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*Research Article*

# A New Strong Convergence Theorem for Equilibrium Problems and Fixed Point Problems in Banach Spaces

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We introduce a new iterative sequence for finding a common element of the set of fixed points of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a Banach space. Then, we study the strong convergence of the sequences. With an appropriate setting, we obtain the corresponding results due to Takahashi-Takahashi and Takahashi-Zembayashi. Some of our results are established with weaker assumptions.

## 1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let $E$ be a Banach space, $E^*$ the dual space of $E$ and $C$ a closed convex subsets of $E$. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The *equilibrium problem* is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. The equilibrium problems include fixed point problems, optimization problems, variational inequality problems, and Nash equilibrium problems as special cases.

Let $E$ be a smooth Banach space and $J$ the normalized duality mapping from $E$ to $E^*$. Alber [1] considered the following functional $\varphi : E \times E \to [0, \infty)$ defined by

$$\varphi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E). \tag{1.2}$$

Using this functional, Matsushita and Takahashi [2, 3] studied and investigated the following mappings in Banach spaces. A mapping $S : C \to E$ is *relatively nonexpansive* if the following properties are satisfied:

(R1) $F(S) \neq \emptyset$,

(R2) $\varphi(p, Sx) \leq \varphi(p, x)$ for all $p \in F(S)$ and $x \in C$,

(R3) $F(S) = \widehat{F}(S)$,

where $F(S)$ and $\widehat{F}(S)$ denote the set of fixed points of $S$ and the set of asymptotic fixed points of $S$, respectively. It is known that $S$ satisfies condition (R3) if and only if $I - S$ is demiclosed at zero, where $I$ is the identity mapping; that is, whenever a sequence $\{x_n\}$ in $C$ converges weakly to $p$ and $\{x_n - Sx_n\}$ converges strongly to 0, it follows that $p \in F(S)$. In a Hilbert space $H$, the duality mapping $J$ is an identity mapping and $\varphi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Hence, if $S : C \to H$ is nonexpansive (i.e., $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$), then it is relatively nonexpansive.

Recently, many authors studied the problems of finding a common element of the set of fixed points for a mapping and the set of solutions of equilibrium problem in the setting of Hilbert space and uniformly smooth and uniformly convex Banach space, respectively (see, e.g., [4–21] and the references therein). In a Hilbert space $H$, S. Takahashi and W. Takahashi [17] introduced the iteration as follows: sequence $\{x_n\}$ generated by $u, x_1 \in C$,

$$\begin{gathered} F(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S(\alpha_n u + (1 - \alpha_n) z_n), \end{gathered} \tag{1.3}$$

for every $n \in \mathbb{N}$, where $S$ is nonexpansive, $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0, 1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that $\{x_n\}$ converges strongly to some element in $F(S) \cap \mathrm{EP}(F)$. In 2009, Takahashi and Zembayashi [19] proposed the iteration in a uniformly smooth and uniformly convex Banach space as follows: a sequence $\{x_n\}$ generated by $u_1 \in E$,

$$\begin{gathered} x_n \in C \text{ such that } F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n \rangle \geq 0, \quad \forall y \in C, \\ u_{n+1} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \end{gathered} \tag{1.4}$$

for every $n \in \mathbb{N}$, $S$ is relatively nonexpansive, $\{\alpha_n\}$ is an appropriate sequence in $[0, 1]$, and $\{r_n\}$ is an appropriate positive real sequence. They proved that if $J$ is weakly sequentially continuous, then $\{x_n\}$ converges *weakly* to some element in $F(S) \cap \mathrm{EP}(F)$.

Motivated by S. Takahashi and W. Takahashi [17] and Takahashi and Zembayashi [19], we prove a strong convergence theorem for finding a common element of the fixed points set of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a uniformly smooth and uniformly convex Banach space.

## 2. Preliminaries

We collect together some definitions and preliminaries which are needed in this paper. We say that a Banach space $E$ is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \qquad x \neq y \text{ imply } \left\|\frac{x+y}{2}\right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x-y\| \geq \varepsilon \text{ imply } \left\|\frac{x+y}{2}\right\| \leq 1-\delta. \tag{2.2}$$

It is known that if $E$ is a uniformly convex Banach space, then $E$ is reflexive and strictly convex. We say that $E$ is *uniformly smooth* if the dual space $E^*$ of $E$ is uniformly convex. A Banach space $E$ is *smooth* if the limit $\lim_{t\to 0}((\|x+ty\|-\|x\|)/t)$ exists for all norm one elements $x$ and $y$ in $E$. It is not hard to show that if $E$ is reflexive, then $E$ is smooth if and only if $E^*$ is strictly convex.

Let $E$ be a smooth Banach space. The function $\varphi : E \times E \to \mathbb{R}$ (see [1]) is defined by

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad (x, y \in E), \tag{2.3}$$

where the *duality mapping* $J : E \to E^*$ is given by

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2 \quad (x \in E). \tag{2.4}$$

It is obvious from the definition of the function $\varphi$ that

$$(\|x\| - \|y\|)^2 \leq \varphi(x,y) \leq (\|x\| + \|y\|)^2, \tag{2.5}$$

$$\varphi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\varphi(x,y) + (1-\lambda)\varphi(x,z), \tag{2.6}$$

for all $\lambda \in [0,1]$ and $x, y, z \in E$. The following lemma is an analogue of Xu's inequality [22, Theorem 2] with respect to $\varphi$.

**Lemma 2.1.** *Let $E$ be a uniformly smooth Banach space and $r > 0$. Then, there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0,\infty)$ such that $g(0) = 0$ and*

$$\varphi\left(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\right) \leq \lambda\varphi(x,y) + (1-\lambda)\varphi(x,z) - \lambda(1-\lambda)g(\|Jy - Jz\|), \tag{2.7}$$

*for all $\lambda \in [0,1]$, $x \in E$, and $y, z \in B_r$.*

It is also easy to see that if $\{x_n\}$ and $\{y_n\}$ are bounded sequences of a smooth Banach space $E$, then $x_n - y_n \to 0$ implies that $\varphi(x_n, y_n) \to 0$.

**Lemma 2.2** (see [23, Proposition 2]). *Let $E$ be a uniformly convex and smooth Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences of $E$ such that $\{x_n\}$ or $\{y_n\}$ is bounded. If $\varphi(x_n, y_n) \to 0$, then $x_n - y_n \to 0$.*

*Remark 2.3.* For any bounded sequences $\{x_n\}$ and $\{y_n\}$ in a uniformly convex and uniformly smooth Banach space $E$, we have

$$\varphi(x_n, y_n) \longrightarrow 0 \iff x_n - y_n \longrightarrow 0 \iff Jx_n - Jy_n \longrightarrow 0. \tag{2.8}$$

Let $C$ be a nonempty closed convex subset of a reflexive, strictly convex, and smooth Banach space $E$. It is known that [1, 23] for any $x \in E$, there exists a unique point $\hat{x} \in C$ such that

$$\varphi(\hat{x}, x) = \min_{y \in C} \varphi(y, x). \tag{2.9}$$

Following Alber [1], we denote such an element $\hat{x}$ by $\Pi_C x$. The mapping $\Pi_C$ is called the *generalized projection* from $E$ onto $C$. It is easy to see that in a Hilbert space, the mapping $\Pi_C$ coincides with the metric projection $P_C$. Concerning the generalized projection, the following are well known.

**Lemma 2.4** (see [23, Propositions 4 and 5]). *Let $C$ be a nonempty closed convex subset of a reflexive, strictly convex and smooth Banach space $E$, $x \in E$, and $\hat{x} \in C$. Then,*

(a) $\hat{x} = \Pi_C x$ *if and only if* $\langle y - \hat{x}, Jx - J\hat{x} \rangle \leq 0$ *for all* $y \in C$,

(b) $\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x)$ *for all* $y \in C$.

*Remark 2.5.* The generalized projection mapping $\Pi_C$ above is relatively nonexpansive and $F(\Pi_C) = C$.

Let $E$ be a reflexive, strictly convex and smooth Banach space. The duality mapping $J^*$ from $E^*$ onto $E^{**} = E$ coincides with the inverse of the duality mapping $J$ from $E$ onto $E^*$, that is, $J^* = J^{-1}$. We make use of the following mapping $V : E \times E^* \to \mathbb{R}$ studied in Alber [1]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2, \tag{2.10}$$

for all $x \in E$ and $x^* \in E^*$. Obviously, $V(x, x^*) = \varphi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. We know the following lemma (see [1] and [24, Lemma 3.2]).

**Lemma 2.6.** *Let $E$ be a reflexive, strictly convex and smooth Banach space, and let $V$ be as in* (2.10). *Then,*

$$V(x, x^*) + 2\left\langle J^{-1}(x^*) - x, y^* \right\rangle \leq V(x, x^* + y^*), \tag{2.11}$$

*for all $x \in E$ and $x^*, y^* \in E^*$.*

**Lemma 2.7** (see [25, Lemma 2.1]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \tag{2.12}$$

*for all $n \in \mathbb{N}$, where the sequences $\{\gamma_n\}$ in $(0,1)$ and $\{\delta_n\}$ in $\mathbb{R}$ satisfy conditions: $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty}\gamma_n = \infty$, and $\limsup_{n\to\infty}\delta_n \leq 0$. Then, $\lim_{n\to\infty}a_n = 0$.*

**Lemma 2.8** (see [26, Lemma 3.1]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,*

$$a_{m_k} \leq a_{m_k+1}, \qquad a_k \leq a_{m_k+1}, \tag{2.13}$$

*for all $k \in \mathbb{N}$. In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.*

For solving the equilibrium problem, we usually assume that a bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$,

(A2) $F$ is monotone, that is, $F(x,y) + F(y,x) \leq 0$, for all $x, y \in C$,

(A3) for all $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$,

(A4) for all $x \in C$, $F(x,\cdot)$ is convex and lower semicontinuous.

The following lemma gives a characterization of a solution of an equilibrium problem.

**Lemma 2.9** (see [19, Lemma 2.8 ]). *Let $C$ be a nonempty closed convex subset of a reflexive, strictly convex, and uniformly smooth Banach space $E$. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions* (A1)–(A4). *For $r > 0$, define a mapping $T_r : E \to C$ so-called the resolvent of $F$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \ \forall y \in C \right\}, \tag{2.14}$$

*for all $x \in E$. Then, the following hold:*

(i) *$T_r$ is single-valued,*

(ii) *$T_r$ is a firmly nonexpansive-type mapping* [27], *that is, for all $x, y \in E$*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle, \tag{2.15}$$

(iii) $F(T_r) = \mathrm{EP}(F)$,

(iv) $\mathrm{EP}(F)$ *is closed and convex,*

**Lemma 2.10** (see [4, Lemma 2.3]). *Let $C$ be a nonempty closed convex subset of a Banach space $E$, $F$ a bifunction from $C \times C \to \mathbb{R}$ satisfying conditions* (A1)–(A4) *and $z \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $F(y,z) \leq 0$ for all $y \in C$.*

*Remark 2.11* (see [27]). Let $C$ be a nonempty subset of a smooth Banach space $E$. If $S : C \to E$ is a firmly nonexpansive-type mapping, then

$$\varphi(z, Sx) \leq \varphi(z, Sx) + \varphi(Sx, x) \leq \varphi(z, x), \tag{2.16}$$

for all $x \in C$ and $z \in F(S)$. In particular, $S$ satisfies condition (R2).

**Lemma 2.12** (see [3, Proposition 2.4]). *Let $C$ be a nonempty closed convex subset of a strictly convex and smooth Banach space $E$ and $S : C \to E$ a relatively nonexpansive mapping. Then, $F(S)$ is closed and convex.*

## 3. Main Results

In this section, we prove a strong convergence theorem for finding a common element of the fixed points set of a relatively nonexpansive mapping and the set of solutions of an equilibrium problem in a uniformly convex and uniformly smooth Banach space.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions* (A1)–(A4) *and $S : C \to E$ a relatively nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \varnothing$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C$, $u_1 \in E$ and*

$$\begin{aligned} &F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall y \in C, \\ &y_n = \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jx_n), \\ &u_{n+1} = J^{-1}(\beta_n Jx_n + (1 - \beta_n)JSy_n), \end{aligned} \tag{3.1}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [a, b] \subset (0,1)$, and $\{r_n\} \subset [c, \infty) \subset (0, \infty)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)\cap \mathrm{EP}(F)}u$.*

*Proof.* Note that $x_n$ can be rewritten as $x_n = T_{r_n}u_n$. Since $F(S) \cap \mathrm{EP}(F)$ is nonempty, closed, and convex, we put $\widehat{u} = \Pi_{F(S)\cap \mathrm{EP}(F)}u$. Since $\Pi_C$, $T_{r_n}$, and $S$ satisfy condition (R2), by (2.6), we get

$$\begin{aligned} \varphi(\widehat{u}, y_n) &\leq \varphi\Big(\widehat{u}, J^{-1}(\alpha_n Ju + (1 - \alpha_n)Jx_n)\Big) \\ &\leq \alpha_n\varphi(\widehat{u}, u) + (1 - \alpha_n)\varphi(\widehat{u}, x_n) \\ &\leq \alpha_n\varphi(\widehat{u}, u) + (1 - \alpha_n)\varphi(\widehat{u}, u_n), \end{aligned} \tag{3.2}$$

and so

$$\begin{aligned}\varphi(\hat{u},u_{n+1}) &\leq \beta_n\varphi(\hat{u},x_n)+(1-\beta_n)\varphi(\hat{u},Sy_n)\\ &\leq \beta_n\varphi(\hat{u},u_n)+(1-\beta_n)\varphi(\hat{u},y_n)\\ &\leq \alpha_n(1-\beta_n)\varphi(\hat{u},u)+(1-\alpha_n(1-\beta_n))\varphi(\hat{u},u_n)\\ &\leq \max\{\varphi(\hat{u},u),\varphi(\hat{u},u_n)\}.\end{aligned} \tag{3.3}$$

By induction, we have

$$\varphi(z,u_{n+1}) \leq \max\{\varphi(\hat{u},u),\varphi(\hat{u},u_1)\}, \tag{3.4}$$

for all $n\in\mathbb{N}$. This implies that $\{u_n\}$ is bounded and so are $\{x_n\}$, $\{y_n\}$, and $\{Sy_n\}$. Put

$$z_n \equiv J^{-1}(\alpha_n Ju+(1-\alpha_n)Jx_n). \tag{3.5}$$

Then, $y_n \equiv \Pi_C z_n$. Using Lemma 2.6 gives

$$\begin{aligned}\varphi(\hat{u},y_n) &\leq \varphi(\hat{u},z_n)=V(\hat{u},Jz_n)\\ &\leq V(\hat{u},Jz_n-\alpha_n(Ju-J\hat{u}))-2\langle z_n-\hat{u},-\alpha_n(Ju-J\hat{u})\rangle\\ &=\varphi\Big(\hat{u},J^{-1}(\alpha_nJ\hat{u}+(1-\alpha_n)Jx_n)\Big)+2\alpha_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle\\ &\leq \alpha_n\varphi(\hat{u},\hat{u})+(1-\alpha_n)\varphi(\hat{u},x_n)+2\alpha_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle\\ &\leq (1-\alpha_n)\varphi(\hat{u},u_n)+2\alpha_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle.\end{aligned} \tag{3.6}$$

Let $g:[0,2r]\to[0,\infty)$ be a function satisfying the properties of Lemma 2.1, where $r=\sup\{\|x_n\|,\|Sy_n\|:n\in\mathbb{N}\}$. Then, by Remark 2.11 and (3.6), we get

$$\begin{aligned}\varphi(\hat{u},u_{n+1}) &\leq \beta_n\varphi(\hat{u},x_n)+(1-\beta_n)\varphi(\hat{u},Sy_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\leq \beta_n(\varphi(\hat{u},u_n)-\varphi(x_n,u_n))+(1-\beta_n)\varphi(\hat{u},y_n)\\ &\quad -\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\leq \beta_n\varphi(\hat{u},u_n)+(1-\beta_n)((1-\alpha_n)\varphi(\hat{u},u_n)+2\alpha_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle)\\ &\quad -\beta_n\varphi(x_n,u_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\end{aligned} \tag{3.7}$$

$$\begin{aligned}&=(1-\gamma_n)\varphi(\hat{u},u_n)+2\gamma_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle\\ &\quad -\beta_n\varphi(x_n,u_n)-\beta_n(1-\beta_n)g(\|Jx_n-JSy_n\|)\\ &\leq (1-\gamma_n)\varphi(\hat{u},u_n)+2\gamma_n\langle z_n-\hat{u},Ju-J\hat{u}\rangle,\end{aligned} \tag{3.8}$$

where $\gamma_n=\alpha_n(1-\beta_n)$ for all $n\in\mathbb{N}$. Notice that $\{\gamma_n\}\subset(0,1)$ satisfying $\lim_{n\to\infty}\gamma_n=0$ and $\sum_{n=1}^{\infty}\gamma_n=\infty$.

The rest of the proof will be divided into two parts.

*Case 1.* Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\varphi(\hat{u}, u_n)\}_{n=n_0}^{\infty}$ is nonincreasing. In this situation, $\{\varphi(\hat{u}, u_n)\}$ is then convergent. Then,

$$\varphi(\hat{u}, u_n) - \varphi(\hat{u}, u_{n+1}) \longrightarrow 0. \tag{3.9}$$

It follows from (3.7) and $\gamma_n \to 0$ that

$$\beta_n \varphi(x_n, u_n) + \beta_n(1-\beta_n) g(\|Jx_n - JSy_n\|) \longrightarrow 0. \tag{3.10}$$

Since $\{\beta_n\} \subset [a,b] \subset (0,1)$,

$$\varphi(x_n, u_n) \longrightarrow 0, \qquad g(\|Jx_n - JSy_n\|) \longrightarrow 0. \tag{3.11}$$

Consequently, by Remark 2.3,

$$x_n - u_n \longrightarrow 0, \qquad Jx_n - JSy_n \longrightarrow 0, \qquad x_n - Sy_n \longrightarrow 0. \tag{3.12}$$

From (2.6) and $\alpha_n \to 0$, we obtain

$$\varphi(x_n, y_n) \le \varphi(x_n, z_n) \le \alpha_n \varphi(x_n, u) + (1-\alpha_n)\varphi(x_n, x_n) = \alpha_n \varphi(x_n, u) \longrightarrow 0. \tag{3.13}$$

This implies that

$$x_n - y_n \longrightarrow 0, \qquad z_n - y_n \longrightarrow 0. \tag{3.14}$$

Therefore,

$$y_n - Sy_n \longrightarrow 0. \tag{3.15}$$

Since $\{y_n\}$ is bounded and $E$ is reflexive, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $y_{n_i} \rightharpoonup z$ and

$$\limsup_{n\to\infty} \langle y_n - \hat{u}, Ju - J\hat{u} \rangle = \lim_{i\to\infty} \langle y_{n_i} - \hat{u}, Ju - J\hat{u} \rangle. \tag{3.16}$$

Then, $x_{n_i} \rightharpoonup z$. Since $x_n - u_n \to 0$ and $r_n \ge c > 0$, by Remark 2.3,

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jx_n - Ju_n\| = 0. \tag{3.17}$$

Notice that

$$F(x_n, y) + \frac{1}{r_n} \langle y - x_n, Jx_n - Ju_n \rangle \ge 0, \quad \forall y \in C. \tag{3.18}$$

Replacing $n$ by $n_i$, we have from (A2) that

$$\frac{1}{r_{n_i}}\langle y - x_{n_i}, Jx_{n_i} - Ju_{n_i}\rangle \geq -F(x_{n_i}, y) \geq F(y, x_{n_i}), \quad \forall y \in C. \tag{3.19}$$

Letting $i \to \infty$, we have from (3.17) and (A4) that

$$F(y, z) \leq 0, \quad \forall y \in C. \tag{3.20}$$

From Lemma 2.10, we have $z \in \mathrm{EP}(F)$. Since $S$ satisfies condition (R3) and (3.15), $z \in F(S)$. It follows that $z \in F(S) \cap \mathrm{EP}(F)$. By Lemma 2.4(a), we immediately obtain that

$$\limsup_{n\to\infty} \langle y_n - \widehat{u}, Ju - J\widehat{u}\rangle = \langle z - \widehat{u}, Ju - J\widehat{u}\rangle \leq 0. \tag{3.21}$$

Since $z_n - y_n \to 0$,

$$\limsup_{n\to\infty} \langle z_n - \widehat{u}, Ju - J\widehat{u}\rangle \leq 0. \tag{3.22}$$

It follows from Lemma 2.7 and (3.8) that $\varphi(\widehat{u}, u_n) \to 0$. Then, $u_n \to \widehat{u}$ and so $x_n \to \widehat{u}$.

*Case 2.* Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\varphi(\widehat{u}, u_{n_i}) < \varphi(\widehat{u}, u_{n_i+1}), \tag{3.23}$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.8, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$,

$$\varphi(\widehat{u}, u_{m_k}) \leq \varphi(\widehat{u}, u_{m_k+1}), \qquad \varphi(\widehat{u}, u_k) \leq \varphi(\widehat{u}, u_{m_k+1}) \tag{3.24}$$

for all $k \in \mathbb{N}$. From (3.7) and $\gamma_n \to 0$, we have

$$\begin{aligned}
&\beta_{m_k}\varphi(x_{m_k}, u_{m_k}) + \beta_{m_k}(1-\beta_{m_k})g(\|Jx_{m_k} - JSy_{m_k}\|) \\
&\quad \leq \left(\varphi(\widehat{u}, u_{m_k}) - \varphi(\widehat{u}, u_{m_k+1})\right) - \gamma_{m_k}\varphi(\widehat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \\
&\quad \leq -\gamma_{m_k}\varphi(\widehat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \longrightarrow 0.
\end{aligned} \tag{3.25}$$

Using the same proof of Case 1, we also obtain

$$\limsup_{k\to\infty} \langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \leq 0. \tag{3.26}$$

From (3.8), we have

$$\varphi(\widehat{u}, u_{m_k+1}) \leq (1-\gamma_{m_k})\varphi(\widehat{u}, u_{m_k}) + 2\gamma_{m_k}\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \tag{3.27}$$

Since $\varphi(\widehat{u}, u_{m_k}) \leq \varphi(\widehat{u}, u_{m_k+1})$, we have

$$\begin{aligned}\gamma_{m_k}\varphi(\widehat{u}, u_{m_k}) &\leq \varphi(\widehat{u}, u_{m_k}) - \varphi(\widehat{u}, u_{m_k+1}) + 2\gamma_{m_k}\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle \\ &\leq 2\gamma_{m_k}\langle y_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle.\end{aligned} \tag{3.28}$$

In particular, since $\gamma_{m_k} > 0$, we get

$$\varphi(\widehat{u}, u_{m_k}) \leq 2\langle z_{m_k} - \widehat{u}, Ju - J\widehat{u}\rangle. \tag{3.29}$$

It follows from (3.26) that $\varphi(\widehat{u}, u_{m_k}) \rightarrow 0$. This together with (3.27) gives

$$\varphi(\widehat{u}, u_{m_k+1}) \longrightarrow 0. \tag{3.30}$$

But $\varphi(\widehat{u}, u_k) \leq \varphi(\widehat{u}, u_{m_k+1})$ for all $k \in \mathbb{N}$, we conclude that $u_k \rightarrow \widehat{u}$, and $x_k \rightarrow \widehat{u}$.

From two cases, we can conclude that $\{u_n\}$ and $\{x_n\}$ converge strongly to $\widehat{u}$ and the proof is finished. □

Applying Theorem 3.1 and [28, Theorem 3.2], we have the following result.

**Theorem 3.2.** *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$, $F : C \times C \rightarrow \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $\{T_i : C \rightarrow E\}_{i=1}^{\infty}$ a sequence of relatively nonexpansive mappings such that $\bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F) \neq \varnothing$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by* (3.1)*, where $S : C \rightarrow E$ is defined by*

$$Sx = J^{-1}\left(\sum_{i=1}^{\infty} \alpha_i J T_i x\right) \quad \text{for each } x \in C. \tag{3.31}$$

*Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{\bigcap_{i=1}^{\infty} F(T_i) \cap \mathrm{EP}(F)} u$.*

Setting $F \equiv 0$ and $r_n \equiv 1$ in Theorem 3.1, we have the following result.

**Corollary 3.3.** *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $S : C \rightarrow E$ a relatively nonexpansive mapping. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C$, $u_1 \in E$ and*

$$\begin{aligned}x_n &= \Pi_C u_n, \\ y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1 - \alpha_n) Jx_n), \\ u_{n+1} &= J^{-1}(\beta_n Jx_n + (1 - \beta_n) JSy_n),\end{aligned} \tag{3.32}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\rightarrow\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{F(S)} u$.*

Letting $S : C \to C$ in Corollary 3.3, we have the following result.

**Corollary 3.4.** *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $S : C \to C$ a relatively nonexpansive mapping. Let $\{x_n\}$ be a sequence in $C$ defined by $u \in C, x_1 \in C$ and*

$$\begin{aligned} y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n), \\ x_{n+1} &= J^{-1}(\beta_n Jx_n + (1-\beta_n)JSy_n), \end{aligned} \tag{3.33}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then $\{x_n\}$ converges strongly to $\Pi_{F(S)}u$.*

Let $S$ be the identity mapping in Theorem 3.1, we also have the following result.

**Corollary 3.5.** *Let $C$ be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space $E$ and $F : C\times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4) such that $\mathrm{EP}(F) \neq \varnothing$. Let $\{u_n\}$ and $\{x_n\}$ be sequences generated by $u \in C, u_1 \in E$ and*

$$\begin{aligned} F(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle &\geq 0, \quad \forall y \in C, \\ y_n &= \Pi_C J^{-1}(\alpha_n Ju + (1-\alpha_n)Jx_n), \\ u_{n+1} &= J^{-1}(\beta_n Jx_n + (1-\beta_n)Jy_n), \end{aligned} \tag{3.34}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{u_n\}$ and $\{x_n\}$ converge strongly to $\Pi_{\mathrm{EP}(F)}u$.*

## 4. Deduced Theorems in Hilbert Spaces

In Hilbert spaces, every nonexpansive mappings are relatively nonexpansive, and $J$ is the identity operator. We obtain the following result.

**Theorem 4.1.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $S : C \to H$ a nonexpansive mapping such that $F(S) \cap \mathrm{EP}(F) \neq \varnothing$. Let $\{x_n\}$ be a sequence in $C$ defined by $u \in C, x_1 \in H$ and*

$$x_{n+1} = \beta_n T_{r_n} x_n + (1-\beta_n) S(\alpha_n u + (1-\alpha_n)T_{r_n}x_n), \tag{4.1}$$

*for all $n \in \mathbb{N}$, where $T_{r_n}$ is the resolvent of $F$, $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{\beta_n\} \subset [a,b] \subset (0,1)$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap \mathrm{EP}(F)}u$.*

*Remark 4.2.* In Theorem 4.1, we have the same conclusion if the mapping $S : C \to H$ is only quasinonexpansive (i.e., $F(S) \neq \varnothing$ and $\|p - Sx\| \leq \|p - x\|$ for all $x \in C$ and $p \in F(S)$) such that $I - T$ is demiclosed at zero.

Letting $F \equiv 0$ in Theorem 4.1, we have the following result.

**Corollary 4.3.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $S : C \to H$ a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ defined by $u \in C, x_1 \in H$ and*

$$x_{n+1} = \beta_n P_C x_n + (1-\beta_n) S(\alpha_n u + (1-\alpha_n) P_C x_n), \tag{4.2}$$

*for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{\beta_n\} \subset [a,b] \subset (0,1)$. Then, $\{x_n\}$ converges strongly to $P_{F(S)}u$.*

Let $S$ be the identity mapping in Theorem 4.1, we have the following result.

**Corollary 4.4.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4). Let $\{x_n\}$ be a sequence in $H$ defined by $u, x_1 \in H$ and*

$$x_{n+1} = \gamma_n u + (1-\gamma_n) T_{r_n} x_n, \tag{4.3}$$

*for all $n \in \mathbb{N}$, where $T_{r_n}$ is the resolvent of $F$, $\{\gamma_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\gamma_n = 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$, and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{EP}(F)}u$.*

*Proof.* We may assume without loss of generality that $\gamma_n < 1/2$ for all $n \in \mathbb{N}$. Setting $\alpha_n = 2\gamma_n$ and $\beta_n = 1/2$ for all $n \in \mathbb{N}$, we get

$$x_{n+1} = \frac{1}{2} T_{r_n} x_n + \frac{1}{2} I(\alpha_n u + (1-\alpha_n) T_{r_n} x_n), \tag{4.4}$$

$\lim_{n\to\infty}\alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Applying Theorem 4.1, $\{x_n\}$ converges strongly to $P_{\mathrm{EP}(F)}u$. □

*Remark 4.5.* Corollary 4.4 improves and extends [29, Corollary 5.3]. More precisely, the conditions $\lim_{n\to\infty}(\gamma_{n+1}/\gamma_n) = 1$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ are removed.

Applying Corollary 4.4 and [30, Theorem 8], we have the following result.

**Corollary 4.6.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, $F : C \times C \to \mathbb{R}$ a bifunction satisfying conditions (A1)–(A4), and $f : C \to C$ a contraction of $H$ into itself. Let $\{x_n\}$ be a sequence in $H$ defined by $u$, $x_1 \in H$ and*

$$x_{n+1} = \gamma_n f(x_n) + (1-\gamma_n) T_{r_n} x_n, \tag{4.5}$$

*for all $n \in \mathbb{N}$, where $T_{r_n}$ is the resolvent of $F$, $\{\gamma_n\} \subset (0,1)$ satisfying $\lim_{n\to\infty}\gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\{r_n\} \subset [c,\infty) \subset (0,\infty)$. Then, $\{x_n\}$ converges strongly to $z = P_{\mathrm{EP}(F)} f(z)$.*

*Remark 4.7.* Corollary 4.6 improves and extends [16, Corollary 3.4]. More precisely, the conditions $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ are removed.

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## References

[1] Y. I. Alber, "Metric and generalized projection operators in Banach spaces: properties and applications," in *Theory and Applications of Nonlinear Operators of Accretive and Monotone Type*, vol. 178 of *Lecture Notes in Pure and Applied Mathematics*, pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[2] S.-Y. Matsushita and W. Takahashi, "Weak and strong convergence theorems for relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, no. 1, pp. 37–47, 2004.

[3] S.-Y. Matsushita and W. Takahashi, "A strong convergence theorem for relatively nonexpansive mappings in a Banach space," *Journal of Approximation Theory*, vol. 134, no. 2, pp. 257–266, 2005.

[4] D. Boonchari and S. Saejung, "Approximation of common fixed points of a countable family of relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 407651, 26 pages, 2010.

[5] L.-C. Ceng and J.-C. Yao, "A hybrid iterative scheme for mixed equilibrium problems and fixed point problems," *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[6] L. C. Ceng, A. Petruşel, and J. C. Yao, "Iterative approaches to solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings," *Journal of Optimization Theory and Applications*, vol. 143, no. 1, pp. 37–58, 2009.

[7] A. Kangtunyakarn and S. Suantai, "A new mapping for finding common solutions of equilibrium problems and fixed point problems of finite family of nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 10, pp. 4448–4460, 2009.

[8] W. Nilsrakoo and S. Saejung, "Weak and strong convergence theorems for countable Lipschitzian mappings and its applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 8, pp. 2695–2708, 2008.

[9] W. Nilsrakoo and S. Saejung, "Weak convergence theorems for a countable family of Lipschitzian mappings," *Journal of Computational and Applied Mathematics*, vol. 230, no. 2, pp. 451–462, 2009.

[10] W. Nilsrakoo and S. Saejung, "Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications," *Journal of Mathematical Analysis and Applications*, vol. 356, no. 1, pp. 154–167, 2009.

[11] W. Nilsrakoo and S. Saejung, "Equilibrium problems and Moudafi's viscosity approximation methods in Hilbert spaces," *Dynamics of Continuous, Discrete & Impulsive Systems. Series A. Mathematical Analysis*, vol. 17, no. 2, pp. 195–213, 2010.

[12] S. Plubtieng and W. Sriprad, "Hybrid methods for equilibrium problems and fixed points problems of a countable family of relatively nonexpansive mappings in Banach spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 962628, 17 pages, 2010.

[13] S. Plubtieng and W. Sriprad, "A viscosity approximation method for finding common solutions of variational inclusions, equilibrium problems, and fixed point problems in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 567147, 20 pages, 2009.

[14] X. Qin, Y. J. Cho, and S. M. Kang, "Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces," *Journal of Computational and Applied Mathematics*, vol. 225, no. 1, pp. 20–30, 2009.

[15] A. Tada and W. Takahashi, "Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem," *Journal of Optimization Theory and Applications*, vol. 133, no. 3, pp. 359–370, 2007.

[16] S. Takahashi and W. Takahashi, "Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 331, no. 1, pp. 506–515, 2007.

[17] S. Takahashi and W. Takahashi, "Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 3, pp. 1025–1033, 2008.

[18] W. Takahashi and K. Zembayashi, "Strong convergence theorem by a new hybrid method for equilibrium problems and relatively nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 528476, 11 pages, 2008.

[19] W. Takahashi and Kei Zembayashi, "Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 1, pp. 45–57, 2009.

[20] K. Wattanawitoon and P. Kumam, "Strong convergence theorems by a new hybrid projection algorithm for fixed point problems and equilibrium problems of two relatively quasi-nonexpansive mappings," *Nonlinear Analysis. Hybrid Systems*, vol. 3, no. 1, pp. 11–20, 2009.

[21] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.

[22] H. K. Xu, "Inequalities in Banach spaces with applications," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 16, no. 12, pp. 1127–1138, 1991.

[23] S. Kamimura and W. Takahashi, "Strong convergence of a proximal-type algorithm in a Banach space," *SIAM Journal on Optimization*, vol. 13, no. 3, pp. 938–945, 2002.

[24] F. Kohsaka and W. Takahashi, "Strong convergence of an iterative sequence for maximal monotone operators in a Banach space," *Abstract and Applied Analysis*, no. 3, pp. 239–249, 2004.

[25] H.-K. Xu, "Another control condition in an iterative method for nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 65, no. 1, pp. 109–113, 2002.

[26] P.-E. Maingé, "Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization," *Set-Valued Analysis*, vol. 16, no. 7-8, pp. 899–912, 2008.

[27] F. Kohsaka and W. Takahashi, "Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces," *SIAM Journal on Optimization*, vol. 19, no. 2, pp. 824–835, 2008.

[28] W. Nilsrakoo and S. Saejung, "On the fixed-point set of a family of relatively nonexpansive and generalized nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 414232, 14 pages, 2010.

[29] Y. Song and Y. Zheng, "Strong convergence of iteration algorithms for a countable family of nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3072–3082, 2009.

[30] T. Suzuki, "Moudafi's viscosity approximations with Meir-Keeler contractions," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 1, pp. 342–352, 2007.