Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2011, Article ID 454093, 14 pages
doi:10.1155/2011/454093

Research Article

On Approximate C^*-Ternary m-Homomorphisms: A Fixed Point Approach

M. Eshaghi Gordji,[1,2] Z. Alizadeh,[1,2] Y. J. Cho,[3] and H. Khodaei[1,2]

[1] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[2] *Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Semnan, Iran*
[3] *Department of Mathematics Education and the RINS, Gyeongsang National University, Chinju 660-701, Republic of Korea*

Correspondence should be addressed to Y. J. Cho, yjcho@gnu.ac.kr

Received 21 November 2010; Accepted 6 March 2011

Academic Editor: Jong Kim

Using fixed point methods, we prove the stability and superstability of C^*-ternary additive, quadratic, cubic, and quartic homomorphisms in C^*-ternary rings for the functional equation $f(2x+y)+f(2x-y)+(m-1)(m-2)(m-3)f(y) = 2^{m-2}[f(x+y)+f(x-y)+6f(x)]$, for each $m = 1,2,3,4$.

1. Introduction

Following the terminology of [1], a nonempty set G with a ternary operation $[\cdot,\cdot,\cdot] : G \times G \times G \to G$ is called a *ternary groupoid*, which is denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is said to be *commutative* if $[x_1,x_2,x_3] = [x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all $x_1,x_2,x_3 \in G$ and all permutations σ of $\{1,2,3\}$. If a binary operation o is defined on G such that $[x,y,z] = (x \circ y) \circ oz$ for all $x,y,z \in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G,[\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is associative, that is, if $[[x,y,z],u,v] = [x,[y,z,u],v] = [x,y,[z,u,v]]$ holds for all $x,y,z,u,v \in G$ (see [2]). Since it is extensively discussed in [3], the full description of a physical system $\mathbb{S}$ implies the knowledge of three basis ingredients: the set of the observables, the set of the states, and the dynamics that describes the time evolution of the system by means of the time dependence of the expectation value of a given observable on a given statue. Originally, the set of the observable was considered to be a C^*-algebra [4]. In many applications, however, it was shown not to be the most convenient choice and the C^*-algebra was replaced by a von

Neumann algebra because the role of the representation turns out to be crucial mainly when long-range interactions are involved (see [5] and references therein). Here we used a different algebraic structure.

A C^*-*ternary ring* is a complex Banach space A, equipped with a ternary product $(x,y,z) \to [x,y,z]$ of A^3 into A, which is C-linear in the outer variables, conjugate C-linear in the middle variable and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$ and satisfies $\|[x,y,z]\| \leq \|x\| \cdot \|y\| \cdot \|z\|$ and $\|[x,y,z]\| = \|x\|^3$.

If a C^*-ternary ring $(A,[\cdot,\cdot,\cdot])$ has an identity, that is, an element $e \in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x \in A$, then it is routine to verify that A, endowed with $x \circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] := x \circ y^* \circ z$ makes A into a C^*-ternary algebra.

Consider the functional equation $\mathfrak{I}_1(f) = \mathfrak{I}_2(f)(\mathfrak{I})$ in a certain general setting. A function g is an approximate solution of $(\mathfrak{I})$ if $\mathfrak{I}_1(g)$ and $\mathfrak{I}_2(g)$ are close in some sense. The Ulam stability problem asks whether or not there exists a true solution of $(\mathfrak{I})$ near g. A functional equation is said to be *superstable* if every approximate solution of the equation is an exact solution of the functional equation. The problem of stability of functional equations originated from a question of Ulam [6] concerning the stability of group homomorphisms.

Let $(G_1,*)$ be a group and $(G_2,\star,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that, if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \star h(y)) < \delta \tag{1.1}$$

for all $x,y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x),H(x)) < \epsilon$ for all $x \in G_1$?

If the answer is affirmative, we say that the equation of homomorphism $H(x * y) = H(x) \star H(y)$ is *stable*. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

In 1941, Hyers [7] gave a first affirmative answer to the question of Ulam for Banach spaces.

Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \tag{1.2}$$

for all $x,y \in X$ and some $\epsilon > 0$. Then there exists a unique additive mapping $T : X \to Y$ such that $\|f(x) - T(x)\| \leq \epsilon$ for all $x \in X$.

A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [8] in 1950 (see also [9]). In 1978, a generalized solution for approximately linear mappings was given by Th. M. Rassias [10]. He considered a mapping $f : X \to Y$ satisfying the condition

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1.3}$$

for all $x,y \in X$, where $\epsilon \geq 0$ and $0 \leq p < 1$. This result was later extended to all $p \neq 1$ and generalized by Gajda [11], Th. M. Rassias and Šemrl [12], and Isac and Th. M. Rassias [13].

In 2000, Lee and Jun [14] have improved the stability problem for approximately additive mappings. The problem when $p = 1$ is not true. Counter examples for the corresponding assertion in the case $p = 1$ were constructed by Gadja [11], Th. M. Rassias and Šemrl [12].

On the other hand, J. M. Rassias [15–17] considered the Cauchy difference controlled by a product of different powers of norm. Furthermore, a generalization of Th. M. Rassias theorems was obtained by Găvruţa [18], who replaced

$$\epsilon\left(\|x\|^p + \|y\|^p\right) \tag{1.4}$$

and $\epsilon\|x\|^p\|y\|^p$ by a general control function $\varphi(x, y)$. In 1949 and 1951, Bourgin [19, 20] is the first mathematician dealing with stability of (ring) homomorphism $f(xy) = f(x)f(y)$. The topic of approximation of functional equations on Banach algebras was studied by a number of mathematicians (see [21–33]).

The functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.5}$$

is related to a symmetric biadditive mapping [34, 35]. It is natural that this equation is called a *quadratic functional equation*. For more details about various results concerning such problems, the readers refer to [36–43].

In 2002, Jun and Kim [44] introduced the following cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1.6}$$

and they established the general solution and the generalized Hyers-Ulam-Rassias stability for the functional equation (1.6). Obviously, the mapping $f(x) = cx^3$ satisfies the functional equation (1.6), which is called the *cubic functional equation*. In 2005, Lee et al. [45] considered the following functional equation

$$f(2x+y) + f(2x-y) = 4f(x+y) + 4f(x-y) + 24f(x) - 6f(y). \tag{1.7}$$

It is easy to see that the mapping $f(x) = dx^4$ is a solution of the functional equation (1.7), which is called the *quartic functional equation*.

2. Preliminaries

In 2007, Park and Cui [46] investigated the generalized stability of a quadratic mapping $f : A \to B$, which is called a C^*-*ternary quadratic mapping* if f is a quadratic mapping satisfies

$$f([x, y, z]) = [f(x), f(y), f(z)] \tag{2.1}$$

for all $x, y, z \in A$. Let $(A, [\cdot, \cdot, \cdot])$ be a C^*-ternary ring derived from a unital commutative C^*-algebra A and let $f : A \to A$ satisfy $f(x) = x^2$ for all $x \in A$. It is easy to show that the mapping $f : A \to A$ is a C^*-ternary quadratic mapping.

Recently, in 2010, Bae and Park [47] investigated the following functional equations

$$f(2x+y)+f(2x-y)=2^{m-2}[f(x+y)+f(x-y)+6f(x)] \tag{2.2}$$

for each $m=1,2,3$, and

$$f(2x+y)+f(2x-y)+6f(y)=4[f(x+y)+f(x-y)+6f(x)] \tag{2.3}$$

and they have obtained the stability of the functional equations (2.2) and (2.3).

We can rewrite the functional equations (2.2) and (2.3) by

$$\begin{aligned} f(2x+y)+f(2x-y)+(m-1)(m-2)(m-3)f(y) \\ =2^{m-2}[f(x+y)+f(x-y)+6f(x)]. \end{aligned} \tag{2.4}$$

Obviously, the monomial $f(x)=ax^m$ $(x\in\mathbb{R})$ is a solution of the functional equation (2.4) for each $m=1,2,3,4$.

For $m=1,2$, Bae and Park [47, 48] showed that the functional equation (2.4) is equivalent to the additive equation and quadratic equation, respectively.

If $m=3$, the functional equation (2.4) is equivalent to the cubic equation [44]. Moreover, Lee et al. [45] solved the solution of the functional equation (2.4) for $m=4$.

In this paper, using the idea of Park and Cui [46], we study the further generalized stability of C^*-ternary additive, quadratic, cubic, and quartic mappings over C^*-ternary algebra via fixed point method for the functional equation (2.4). Moreover, we establish the superstability of this functional equation by suitable control functions.

Definition 2.1. Let A and B be two C^*-ternary algebras.

(1) A mapping $f: A\to B$ is called a *C^*-ternary additive homomorphism* (briefly, C^*-ternary 1-homomorphism) if f is an additive mapping satisfying (2.1) for all $x,y,z\in A$.

(2) A mapping $f: A\to B$ is called a *C^*-ternary quadratic mapping* (briefly, C^*-ternary 2-homomorphism) if f is a quadratic mapping satisfying (2.1) for all $x,y,z\in A$.

(3) A mapping $f: A\to B$ is called a *C^*-ternary cubic mapping* (briefly, C^*-ternary 3-homomorphism) if f is a cubic mapping satisfying (2.1) for all $x,y,z\in A$.

(4) A mapping $f: A\to B$ is called a *C^*-ternary quartic homomorphism* (briefly, C^*-ternary 4-homomorphism) if f is a quartic mapping satisfying (2.1) for all $x,y,z\in A$.

Now, we state the following notion of fixed point theorem. For the proof, refer to [49] (see also Chapter 5 in [50] and [51, 52]). In 2003, Radu [53] proposed a new method for obtaining the existence of exact solutions and error estimations, based on the fixed point alternative (see also [54–57]).

Let (X,d) be a generalized metric space. We say that a mapping $T: X\to X$ satisfies a Lipschitz condition if there exists a constant $L\geq 0$ such that $d(Tx,Ty)\leq Ld(x,y)$ for all $x,y\in X$, where the number L is called the Lipschitz constant. If the Lipschitz constant

L is less than 1, then the mapping T is called a *strictly contractive mapping*. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

The following theorem was proved by Diaz and Margolis [49] and Radu [53].

Theorem 2.2. *Suppose that (Ω, d) is a complete generalized metric space and $T : \Omega \to \Omega$ is a strictly contractive mapping with the Lipschitz constant L. Then, for any $x \in \Omega$, either*

$$d\left(T^m x, T^{m+1} x\right) = \infty, \quad \forall m \geq 0, \tag{2.5}$$

or there exists a natural number m_0 such that

(1) $d(T^m x, T^{m+1} x) < \infty$ *for all* $m \geq m_0$*;*

(2) *the sequence* $\{T^m x\}$ *is convergent to a fixed point* y^* *of* T*;*

(3) y^* *is the unique fixed point of* T *in* $\Lambda = \{y \in \Omega : d(T^{m_0} x, y) < \infty\}$*;*

(4) $d(y, y^*) \leq (1/(1-L)) d(y, Ty)$ *for all* $y \in \Lambda$.

3. Approximation of C^*-Ternary m-Homomorphisms between C^*-Ternary Algebras

In this section, we investigate the generalized stability of C^*-ternary m-homomorphism between C^*-ternary algebras for the functional equation (2.4).

Throughout this section, we suppose that X and Y are two C^*-ternary algebras. For convenience, we use the following abbreviation: for any function $f : X \to Y$,

$$\begin{aligned} \Delta_m f(x, y) &= f(2x + y) + f(2x - y) + (m-1)(m-2)(m-3) f(y) \\ &\quad - 2^{m-2}\left[f(x+y) + f(x-y) + 6 f(x)\right] \end{aligned} \tag{3.1}$$

for all $x, y \in X$.

From now on, let m be a positive integer less than 5.

Theorem 3.1. *Let $f : X \to Y$ be a mapping for which there exist functions $\varphi_m : X \times X \to [0, \infty)$ and $\psi_m : X \times X \times X \to [0, \infty)$ such that*

$$\left\| \Delta_m f(x, y) \right\| \leq \varphi_m(x, y), \tag{3.2}$$

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \leq \psi_m(x, y, z) \tag{3.3}$$

for all $x, y, z \in X$. If there exists a constant $0 < L < 1$ such that

$$\begin{aligned} \varphi_m\left(\frac{x}{2}, \frac{y}{2}\right) &\leq \frac{L}{2^m} \varphi_m(x, y), \\ \psi_m\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) &\leq \frac{L}{2^{3m}} \psi_m(x, y, z) \end{aligned} \tag{3.4}$$

for all $x, y, z \in X$, then there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ such that

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{L}{2^{m+1}(1-L)}\varphi_m(x, 0) \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (3.4) that

$$\lim_{n\to\infty} 2^{mn}\varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0, \tag{3.6}$$

$$\lim_{n\to\infty} 2^{3mn}\psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.7}$$

for all $x, y, z \in X$. By (3.6), $\lim_{n\to\infty} 2^{mn}\varphi_m(0,0) = 0$ and so $\varphi_m(0,0) = 0$. Letting $x = y = 0$ in (3.2), we get $f(0) \leq \varphi_m(0,0) = 0$ and so $f(0) = 0$.

Let $\Omega = \{g : g : X \to Y,\ g(0) = 0\}$. We introduce a generalized metric on Ω as follows:

$$d(g,h) = d_{\varphi_m}(g,h) = \inf\{K \in (0,\infty) : \|g(x) - h(x)\| \leq K\varphi_m(x,0),\ \forall x \in X\}. \tag{3.8}$$

It is easy to show that (Ω, d) is a generalized complete metric space [55].

Now, we consider the mapping $T : \Omega \to \Omega$ defined by $Tg(x) = 2^m g(x/2)$ for all $x \in X$ and $g \in \Omega$. Note that, for all $g, h \in \Omega$ and $x \in X$,

$$\begin{aligned} d(g,h) < K &\Longrightarrow \|g(x) - h(x)\| \leq K\varphi_m(x,0) \\ &\Longrightarrow \left\|2^m g\left(\frac{x}{2}\right) - 2^m h\left(\frac{x}{2}\right)\right\| \leq 2^m K\varphi_m\left(\frac{x}{2}, 0\right) \\ &\Longrightarrow \left\|2^m g\left(\frac{x}{2}\right) - 2^m h\left(\frac{x}{2}\right)\right\| \leq LK\varphi_m(x,0) \\ &\Longrightarrow d(Tg, Th) \leq LK. \end{aligned} \tag{3.9}$$

Hence we see that

$$d(Tg, Th) \leq Ld(g,h) \tag{3.10}$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L. Putting $y = 0$ in (3.2), we have

$$\left\|2f(2x) - 2^{m+1}f(x)\right\| \leq \varphi_m(x, 0) \tag{3.11}$$

for all $x \in X$ and so

$$\left\| f(x) - 2^m f\left(\frac{x}{2}\right) \right\| \le \frac{1}{2}\varphi_m\left(\frac{x}{2}, 0\right) \le \frac{L}{2^{m+1}}\varphi_m(x, 0) \tag{3.12}$$

for all $x \in X$, that is, $d(f, Tf) \le L/2^{m+1} < \infty$.

Now, from Theorem 2.2, it follows that there exists a fixed point $\mathfrak{F}$ of T in Ω such that

$$\mathfrak{F}(x) = \lim_{n\to\infty} 2^{mn} f\left(\frac{x}{2^n}\right) \tag{3.13}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, \mathfrak{F}) = 0$.

On the other hand, it follows from (3.2), (3.6), and (3.13) that

$$\left\| \Delta_m \mathfrak{F}(x, y) \right\| = \lim_{n\to\infty} 2^{mn} \left\| \Delta_m f\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \right\| \le \lim_{n\to\infty} 2^{mn} \varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \tag{3.14}$$

for all $x, y \in X$ and so $\Delta_m \mathfrak{F}(x, y) = 0$. By the result in [44, 45, 47], $\mathfrak{F}$ is m-mapping and so it follows from the definition of $\mathfrak{F}$, (3.3) and (3.7) that

$$\begin{aligned} \left\| \mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)] \right\| &= \lim_{n\to\infty} 2^{3mn} \left\| f\left(\frac{[x, y, z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right) \right] \right\| \\ &\le \lim_{n\to\infty} 2^{3mn} \varphi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) \\ &= 0 \end{aligned} \tag{3.15}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)]$.

According to Theorem 2.2, since $\mathfrak{F}$ is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, $\mathfrak{F}$ is the unique mapping such that

$$\left\| f(x) - \mathfrak{F}(x) \right\| \le K\varphi_m(x, 0) \tag{3.16}$$

for all $x \in X$ and $K > 0$. Again, using Theorem 2.2, we have

$$d(f, \mathfrak{F}) \le \frac{1}{1 - L} d(f, Tf) \le \frac{L}{2^{m+1}(1 - L)} \tag{3.17}$$

and so

$$\left\| f(x) - \mathfrak{F}(x) \right\| \le \frac{L}{2^{m+1}(1 - L)} \varphi_m(x, 0) \tag{3.18}$$

for all $x \in X$. This completes the proof. □

Corollary 3.2. *Let θ, r, p be nonnegative real numbers with $r, p > m$ and $(3p - r)/2 \geq m$. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\|\Delta_m f(x, y)\right\| \leq \theta\left(\|x\|^r + \left\|y\right\|^r\right), \tag{3.19}$$

$$\left\|f([x, y, z]) - [f(x), f(y), f(z)]\right\| \leq \theta\left(\|x\|^p \cdot \left\|y\right\|^p \cdot \|z\|^p\right) \tag{3.20}$$

for all $x, y, z \in X$. Then there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ satisfying

$$\left\|f(x) - \mathfrak{F}(x)\right\| \leq \frac{\theta}{2(2^r - 2^m)}\|x\|^r \tag{3.21}$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi_m(x, y) := \theta\left(\|x\|^r + \left\|y\right\|^r\right), \qquad \psi_m(x, y, z) := \theta\left(\|x\|^p \cdot \left\|y\right\|^p \cdot \|z\|^p\right) \tag{3.22}$$

for all $x, y, z \in X$. Then we can choose $L = 2^{m-r}$ and so the desired conclusion follows. □

Remark 3.3. Let $f : X \to Y$ be a mapping with $f(0) = 0$ such that there exist functions $\varphi_m : X \times X \to [0, \infty)$ and $\psi_m : X \times X \times X \to [0, \infty)$ satisfying (3.2) and (3.3). Let $0 < L < 1$ be a constant such that

$$\varphi_m(2x, 2y) \leq 2^m L\varphi_m(x, y), \qquad \psi_m(2x, 2y, 2z) \leq 2^{3m} L\psi_m(x, y, z) \tag{3.23}$$

for all $x, y, z \in X$. By the similar method as in the proof of Theorem 3.1, one can show that there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ satisfying

$$\left\|f(x) - \mathfrak{F}(x)\right\| \leq \frac{1}{2^{m+1}(1 - L)}\varphi_m(x, 0) \tag{3.24}$$

for all $x \in X$. For the case

$$\varphi_m(x, y) := \delta + \theta\left(\|x\|^r + \left\|y\right\|^r\right), \qquad \psi_m(x, y, z) := \delta + \theta\left(\|x\|^p \cdot \left\|y\right\|^p \cdot \|z\|^p\right), \tag{3.25}$$

where θ, δ are nonnegative real numbers and $0 < r, p < m$ and $(3p - r)/2 \leq m$, there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ satisfying

$$\left\|f(x) - \mathfrak{F}(x)\right\| \leq \frac{\delta}{2(2^m - 2^r)} + \frac{\theta}{2(2^m - 2^r)}\|x\|^r \tag{3.26}$$

for all $x \in X$.

In the following, we formulate and prove a theorem in superstability of C^*-ternary m-homomorphism in C^*-ternary rings for the functional equation (2.4).

Theorem 3.4. *Suppose that there exist functions $\varphi_m : X \times X \to [0,\infty)$, $\psi_m : X \times X \times X \to [0,\infty)$ and a constant $0 < L < 1$ such that*

$$\varphi_m\left(0, \frac{y}{2}\right) \leq \frac{L}{2^m}\varphi_m(0,y),$$
$$\psi_m\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\right) \leq \frac{L}{2^{3m}}\psi_m(x,y,z) \tag{3.27}$$

for all $x, y, z \in X$. Moreover, if $f : X \to Y$ is a mapping such that

$$\|\Delta_m f(x,y)\| \leq \varphi_m(0,y), \tag{3.28}$$

$$\|f([x,y,z]) - [f(x), f(y), f(z)]\| \leq \psi_m(x,y,z) \tag{3.29}$$

for all $x, y, z \in X$, then f is a C^*-ternary m-homomorphism.

Proof. It follows from (3.27) that

$$\lim_{n\to\infty} 2^{mn}\varphi_m\left(0, \frac{y}{2^n}\right) = 0, \tag{3.30}$$

$$\lim_{n\to\infty} 2^{3mn}\psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{3.31}$$

for all $x, y, z \in X$. We have $f(0) = 0$ since $\varphi_m(0,0) = 0$. Letting $y = 0$ in (3.28), we get $f(2x) = 2^m f(x)$ for all $x \in X$. By using induction, we obtain

$$f(2^n x) = 2^{mn} f(x) \tag{3.32}$$

for all $x \in X$ and $n \in \mathbb{N}$ and so

$$f(x) = 2^{mn} f\left(\frac{x}{2^n}\right) \tag{3.33}$$

for all $x \in X$ and $n \in \mathbb{N}$. It follows from (3.29) and (3.33) that

$$\begin{aligned}
&\|f([x,y,z]) - [f(x), f(y), f(z)]\| \\
&\quad = 2^{3mn}\left\| f\left(\frac{[x,y,z]}{2^{3n}}\right) - \left[f\left(\frac{x}{2^n}\right), f\left(\frac{y}{2^n}\right), f\left(\frac{z}{2^n}\right)\right]\right\| \\
&\quad \leq 2^{3mn}\psi_m\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right)
\end{aligned} \tag{3.34}$$

for all $x, y, z \in X$, and $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in (3.34) and using (3.31), we have $f([x,y,z]) = [f(x), f(y), f(z)]$ for all $x, y, z \in X$.

On the other hand, we have

$$\|\Delta_m f(x,y)\| = 2^{mn}\left\|\Delta_m f\left(\frac{x}{2^n}, \frac{y}{2^n}\right)\right\| \leq 2^{mn}\varphi_m\left(0, \frac{y}{2^n}\right) \tag{3.35}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Thus, letting $n \to \infty$ in (3.35) and using (3.30), we have $\Delta_m f(x, y) = 0$ for all $x, y \in X$. Therefore, f is a C^*-ternary m-homomorphism. This completes the proof. □

Corollary 3.5. *Let θ, r, s be nonnegative real numbers with $r > m$ and $s > 3m$. If $f : X \to Y$ is a function such that*

$$\|\Delta_m f(x, y)\| \leq \theta\|y\|^r, \qquad \|f([x, y, z]) - [f(x), f(y), f(z)]\| \leq \theta(\|x\|^s + \|y\|^s + \|z\|^s) \tag{3.36}$$

for all $x, y, z \in X$, then f is a C^*-ternary m-homomorphism.

Remark 3.6. Let θ, r be nonnegative real numbers with $r < m$. Suppose that there exists a function $\varphi_m : X \times X \times X \to [0, \infty)$ and a constant $0 < L < 1$ such that

$$\varphi_m(2x, 2y, 2z) \leq 2^{3m} L \varphi_m(x, y, z) \tag{3.37}$$

for all $x, y, z \in X$. Moreover, if $f : X \to Y$ is a mapping such that

$$\|\Delta_m f(x, y)\| \leq \theta\|y\|^r, \qquad \|f([x, y, z]) - [f(x), f(y), f(z)]\| \leq \varphi_m(x, y, z) \tag{3.38}$$

for all $x, y, z \in X$, then f is a C^*-ternary m-homomorphism.

In the rest of this section, assume that X is a unital C^*-ternary algebra with the unit e and Y is a C^*-ternary algebra with the unit e'.

Theorem 3.7. *Let θ, r, p be positive real numbers with $r > m, p > m$ and $(3p-r)/2 \geq m$ (resp. $(3p-r)/2 \leq m$). Suppose that $f : X \to Y$ is a mapping satisfying* (3.19) *and* (3.20). *If there exist a real number $\lambda > 1$ and $x_0 \in X$ such that $\lim_{n\to\infty} \lambda^{mn} f(x_0/\lambda^n) = e'$ (resp. $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$), then the mapping $f : X \to Y$ is a C^*-ternary m-homomorphism.*

Proof. By Corollary 3.2, there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ such that

$$\|f(x) - \mathfrak{F}(x)\| \leq \frac{\theta}{2(2^r - 2^m)} \|x\|^r \tag{3.39}$$

for all $x \in X$. It follows from (3.39) that

$$\mathfrak{F}(x) = \lim_{n\to\infty} \lambda^{mn} f\left(\frac{x}{\lambda^n}\right) \quad \left(\mathfrak{F}(x) = \lim_{n\to\infty} \frac{1}{\lambda^{mn}} f(\lambda^n x)\right) \tag{3.40}$$

for all $x \in X$ and $\lambda > 1$. Therefore, by the assumption, we get that $\mathfrak{F}(x_0) = e'$.

Let $\lambda > 1$ and $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$. It follows from (3.20) that

$$\begin{aligned}
&\left\| [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)] - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \left\| \mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \lim_{n\to\infty} \lambda^{2mn} \left\| f\left(\left[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\right]\right) - \left[f\left(\frac{x}{\lambda^n}\right), f\left(\frac{y}{\lambda^n}\right), f(z)\right] \right\| \\
&\quad \le \theta \lim_{n\to\infty} \lambda^{2mn} \left[\frac{1}{\lambda^{2np}} \left(\|x\|^p \cdot \|y\|^p\right) \cdot \|z\|^p\right] \\
&\quad = 0
\end{aligned} \tag{3.41}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), f(z)]$ for all $x, y, z \in X$. Letting $x = y = x_0$ in the last equality, we get $f(z) = \mathfrak{F}(z)$ for all $z \in X$.

Similarly, one can show that $f(z) = \mathfrak{F}(z)$ for all $z \in X$ when $\lambda > 1$ and $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$. Therefore, the mapping $f : X \to Y$ is a C^*-ternary m-homomorphism. This completes the proof. □

Theorem 3.8. *Let θ, r, p be positive real numbers with $r > m$ and $p > 2m$ and $(3p - r)/2 \ge m$ (resp. $(3p - r)/2 \le m$). Suppose that $f : X \to Y$ is a mapping satisfying* (3.19) *and*

$$\left\| f([x, y, z]) - [f(x), f(y), f(z)] \right\| \le \theta\left(\|x\|^p \cdot \|y\|^p + \|y\|^p \cdot \|z\|^p + \|x\|^p \cdot \|z\|^p\right) \tag{3.42}$$

for all $x, y, z \in X$. If there exist a real number $\lambda > 1$ and $x_0 \in X$ such that $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$ (resp. $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x_0) = e'$), then the mapping $f : X \to Y$ is a C^*-ternary m-homomorphism.

Proof. By Theorem 3.1 there exists a unique C^*-ternary m-homomorphism $\mathfrak{F} : X \to Y$ such that

$$\left\| f(x) - \mathfrak{F}(x) \right\| \le \frac{\theta}{2(2^r - 2^m)} \|x\|^r \tag{3.43}$$

for all $x \in X$. It follows from (3.43) that

$$\mathfrak{F}(x) = \lim_{n\to\infty} \lambda^{mn} f\left(\frac{x}{\lambda^n}\right) \qquad \left(\mathfrak{F}(x) = \lim_{n\to\infty} \frac{1}{\lambda^{mn}} f(\lambda^n x)\right) \tag{3.44}$$

for all $x \in X$ and $\lambda > 1$. Therefore, by the assumption, we get that $\mathfrak{F}(x_0) = e'$.

Let $\lambda > 1$ and $\lim_{n\to\infty}\lambda^{mn} f(x_0/\lambda^n) = e'$. It follows from (3.20) that

$$\begin{aligned}
&\left\| [\mathfrak{F}(x), \mathfrak{F}(y), \mathfrak{F}(z)] - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \left\| \mathfrak{F}([x, y, z]) - [\mathfrak{F}(x), \mathfrak{F}(y), f(z)] \right\| \\
&\quad = \lim_{n\to\infty} \lambda^{2mn} \left\| f\left(\left[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\right]\right) - \left[f\left(\frac{x}{\lambda^n}\right), f\left(\frac{y}{\lambda^n}\right), f(z)\right] \right\| \\
&\quad \le \theta \lim_{n\to\infty} \lambda^{2mn} \left[\frac{1}{\lambda^{2np}} \|x\|^p \cdot \|y\|^p + \frac{1}{\lambda^{np}} \|y\|^p \cdot \|z\|^p + \frac{1}{\lambda^{np}} \|x\|^p \cdot \|z\|^p \right] \\
&\quad = 0
\end{aligned} \tag{3.45}$$

for all $x, y, z \in X$ and so $\mathfrak{F}([x, y, z]) = [\mathfrak{F}(x), \mathfrak{F}(y), f(z)]$ for all $x, y, z \in X$. Letting $x = y = x_0$ in the last equality, we get $f(z) = \mathfrak{F}(z)$ for all $z \in X$.

Similarly, one can show that $f(z) = \mathfrak{F}(z)$ for all $z \in X$ when $\lambda > 1$ and $\lim_{n\to\infty}(1/\lambda^{mn}) f(\lambda^n x) = e'$. Therefore, the mapping $f : X \to Y$ is a C^*-ternary m-homomorphism. This completes the proof. □

Acknowledgment

This work was supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00050).

References

[1] S. Duplij, "Ternary Hopf algebras," in *Symmetry in Nonlinear Mathematical Physics*, vol. 2 of *Natsīonal'noï Akademīï Nauk Ukraïni Mat. Zastos., 43, Part 1, 2; Natsīonal'noï Akademīï Nauk Ukraïni, Institute of Mathematics, Kyiv*, pp. 439–448, Natsīonal'noï Akademīï Nauk Ukraïni, Kiev, Ukraine, 2002.

[2] N. Bazunova, A. Borowiec, and R. Kerner, "Universal differential calculus on ternary algebras," *Letters in Mathematical Physics*, vol. 67, no. 3, pp. 195–206, 2004.

[3] G. L. Sewell, *Quantum Mechanics and Its Emergent Macrophysics*, Princeton University Press, Princeton, NJ, USA, 2002.

[4] R. Haag and D. Kastler, "An algebraic approach to quantum field theory," *Journal of Mathematical Physics*, vol. 5, pp. 848–861, 1964.

[5] F. Bagarello and G. Morchio, "Dynamics of mean-field spin models from basic results in abstract differential equations," *Journal of Statistical Physics*, vol. 66, no. 3-4, pp. 849–866, 1992.

[6] S. M. Ulam, *Problems in Modern Mathematics*, John Wiley & Sons, New York, NY, USA, 1964.

[7] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[8] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[9] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[10] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[11] Z. Gajda, "On stability of additive mappings," *International Journal of Mathematics and Mathematical Sciences*, vol. 14, no. 3, pp. 431–434, 1991.

[12] Th. M. Rassias and P. Šemrl, "On the behavior of mappings which do not satisfy Hyers-Ulam stability," *Proceedings of the American Mathematical Society*, vol. 114, no. 4, pp. 989–993, 1992.

[13] G. Isac and Th. M. Rassias, "On the Hyers-Ulam stability of ψ-additive mappings," *Journal of Approximation Theory*, vol. 72, no. 2, pp. 131–137, 1993.

[14] Y.-H. Lee and K.-W. Jun, "On the stability of approximately additive mappings," *Proceedings of the American Mathematical Society*, vol. 128, no. 5, pp. 1361–1369, 2000.

[15] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Journal of Functional Analysis*, vol. 46, no. 1, pp. 126–130, 1982.

[16] J. M. Rassias, "On approximation of approximately linear mappings by linear mappings," *Bulletin des Sciences Mathématiques*, vol. 108, no. 4, pp. 445–446, 1984.

[17] J. M. Rassias, "Solution of a problem of Ulam," *Journal of Approximation Theory*, vol. 57, no. 3, pp. 268–273, 1989.

[18] P. Găvruţa, "A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings," *Journal of Mathematical Analysis and Applications*, vol. 184, no. 3, pp. 431–436, 1994.

[19] D. G. Bourgin, "Classes of transformations and bordering transformations," *Bulletin of the American Mathematical Society*, vol. 57, pp. 223–237, 1951.

[20] D. G. Bourgin, "Approximately isometric and multiplicative transformations on continuous function rings," *Duke Mathematical Journal*, vol. 16, pp. 385–397, 1949.

[21] R. Badora, "On approximate ring homomorphisms," *Journal of Mathematical Analysis and Applications*, vol. 276, no. 2, pp. 589–597, 2002.

[22] J. Baker, J. Lawrence, and F. Zorzitto, "The stability of the equation $f(x+y) = f(x)f(y)$," *Proceedings of the American Mathematical Society*, vol. 74, pp. 242–246, 1979.

[23] M. B. Savadkouhi, M. E. Gordji, J. M. Rassias, and N. Ghobadipour, "Approximate ternary Jordan derivations on Banach ternary algebras," *Journal of Mathematical Physics*, vol. 50, no. 4, article 042303, 9 pages, 2009.

[24] M. Eshaghi Gordji and A. Najati, "Approximately J*-homomorphisms: a fixed point approach," *Journal of Geometry and Physics*, vol. 60, no. 5, pp. 809–814, 2010.

[25] C.-G. Park, "Homomorphisms between Lie JC^*-algebras and Cauchy-Rassias stability of Lie JC^*-algebra derivations," *Journal of Lie Theory*, vol. 15, no. 2, pp. 393–414, 2005.

[26] M. Eshaghi Gordji, M. B. Ghaemi, S. Kaboli Gharetapeh, S. Shams, and A. Ebadian, "On the stability of J^*-derivations," *Journal of Geometry and Physics*, vol. 60, no. 3, pp. 454–459, 2010.

[27] M. Eshaghi Gordji, S. Kaboli Gharetapeh, E. Rashidi, T. Karimi, and M. Aghaei, "Ternary Jordan C^*-derivations in C*-ternary algebras," *Journal of Computational Analysis and Applications*, vol. 12, no. 2, pp. 463–470, 2010.

[28] C. Park and M. E. Gordji, "Comment on "Approximate ternary Jordan derivations on Banach ternary algebras" [Bavand Savadkouhi et al., Journal of Mathematical Physics, vol. 50, article 042303, 2009]," *Journal of Mathematical Physics*, vol. 51, no. 4, article 044102, p. 7, 2010.

[29] C. Park and A. Najati, "Generalized additive functional inequalities in Banach algebras," *International Journal of Nonlinear Analysis and Applications*, vol. 1, pp. 54–62, 2010.

[30] C. Park and Th. M. Rassias, "Isomorphisms in unital C^*-algebras," *International Journal of Nonlinear Analysis and Applications*, vol. 1, pp. 1–10, 2010.

[31] M. Eshaghi Gordji, T. Karimi, and S. Kaboli Gharetapeh, "Approximately n-Jordan homomorphisms on Banach algebras," *Journal of Inequalities and Applications*, vol. 2009, Article ID 870843, 8 pages, 2009.

[32] M. Eshaghi Gordji, J. M. Rassias, and N. Ghobadipour, "Generalized Hyers-Ulam stability of generalized (n,k)-derivations," *Abstract and Applied Analysis*, vol. 2009, Article ID 437931, 8 pages, 2009.

[33] M. Eshaghi Gordji and M. B. Savadkouhi, "Approximation of generalized homomorphisms in quasi-Banach algebras," *Analele Stiintifice ale Universitatii Ovidius Constanta*, vol. 17, no. 2, pp. 203–213, 2009.

[34] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, vol. 31, Cambridge University Press, Cambridge, UK, 1989.

[35] P. Kannappan, "Quadratic functional equation and inner product spaces," *Results in Mathematics. Resultate der Mathematik*, vol. 27, no. 3-4, pp. 368–372, 1995.

[36] S. Czerwik, "On the stability of the quadratic mapping in normed spaces," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 62, pp. 59–64, 1992.

[37] M. Eshaghi Gordji and H. Khodaei, "Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi-Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 11, pp. 5629–5643, 2009.

[38] M. Eshaghi Gordji and H. Khodaei, "On the generalized Hyers-Ulam-Rassias stability of quadratic functional equations," *Abstract and Applied Analysis*, vol. 2009, Article ID 923476, 11 pages, 2009.

[39] G. L. Forti, "An existence and stability theorem for a class of functional equations," *Stochastica*, vol. 4, no. 1, pp. 23–30, 1980.

[40] G.-L. Forti, "Elementary remarks on Ulam-Hyers stability of linear functional equations," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 1, pp. 109–118, 2007.

[41] S.-M. Jung, "Hyers-Ulam-Rassias stability of Jensen's equation and its application," *Proceedings of the American Mathematical Society*, vol. 126, no. 11, pp. 3137–3143, 1998.

[42] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, Fla, USA, 2001.

[43] H. Khodaei and Th. M. Rassias, "Approximately generalized additive functions in several variables," *International Journal of Nonlinear Analysis*, vol. 1, pp. 22–41, 2010.

[44] K.-W. Jun and H.-M. Kim, "The generalized Hyers-Ulam-Rassias stability of a cubic functional equation," *Journal of Mathematical Analysis and Applications*, vol. 274, no. 2, pp. 867–878, 2002.

[45] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.

[46] C. Park and J. Cui, "Generalized stability of C^*-ternary quadratic mappings," *Abstract and Applied Analysis*, vol. 2007, Article ID 23282, 6 pages, 2007.

[47] J.-H. Bae and W.-G. Park, "A functional equation having monomials as solutions," *Applied Mathematics and Computation*, vol. 216, no. 1, pp. 87–94, 2010.

[48] J.-H. Bae and W.-G. Park, "On a cubic equation and a Jensen-quadratic equation," *Abstract and Applied Analysis*, vol. 2007, Article ID 45179, 10 pages, 2007.

[49] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

[50] I. A. Rus, *Principles and Applications of Fixed Point Theory*, Cluj-Napoca, 1979.

[51] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser Boston Inc., Boston, Mass, USA, 1998.

[52] E. Zeidler, "Nonlinear functional analysis and its applications," in *Fixed-Point Theorems. 2*, vol. 1, p. xxiii +909, Springer, 1993.

[53] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory*, vol. 4, no. 1, pp. 91–96, 2003.

[54] L. Cădariu and V. Radu, "Fixed points and the stability of Jensen's functional equation," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 4, no. 1, article 4, p. 7, 2003.

[55] L. Cădariu and V. Radu, "On the stability of the Cauchy functional equation: a fixed point approach," in *Iteration Theory*, vol. 346 of *Grazer Mathematische Berichte*, pp. 43–52, Karl-Franzens-Universitaet Graz, Graz, Austria, 2004.

[56] A. Ebadian, N. Ghobadipour, and M. E. Gordji, "A fixed point method for perturbation of bimultipliers and Jordan bimultipliers in C^*-ternary algebras," *Journal of Mathematical Physics*, vol. 51, 10 pages, 2010.

[57] M. Eshaghi Gordji and H. Khodaei, "The fixed point method for fuzzy approximation of a functional equation associated with inner product spaces," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 140767, 15 pages, 2010.