0)$.

Lemma 2.8 (see [28]). *Let X and Y be two Hausdorff topological vector spaces and $T : X \to 2^Y$ a multivalued map. If T is closed and $\overline{T(X)}$ is compact, then T is u.s.c., where $T(X) = \bigcup_{x \in X} T(x)$ and $\overline{A}$ denotes the closure of the set A.*

Lemma 2.9 (see [32]). *Let E be a metric space and $A, A_n \in E$ $(n = 1, 2, \ldots)$ be compact subsets. If, for any open set O with $A \subseteq O$, there exists n_0 such that $A_n \subseteq O$ for all $n \geq n_0$, then any sequence $\{x_n\}$, satisfying $x_n \in A_n$, for $n = 1, 2, \ldots$, has some subsequence which converges to some point of A.*

3. Existence of Solutions for IMVEP

In this section, we will apply the generalized Fan-Browder fixed point theorem to establish some existence results of solutions for IMVEP. First of all, we have the following lemma.

Lemma 3.1. *Let X, Y, and Z be topological vector spaces, and K a nonempty convex subset of X, and D a nonempty subset of Z. Let $C : X \to 2^Y$ be a multivalued map such that, for any $x \in X$, $C(x)$ is a proper, closed, and convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$. Given two vector-valued maps $g : K \times D \to X$, $h : K \times K \to X$, and two multivalued maps $F : X \times X \to 2^Y$, $T : K \to 2^D$. Consider the following problems.*

(I) *Find $\overline{x} \in K$ such that, $\forall y \in K$, $\exists u \in T(\overline{x}) : F(g(\overline{x}, u), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x})$;*

(II) *Find $\overline{x} \in K$ such that, $\forall y \in K$, $\exists v \in T(y) : F(g(y, v), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x})$;*

(III) *Find $\overline{x} \in K$ such that, $\forall y \in K$, $\forall v \in T(y) : F(g(y, v), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x})$.*

Then,

(i) *Problem (I) implies Problem (II) if T is weakly $C_x - h$-pseudomonotone with respect to F and g on K, moreover, implies Problem (III) if T is $C_x - h$-pseudomonotone with respect to F and g on K;*

(ii) *Problem (II) implies Problem (I) if T is $V - h$-hemicontinuous with respect to F and g on K and, for any $x, y \in K$ and any $v \in T(y)$, $F(g(y, v), h(x, \cdot))$ is $C(x)$-convex and $F(g(y, v), h(x, x)) \subseteq -C(x)$;*

(iii) *Problem (III) implies Problem (II).*

Proof. (i) It follows from the weakly $C_x - h$-pseudomonotone with respect to F and g on K and $C_x - h$-pseudomonotone with respect to F and g on K, respectively.

(ii) Let $\overline{x}$ be a solution of (II). Then, $\forall y \in K$, $\exists v \in T(y)$ such that

$$F(g(y, v), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.1}$$

Let $y_\beta = \overline{x} + \beta(y - \overline{x})$, for all $\beta \in (0, 1)$. Since K is convex, we have $y_\beta \in K$. Then, there exists $v_\beta \in T(y_\beta)$ such that

$$F(g(y_\beta, v_\beta), h(\overline{x}, y_\beta)) \not\subseteq -\operatorname{int} C(\overline{x}), \quad \forall \beta. \tag{3.2}$$

Since for any $x, y \in K$ and any $v \in T(y)$, $F(g(y, v), h(x, \cdot))$ is $C(x)$-convex, we have

$$F(g(y_\beta, v_\beta), h(\overline{x}, y_\beta)) \subseteq \beta F(g(y_\beta, v_\beta), h(\overline{x}, y)) + (1 - \beta) F(g(y_\beta, v_\beta), h(\overline{x}, \overline{x})) - C(\overline{x}). \tag{3.3}$$

Notice that for all $x, y \in K$ and any $v \in T(y)$, $F(g(y, v), h(x, x)) \subseteq -C(x)$. Then, we can obtain that

$$F(g(y_\beta, v_\beta), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x}), \quad \forall \beta. \tag{3.4}$$

Indeed, suppose to the contrary, that $F(g(y_\beta,v_\beta),h(\overline{x},y))\subseteq -\operatorname{int}C(\overline{x})$ for some β, then

$$\begin{aligned}F(g(y_\beta,v_\beta),h(\overline{x},y_\beta))&\subseteq \beta F(g(y_\beta,v_\beta),h(\overline{x},y))+(1-\beta)F(g(y_\beta,v_\beta),h(\overline{x},\overline{x}))-C(\overline{x})\\&\subseteq -\operatorname{int}C(\overline{x})-C(\overline{x})-C(\overline{x})\\&\subseteq -\operatorname{int}C(\overline{x}),\end{aligned}\tag{3.5}$$

which contradicts (3.2), and so (3.4) holds.

We claim that there exists $u\in T(\overline{x})$ such that

$$F(g(\overline{x},u),h(\overline{x},y))\nsubseteq -\operatorname{int}C(\overline{x}),\tag{3.6}$$

and so $\overline{x}$ is a solution of Problem (I).

In fact, if it is not the case, then we have, for any $u\in T(\overline{x})$, $F(g(\overline{x},u),h(\overline{x},y))\subseteq -\operatorname{int}C(\overline{x})$. And then it follows from the fact that T is $V-h$-hemicontinuous with respect to F and g on K that there exists $\overline{u}\in T(\overline{x})$ and $\beta_0\in(0,1)$ such that $F(g(\overline{x},\overline{u}),h(\overline{x},y))\subseteq -\operatorname{int}C(\overline{x})$ and

$$\forall\beta\in(0,\beta_0],\quad F(g(y_\beta,v_\beta),h(\overline{x},y))\subseteq -\operatorname{int}C(\overline{x}).\tag{3.7}$$

(3.7) contradicts (3.4), and so (3.6) holds.

(iii) is obvious.

This completes the proof. □

Now, we are ready to prove some existence theorems for IMVEP under suitable pseudomonotonicity assumptions.

Theorem 3.2. *Let X, Y, and Z be topological vector spaces, and K a nonempty convex subset of X and D a nonempty subset of Z. Let $C:X\to 2^Y$ be a multivalued map such that, for any $x\in X$, $C(x)$ is a proper, closed, and convex cone in Y with $\operatorname{int}C(x)\neq\emptyset$. Given two maps $g:K\times D\to X$, $h:K\times K\to X$, and two multivalued maps $F:X\times X\to 2^Y$, $T:K\to 2^D$. Suppose the following conditions are satisfied:*

(i) *h is continuous in the first variable;*

(ii) *T is C_x-h-pseudomonotone and $V-h$-hemicontinuous with respect to F and g on K;*

(iii) *the multivalued map $W:K\to 2^Y$, defined by $W(x)=Y\setminus\{-\operatorname{int}C(x)\}$ is closed;*

(iv) *F is u.s.c. and compact-valued, and it satisfies the following conditions:*

  (a) *for any $x,y\in K$ and $v\in T(y)$, $F(g(y,v),h(x,\cdot))$ is $C(x)$-convex and $F(g(y,v),h(x,x))\subseteq -C(x)$;*

  (b) *for any $x\in K$, there exists $u\in T(x)$ such that $0\in F(g(x,u),h(x,x))$;*

(v) *there exists a nonempty, compact and closed subset $A\subseteq K$ and $\overline{y}\in A$ such that, for all $x\in K\setminus A$, one has*

$$\forall u\in T(x),\quad F(g(x,u),h(x,\overline{y}))\subseteq -\operatorname{int}C(x).\tag{3.8}$$

Then, IMVEP is solvable, that is, there exists $\overline{x} \in K$ *such that*

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F(g(\overline{x}, u), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.9}$$

Proof. Define two multivalued maps $H,\ G : K \to 2^K$ as follows, for any $x \in K$,

$$\begin{aligned} H(x) &= \{y \in K : \exists v \in T(y),\ F(g(y, v), h(x, y)) \subseteq -\operatorname{int} C(x)\}, \\ G(x) &= \{y \in K : \forall u \in T(x),\ F(g(x, u), h(x, y)) \subseteq -\operatorname{int} C(x)\}. \end{aligned} \tag{3.10}$$

The proof is divided into the following steps.
(I) For all $x \in K$, $H(x) \subseteq G(x)$.
Indeed, let $y \in H(x)$, then there exists $v \in T(y)$ such that

$$F(g(y, v), h(x, y)) \subseteq -\operatorname{int} C(x). \tag{3.11}$$

If $y \notin G(x)$, then there exists $u \in T(x)$ such that

$$F(g(x, u), h(x, y)) \not\subseteq -\operatorname{int} C(x). \tag{3.12}$$

Since T is $C_x - h$-pseudomonotone with respect to F and g on K, then, by (3.12), we have for all $v \in T(y)$,

$$F(g(y, v), h(x, y)) \not\subseteq -\operatorname{int} C(x), \tag{3.13}$$

which contradicts (3.11). Thus, $y \in G(x)$, and so $H(x) \subseteq G(x)$.

(II) For all $x \in K$, $G(x)$ is convex.

In fact, for any $y_1, y_2 \in G(x)$ and $t \in (0, 1)$, by the definition of G, we have, for each $u \in T(x)$,

$$\begin{aligned} &F(g(x, u), h(x, y_1)) \subseteq -\operatorname{int} C(x), \\ &F(g(x, u), h(x, y_2)) \subseteq -\operatorname{int} C(x). \end{aligned} \tag{3.14}$$

Let $y_t = ty_1 + (1 - t)y_2$. Since K is convex, we have $y_t \in K$. Noting that $F(g(x, u), h(x, \cdot))$ is $C(x)$-convex, we have

$$\begin{aligned} F(g(x, u), h(x, y_t)) &\subseteq tF(g(x, u), h(x, y_1)) + (1 - t)F(g(x, u), h(x, y_2)) - C(x) \\ &\subseteq -\operatorname{int} C(x) - \operatorname{int} C(x) - C(x) \\ &\subseteq -\operatorname{int} C(x). \end{aligned} \tag{3.15}$$

By the arbitrary of u, we have $y_t \in G(x)$, and so $G(x)$ is convex.

(III) For any $y \in K$, $H^{-1}(y)$ is compactly open.

Indeed, for any given compact subset $L \subseteq K$, let $P = H^{-1}(y) \cap L$. We will show that P is open in L by proving that P^C is closed in L. Let $\{x_\alpha\} \subseteq P^C$ be an arbitrary net such that $x_\alpha \to x_0 \in L$. Then, for each α, $x_\alpha \in P^C$, that is, $y \notin H(x_\alpha)$, thus, for any $v \in T(y)$,

$$F(g(y,v), h(x_\alpha, y)) \not\subseteq -\operatorname{int} C(x_\alpha), \quad \forall \alpha. \tag{3.16}$$

It follows that, for each α, there exists $\omega_\alpha \in F(g(y,v), h(x_\alpha, y))$ such that $\omega_\alpha \notin -\operatorname{int} C(x_\alpha)$, that is,

$$\omega_\alpha \in W(x_\alpha) = Y \setminus \{-\operatorname{int} C(x_\alpha)\}, \quad \forall \alpha. \tag{3.17}$$

Since h is continuous in the first variable and F is u.s.c. and compact-valued, it follows from Lemma 2.6 that there exists $\omega_0 \in F(g(y,v), h(x_0, y))$ and a subnet of $\{\omega_\alpha\}$, we still denote this subnet by $\{\omega_\alpha\}$, such that $\omega_\alpha \to \omega_0$. Notice that W is closed. We have $\omega_0 \in W(x_0) = Y \setminus \{-\operatorname{int} C(x_0)\}$. It follows that $\omega_0 \notin -\operatorname{int} C(x_0)$, and so

$$F(g(y,v), h(x_0, y)) \not\subseteq -\operatorname{int} C(x_0). \tag{3.18}$$

Then, by the arbitrary of v, we have $y \notin H(x_0)$, that is, $x_0 \notin H^{-1}(y)$. Since $x_0 \in L$, we know that $x_0 \in P^C$, and so P^C is closed in L.

(IV) By the assumption (v), there exists a nonempty, compact, and closed subset $A \subseteq K$ and $\overline{y} \in A$ such that, for all $x \in K \setminus A$, we have

$$\forall u \in T(x), \quad F(g(x,u), h(x, \overline{y})) \subseteq -\operatorname{int} C(x). \tag{3.19}$$

This implies that $\overline{y} \in G(x)$, that is, $x \in G^{-1}(\overline{y})$. And thus $K \setminus A \subseteq G^{-1}(\overline{y})$.

(V) G has no fixed point in K.

Suppose that it is not the case, then there exists $x_0 \in K$ such that $x_0 \in G(x_0)$, that is,

$$\forall u \in T(x_0), \quad F(g(x_0,u), h(x_0, x_0)) \subseteq -\operatorname{int} C(x_0). \tag{3.20}$$

By the assumption (iv), we have $0 \in F(g(x_0,u), h(x_0,x_0))$ for some $u \in T(x_0)$, and it follows that $0 \in -\operatorname{int} C(x_0)$. This implies that $C(x_0)$ is an absorbing set in Y, which contradicts the assumption that $C(x_0)$ is proper in Y. Therefore, G has no fixed point.

Since G has no fixed point in K, it follows from Lemma 2.7 that there exists $\overline{x} \in K$ such that $H(\overline{x}) = \emptyset$, that is,

$$\forall y \in K, \quad \forall v \in T(y), \quad F(g(y,v), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.21}$$

From Lemma 3.1, we have $\overline{x} \in K$ such that

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F(g(\overline{x},u), h(\overline{x}, y)) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.22}$$

This completes the proof. □

Remark 3.3. Theorem 3.2 is a multivalued extension of [7, Theorem 3].

Remark 3.4. The condition (v) of Theorem 3.2 is satisfied automatically if K is compact.

We now give an example to illustrate Theorem 3.2.

Example 3.5. Let $X, Y, Z, K, D, C, g, h, T$, and F be as in Example 2.5. We will show that all conditions of Theorem 3.2 are satisfied.

(I) It follows from Example 2.5 that the condition (ii) of Theorem 3.2 is satisfied. And it is obvious that F is compact valued and W is a closed mapping.

(II) We will show that F is u.s.c. on $X \times X$.

Let $x, y \in X$, for any set V with $F(x, y) \subseteq V$.

(1) If $x = y$, then $F(x, y) = \{x\} \subseteq V$, and then there exists $r > 0$ such that $[x-r, x+r] \subseteq V$. Taking $U_y = U_x = [x-r, x+r]$, then for any $x' \in U_x$ and $y' \in U_y$, we have

$$F(x', y') = \begin{cases} [x', y'] \subseteq [x-r, x+r] \subseteq V, & \text{if } x' \geq y', \\ [y', x'] \subseteq [x-r, x+r] \subseteq V, & \text{otherwise.} \end{cases} \tag{3.23}$$

(2) If $x < y$, then $F(x, y) = [x, y] \subseteq V$, and then there exists $r : 0 < r < (y-x)/2$ such that

$$[x-r, y+r] \subseteq V. \tag{3.24}$$

Taking $U_x = [x-r, x+r]$ and $U_y = [y-r, y+r]$, we have

$$x + r \leq x + \frac{y-x}{2} = \frac{y+x}{2} = y - \frac{y-x}{2} = y - r. \tag{3.25}$$

It follows that for any $x' \in U_x$ and any $y' \in U_y$, then we have $x' \leq y'$. Thus,

$$F(x', y') = [x', y'] \subseteq [x-r, y+r] \subseteq V. \tag{3.26}$$

(3) If $x > y$, the argument is similar to (2).

Hence, F is u.s.c. on $X \times X$.

(III) We will show that for any $x, y \in K$ and $v \in T(y)$, $F(g(y, v), h(x, \cdot))$ is $C(x)$-convex.

For any $x, y \in K$ and $v \in T(y)$. Let $u_t = tu_1 + (1-t)u_2$ for each $t \in (0,1)$ and $u_1, u_2 \in K$. Since

$$\begin{aligned} u_t - x &= tu_1 + (1-t)u - 2 - x \geq 0 - x = -x, \\ -(1 + y + v) &\leq -1 \leq -x \leq u_i - x, \quad i = 1, 2, \end{aligned} \tag{3.27}$$

it follows that

$$F(g(y,v),h(x,u_i)) = F(-(1+y+v),u_i - x) = \left[-(1+y+v),u_i - x\right], \quad i=1,2. \tag{3.28}$$

Thus, we have

$$\begin{aligned}
F(g(y,v),h(x,u_t)) &= F(-(1+y+v),u_t - x) \\
&= \left[-(1+y+v),u_t - x\right] \\
&= \left[-(1+y+v),t(u_1 - x)+(1-t)(u_2 - x)\right] \\
&= t\left[-(1+y+v),u_1 - x\right]+(1-t)\left[-(1+y+v),u_2 - x\right] \\
&= tF(g(\mathrm{y},v),h(x,u_1))+(1-t)F(g(y,v),h(x,u_2)) \\
&\subseteq tF(g(y,v),h(x,u_1))+(1-t)F(g(y,v),h(x,u_2)) - C(x).
\end{aligned} \tag{3.29}$$

This shows that $F(g(y,v),h(x,\cdot))$ is $C(x)$-convex.

(IV) Obviously, for any $x,y \in K$ and any $y \in T(y)$,

$$\begin{aligned}
F(g(y,v),h(x,x)) &= F(-(1+y+v),x-x) \\
&= \left[-(1+y+v),0\right] \subseteq -R_+ = -C(x),
\end{aligned} \tag{3.30}$$

that is, for any $x,y \in K$ and $y \in T(y)$, $F(g(y,v),h(x,x)) \subseteq -C(x)$.

For any $x \in K$ and $u \in T(x)$, we have

$$F(g(x,u),h(x,x)) = F(-(1+x+u),x-x) = [-(1+x+u),0], \tag{3.31}$$

thus, $0 \in F(g(x,u),h(x,x))$.

By the above arguments, we know that all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, IMVEP is solvable.

Indeed, let $\overline{x} = 0 \in [0,1]$, then for any $y \in [0,1]$, there exists $u = 0 \in T(\overline{x})$ such that

$$F(g(\overline{x},u),h(\overline{x},y)) = F(g(0,0),h(0,y)) = F(-1,y) = \left[-1,y\right] \not\subseteq -\operatorname{int} R_+. \tag{3.32}$$

Thus, $\overline{x}$ is a solution of IMVEP.

We now obtain an existence theorem for IMVEP for weakly $C_x - h$-pseudomonotone maps with respect to F and g under additional assumptions.

Theorem 3.6. *Let X, Y, Z, K, D, C, g, h, F, and T be as in Theorem 3.2. Assume that the conditions (iii)–(v) of Theorem 3.2 and the following conditions are satisfied:*

(i)$'$ *g is continuous in the second variable and h is continuous in the first variable;*

(ii)$'$ *T is compact-valued, weakly C_x–h-pseudomonotone and V–h-hemicontinuous with respect to F and g on K.*

Then, IMVEP is solvable, that is, there exists $\overline{x} \in K$ such that

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F(g(\overline{x}, u), h(\overline{x}, y)) \nsubseteq -\operatorname{int} C(\overline{x}). \tag{3.33}$$

Proof. Define two multivalued maps $H, G : K \to 2^K$ as follows, for any $x \in K$,

$$\begin{aligned} H(x) &= \{y \in K : \forall v \in T(y), F(g(y, v), h(x, y)) \subseteq -\operatorname{int} C(x)\}, \\ G(x) &= \{y \in K : \forall u \in T(x), F(g(x, u), h(x, y)) \subseteq -\operatorname{int} C(x)\}. \end{aligned} \tag{3.34}$$

By using the same arguments as in the proof (I) of Theorem 3.2 and weakly $C_x - h$-pseudomonotonicity with respect to F and g on K of T, we see that for any $x \in K$, $H(x) \subseteq G(x)$.

We have already seen in the proof of Theorem 3.2 that for each $x \in K$, $G(x)$ is convex and the multivalued map G has no fixed point. Moreover, there exists a nonempty, compact, and closed subset $A \subseteq K$ and $\overline{y} \in A$ such that $K \setminus A \subseteq G^{-1}(\overline{y})$.

Next, we will show that, for each $y \in K$, $H^{-1}(y)$ is compactly open.

Indeed, for any given compact subset $L \subseteq K$, let $P = H^{-1}(y) \cap L$. We will show that P is open in L by proving that P^C is closed in L. Let $\{x_\alpha\} \subseteq P^C$ be an arbitrary net such that $x_\alpha \to x_0 \in L$. Then, for each α, $x_\alpha \in P^C$, that is, $y \notin H(x_\alpha)$. Thus, for each α, there exists $v_\alpha \in T(y)$ such that

$$F(g(y, v_\alpha), h(x_\alpha, y)) \nsubseteq -\operatorname{int} C(x_\alpha). \tag{3.35}$$

Since T is compact valued, there exists a subnet $\{v_\beta\}$ of $\{v_\alpha\}$ such that $v_\beta \to v_0 \in T(y)$. Then, by virtue of (3.35), for each β, there exists $\omega_\beta \in F(g(y, v_\beta), h(x_\beta, y))$ such that $\omega_\beta \notin -\operatorname{int} C(x_\beta)$, that is,

$$\omega_\beta \in W(x_\beta) = Y \setminus \{-\operatorname{int} C(x_\beta)\}, \quad \forall \beta. \tag{3.36}$$

Since g is continuous in the second variable and h is continuous in the first variable, we have

$$g(y, v_\beta) \longrightarrow g(y, v_0), \qquad h(x_\beta, y) \longrightarrow h(x_0, y). \tag{3.37}$$

In addition, F is u.s.c. and compact valued, it follows from Lemma 2.6 that there exist $\omega_0 \in F(g(y, v_0), h(x_0, y))$ and a subnet of $\{\omega_\beta\}$, we still denote this subnet by $\{\omega_\beta\}$, such that $\omega_\beta \to \omega_0$. Notice that W is closed. We have $\omega_0 \in W(x_0) = Y \setminus \{-\operatorname{int} C(x_0)\}$. It follows that $\omega_0 \notin -\operatorname{int} C(x_0)$, and so

$$F(g(y, v_0), h(x_0, y)) \nsubseteq -\operatorname{int} C(x_0). \tag{3.38}$$

Thus, $y \notin H(x_0)$, that is, $x_0 \notin H^{-1}(y)$. Since $x_0 \in L$, we know that $x_0 \in P^C$, and so P^C is closed in L.

Hence, as in the proof of Theorem 3.2, there exists $\overline{x} \in K$ such that $H(\overline{x}) = \emptyset$, that is,

$$\forall y \in K, \ \exists v \in T(y), \quad F(g(y, v), h(\overline{x}, y)) \nsubseteq -\operatorname{int} C(\overline{x}). \tag{3.39}$$

From Lemma 3.1, we have $\overline{x} \in K$ such that

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F\big(g(\overline{x}, u), h(\overline{x}, y)\big) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.40}$$

This completes the proof. □

Remark 3.7. Theorem 3.6 is a multivalued extension of [7, Theorem 4].

Next, we will prove an existence result for IMVEP without any kind of pseudomonotonicity assumption.

Theorem 3.8. *Let X, Y, Z, K, D, C, g, h, F, and T be as in Theorem 3.2. Assume that the conditions (iii)–(v) of Theorem 3.2 and the following conditions are satisfied:*

(i)″ *g is continuous in both variables and h is continuous in the first variable;*

(ii)″ *T is u.s.c. and compact-valued.*

Then, IMVEP is solvable, that is, there exists $\overline{x} \in K$ such that

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F\big(g(\overline{x}, u), h(\overline{x}, y)\big) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{3.41}$$

Proof. Define a multivalued map $G : K \to 2^K$ as in the proof of Theorem 3.2. As we have seen in the proof in Theorem 3.2 that, for each $x \in K$, $G(x)$ is convex and the multivalued map G has no fixed point. Moreover, there exists a nonempty, compact, and closed subset $A \subseteq K$ and $\overline{y} \in A$ such that $K \setminus A \subseteq G^{-1}(\overline{y})$.

Now, we have only to show that, for any $y \in K$, $G^{-1}(y)$ is compactly open.

Indeed, for any given compact subset $L \subseteq K$, let $P = G^{-1}(y) \cap L$. We will show that P is open in L by proving that P^C is closed in L. Let $\{x_\alpha\} \subseteq P^C$ be an arbitrary net such that $x_\alpha \to x_0 \in L$. Then, for each α, $x_\alpha \in P^C$, that is, $y \notin G(x_\alpha)$. Thus, for each α, there exists $u_\alpha \in T(x_\alpha)$ such that

$$F\big(g(x_\alpha, u_\alpha), h(x_\alpha, y)\big) \not\subseteq -\operatorname{int} C(x_\alpha). \tag{3.42}$$

Since T is u.s.c. and compact-valued, it follow from Lemma 2.6 that there exists $u_0 \in T(x_0)$ and a subnet $\{u_\beta\} \subseteq \{u_\alpha\}$ such that $u_\beta \to u_0$. Then, by the assumption (i), we have

$$g(x_\beta, u_\beta) \longrightarrow g(x_0, u_0), \qquad h(x_\beta, y) \longrightarrow h(x_0, y). \tag{3.43}$$

Furthermore, by virtue of (3.42), for each β, there exists some $\omega_\beta \in F(g(x_\beta, u_\beta), h(x_\beta, y))$ such that $\omega_\beta \notin -\operatorname{int} C(x_\beta)$. It follows that

$$\omega_\beta \in W(x_\beta) = Y \setminus \{-\operatorname{int} C(x_\beta)\}, \quad \forall \beta. \tag{3.44}$$

Since F is u.s.c. and compact-valued, it follows from Lemma 2.6 that there exist $\omega_0 \in F(g(x_0, u_0), h(x_0, y))$ and a subnet of $\{\omega_\beta\}$, we still denote this subnet by $\{\omega_\beta\}$ such that

$\omega_\beta \to \omega_0$. Notice that W is closed. We have $\omega_0 \in W(x_0) = Y \setminus \{-\operatorname{int} C(x_0)\}$, that is, $\omega_0 \notin -\operatorname{int} C(x_0)$, and so

$$F(g(x_0, u_0), h(x_0, y)) \nsubseteq -\operatorname{int} C(x_0). \tag{3.45}$$

Thus, $y \notin G(x_0)$, that is, $x_0 \notin G^{-1}(y)$. Since $x_0 \in L$, we know that $x_0 \in P^C$, and so P^C is closed in L.

Thus, as in the proof of Theorem 3.2, there exists $\overline{x} \in K$ such that $G(\overline{x}) = \emptyset$, that is,

$$\forall y \in K,\ \exists u \in T(\overline{x}), \quad F(g(\overline{x}, u), h(\overline{x}, y)) \nsubseteq -\operatorname{int} C(\overline{x}). \tag{3.46}$$

This completes the proof. □

Remark 3.9. From the proof of Theorem 3.8, we can see that, if D is compact, then the condition (ii)$''$ can be replaced by the following condition (ii)$'''$ T is closed.

4. Stability of Solution Sets for IMVEP

In this section, we discuss the stability of solutions for IMVEP.

Throughout this section, let X and Z be Banach spaces, and Y a topological vector space, $K \subseteq X$ a nonempty compact convex subset and $D \subseteq Z$ a nonempty subset, and $C : X \to 2^Y$ a multivalued map such that, for all $x \in X$, $C(x)$ is a proper, closed convex cone in Y with $\operatorname{int} C(x) \neq \emptyset$.

Let

$$\begin{aligned} E = \Big\{ (g, h, T) :\ & g : K \times D \longrightarrow X \text{ is continuous in both variables;} \\ & h : K \times K \longrightarrow X \text{ is continuous in the first variable;} \\ & T : K \longrightarrow 2^D \text{ is u.s.c. and compact valued} \Big\}. \end{aligned} \tag{4.1}$$

Let A, B be two compact sets in a normed space $(U, \|\cdot\|)$. Recall the Hausdorff metric defined by

$$H(A, B) = \max\left\{ \sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\| \right\}. \tag{4.2}$$

For any given $(g, h, T), (g', h', T') \in E$, let

$$\begin{aligned} \rho((g, h, T), (g', h', T')) = & \sup_{(x,y) \in K \times D} \|g(x, y) - g'(x, y)\| + \sup_{(x,y) \in K \times K} \|h(x, y) - h'(x, y)\| \\ & + \sup_{x \in K} H(T(x), T'(x)), \end{aligned} \tag{4.3}$$

where H is the Hausdorff metric. Then, it is easy to verify that (E, ρ) is a metric space.

Assume that the multivalued maps F and W satisfy all the conditions of Theorem 3.2. Then, it follows from Theorem 3.8 that for any $(g,h,T) \in E$, IMVEP has a solution, that is, there exists $\overline{x} \in K$ such that

$$\forall y \in K, \quad \exists u \in T(\overline{x}) : F(g(\overline{x},u),h(\overline{x},y)) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{4.4}$$

Let

$$\varphi(g,h,T) = \{x \in K : \forall y \in K,\ \exists u \in T(x),\ F(g(x,u),h(x,y)) \not\subseteq -\operatorname{int} C(x)\}. \tag{4.5}$$

Then $\varphi(g,h,T) \neq \emptyset$, which implies that φ is a multivalued map from E into K.

Theorem 4.1. *$\varphi : E \to 2^K$ is an upper semicontinuous map with nonempty compact values.*

Proof. Since K is compact, it follows from Lemma 2.8 that we need only to show that φ is closed. Let $((g_n,h_n,T_n),x_n) \in \operatorname{Gr}(\varphi)$ and $((g_n,h_n,T_n),x_n) \to ((g,h,T),\overline{x})$. We will show that $((g,h,T),\overline{x}) \in \operatorname{Gr}(\varphi)$.

Indeed, for any $y \in K$ and n, since $x_n \in \varphi(g_n,h_n,T_n)$, there exists $u_n \in T_n(x_n)$ such that

$$F(g_n(x_n,u_n),h_n(x_n,y)) \not\subseteq -\operatorname{int} C(x_n). \tag{4.6}$$

Notice that T and T_n are compact-valued. Then, for any open set O with $T(\overline{x}) \subseteq O$, there exists $\varepsilon > 0$ such that

$$\{u \in Z : d(u,T(\overline{x})) < \varepsilon\} \subseteq O, \tag{4.7}$$

where $d(u,T(\overline{x})) = \inf_{u' \in T(\overline{x})} \|u - u'\|$. Since $\rho((g_n,h_n,T_n),(g,h,T)) \to 0$ and T is u.s.c., there exists n_0 such that, for all $n \geq n_0$,

$$\sup_{x \in K} H(T(x),T_n(x)) < \frac{\varepsilon}{2}, \tag{4.8}$$

$$T(x_n) \subseteq \left\{u \in Z : d(u,T(\overline{x})) < \frac{\varepsilon}{2}\right\}. \tag{4.9}$$

It follows from (4.8) that

$$H(T(x_n),T_n(x_n)) < \frac{\varepsilon}{2}. \tag{4.10}$$

By (4.10), (4.9), and (4.7), for all $n \geq n_0$, we have

$$\begin{aligned} T_n(x_n) &\subseteq \left\{u \in Z : d(u,T(x_n)) < \frac{\varepsilon}{2}\right\} \\ &\subseteq \{u \in Z : d(u,T(\overline{x})) < \varepsilon\} \\ &\subseteq O. \end{aligned} \tag{4.11}$$

Since $T(\overline{x}) \subseteq O$ and for all $n \geq n_0$, $u_n \in T_n(x_n) \subseteq O$, by Lemma 2.9, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$u_{n_k} \longrightarrow \overline{u} \in T(\overline{x}). \tag{4.12}$$

Since g_{n_k} is continuous, h_{n_k} is continuous in the first variable, and $g_{n_k} \to g, h_{n_k} \to h$, we have

$$g_{n_k}(x_{n_k}, u_{n_k}) \longrightarrow g(\overline{x}, \overline{u}), \qquad h_{n_k}(x_{n_k}, y) \longrightarrow h(\overline{x}, y). \tag{4.13}$$

Then, it follows from (4.6) that

$$F\left(g_{n_k}(x_{n_k}, u_{n_k}), h_{n_k}(x_{n_k}, y)\right) \not\subseteq -\operatorname{int} C(x_{n_k}). \tag{4.14}$$

Thus, there exist $\omega_{n_k} \in F(g_{n_k}(x_{n_k}, u_{n_k}), h_{n_k}(x_{n_k}, y))$ such that $\omega_{n_k} \notin -\operatorname{int} C(x_{n_k})$, that is, $\omega_{n_k} \in W(x_{n_k}) = Y \setminus \{-\operatorname{int} C(x_{n_k})\}$. Since F is u.s.c. and compact valued, there exist $\overline{\omega} \in F(g(\overline{x}, \overline{u}), h(\overline{x}, y))$ and a subsequence of $\{\omega_{n_k}\}$, we still denote this subsequence by $\{\omega_{n_k}\}$, such that $\omega_{n_k} \to \overline{\omega}$. Notice that W is closed. We have $\overline{\omega} \in W(\overline{x}) = Y \setminus \{-\operatorname{int} C(\overline{x})\}$, that is, $\overline{\omega} \in -\operatorname{int} C(\overline{x})$, and so

$$F\left(g(\overline{x}, \overline{u}), h(\overline{x}, y)\right) \not\subseteq -\operatorname{int} C(\overline{x}). \tag{4.15}$$

This implies that $((g, h, T), \overline{x}) \in \operatorname{Gr}(\varphi)$, and so φ is closed.

This completes the proof. □

5. Conclusions

In this paper, the existence and stability of solutions for a class of implicit multivalued vector equilibrium problems are studied. By using the generalized Fan-Browder fixed point theorem [27], some existence results of solutions for the implicit multivalued vector equilibrium problems are obtained under some suitable assumptions. These results generalize and extend some corresponding results of Ansari et al. [7]. Also, in Section 4 of this paper, a stability result of solutions for the implicit multivalued vector equilibrium problems is obtained. It is worth mentioning that, up till now, there is no paper to consider the stability of solutions for the implicit multivalued vector equilibrium problems. So, the stability result obtained in Section 4 of this paper is new and interesting.

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