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Letter to the Editor

A Counterexample to "An Extension of Gregus Fixed Point Theorem"

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In the paper by Olaleru and Akewe (2007), the authors tried to generalize Gregus fixed point theorem. In this paper we give a counterexample on their main statement.

1. Introduction

Let X be a Banach space and C be a closed convex subset of X. In 1980 Greguš [1] proved the following results.

Theorem 1.1. *Let $T : C \to C$ be a mapping satisfying the inequality*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Tx\| + c\|y - Ty\|, \tag{1.1}$$

for all $x, y \in C$, where $0 < a < 1,\ b, c \geq 0$, and $a + b + c = 1$. Then T has a unique fixed point.

Several papers have been written on the Gregus fixed point theorem. For example, see [2–6]. We can combine the Gregus condition by the following inequality, where T is a mapping on metric space (X, d):

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty), \tag{1.2}$$

for all $x, y \in X$, where $0 < a < 1,\ b, c, e, f \geq 0$, and $a + b + c + e + f = 1$.

Definition 1.2. Let X be a topological vector space on $\mathbb{K}(=\mathbb{C}$ *or* $\mathbb{R})$. The mapping $F: X \to \mathbb{R}$ is said to be an *F-norm* such that for all $x, y \in X$

(i) $F(x) \geq 0$,

(ii) $F(x) = 0 \to x = 0$,

(iii) $F(x+y) \leq F(x) + F(y)$,

(iv) $F(\lambda x) \leq F(x)$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$,

(v) if $\lambda_n \to 0$ and $\lambda_n \in \mathbb{K}$, then $F(\lambda_n x) \to 0$.

In 2007, Olaleru and Akewe [7] considered the existence of fixed point of T when T is defined on a closed convex subset C of a complete metrizable topological vector space X and satisfies condition (1.2) and extended the Gregus fixed point.

Theorem 1.3. *Let C be a closed convex subset of a complete metrizable topological vector space X and $T: C \to C$ a mapping that satisfies*

$$F(Tx - Ty) \leq aF(x-y) + bF(x-Tx) + cF(y-Ty) + eF(y-Tx) + fF(x-Ty) \tag{1.3}$$

for all $x, y \in X$, where F is an F-norm on X, $0 < a < 1$, $b, c, e, f \geq 0$, and $a+b+c+e+f = 1$. Then T has a unique fixed point.

Here, we give an example to show that the above mentioned theorem is not correct.

2. Counterexample

Example 2.1. Let $X = \mathbb{R}$ endowed with the Euclidean metric and $C = X$. Let $T: C \to C$ defined by $Tx = x+1$. Let $0 < a < 1$ and $e > 0$ such that $a + 2e = 1$. Then for all $x \in C$ such that $y > x$, we have that

$$\begin{aligned}
&|Tx - Ty| \leq a|x-y| + e|y-Tx| + e|x-Ty| \\
&\Longleftrightarrow y - x \leq a(y-x) + e|y-x-1| + e|x-y-1| \\
&\Longleftrightarrow y - x \leq a(y-x) + e|y-x-1| + e(y+1-x) \\
&\Longleftrightarrow e(y-x) = (1-a-e)(y-x) \leq e|y-x-1| + e \\
&\Longleftrightarrow y - x \leq |y-x-1| + 1.
\end{aligned} \tag{2.1}$$

We have two cases, $y > x+1$ or $y \leq x+1$.

If $y > x+1$, then $y - x = y - x - 1 + 1$, and hence inequality (2.1) is true. If $y \leq x+1$, then $0 < y - x \leq 1$, and so $y - x \leq |y-x-1| + 1$, and hence inequality (2.1) is true. So condition (1.3) holds for $b = c = 0$ and $e = f$, but T has not fixed point.

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