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Research Article

Finding Common Solutions of a Variational Inequality, a General System of Variational Inequalities, and a Fixed-Point Problem via a Hybrid Extragradient Method

Lu-Chuan Ceng,[1] Sy-Ming Guu,[2] and Jen-Chih Yao[3]

[1] *Department of Mathematics, Shanghai Normal University, Scientific Computing Key Laboratory of Shanghai Universities, Shanghai 200234, China*
[2] *Department of Business Administration, College of Management, Yuan-Ze University, Taoyuan Hsien, Chung-Li City 330, Taiwan*
[3] *Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 804, Taiwan*

Correspondence should be addressed to Sy-Ming Guu, iesmguu@saturn.yzu.edu.tw

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We propose a hybrid extragradient method for finding a common element of the solution set of a variational inequality problem, the solution set of a general system of variational inequalities, and the fixed-point set of a strictly pseudocontractive mapping in a real Hilbert space. Our hybrid method is based on the well-known extragradient method, viscosity approximation method, and Mann-type iteration method. By constrasting with other methods, our hybrid approach drops the requirement of boundedness for the domain in which various mappings are defined. Furthermore, under mild conditions imposed on the parameters we show that our algorithm generates iterates which converge strongly to a common element of these three problems.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by P_C the metric projection of H onto C. Moreover, we also denote by $\mathbf{R}$ the set of all real numbers. For a given nonlinear operator $A : C \to H$, we consider the following variational inequality problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of the variational inequality (1.1) is denoted by $\mathrm{VI}(A, C)$. Variational inequality theory has been studied quite extensively and has emerged as an important tool in the study of a wide class of obstacle, unilateral, free, moving, equilibrium problems. See, for example, [1–21] and the references therein.

For finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$ when C is closed and convex, S is nonexpansive and A is α-inverse strongly monotone, Takahashi and Toyoda [22] introduced the following Mann-type iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \geq 0, \tag{1.2}$$

where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$. Nadezhkina and Takahashi [23] and Zeng and Yao [24] proposed extragradient methods motivated by Korpelevič [25] for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of a variational inequality problem. Further, these iterative methods were extended in [26] to develop a new iterative method for finding elements in $\mathrm{Fix}(S) \cap \mathrm{VI}(A, C)$.

Let $B_1, B_2 : C \to H$ be two mappings. Now we consider the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.3}$$

which is called a general system of variational inequalities where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. The set of solutions of problem (1.3) is denoted by $\mathrm{GSVI}(B_1, B_2, C)$. In particular, if $B_1 = B_2 = A$, then problem (1.3) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \mu_1 A y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \mu_2 A x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{1.4}$$

which was defined by Verma [27] (see also [28]) and is called the new system of variational inequalities. Further, if $x^* = y^*$ additionally, then problem (1.4) reduces to the classical variational inequality problem (1.1).

Ceng et al. [29] studied the problem (1.3) by transforming it into a fixed-point problem. Precisely and for easy reference, we state their results in the following lemma and theorem.

Lemma CWY (see [29]). *For given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of problem* (1.3) *if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C \left[P_C (x - \mu_2 B_2 x) - \mu_1 B_1 P_C (x - \mu_2 B_2 x) \right], \quad \forall x \in C, \tag{1.5}$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$. In particular, if the mapping $B_i : C \to H$ is μ_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\mu_i)$ for $i = 1, 2$.

Throughout this paper, the fixed-point set of the mapping G is denoted by Γ. Utilizing Lemma CWY, they introduced and studied a relaxed extragradient method for solving problem (1.3).

Theorem CWY (see [29, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Suppose $x_1 = u \in C$ and $\{x_n\}$ is generated by*

$$\begin{gathered} y_n = P_C(x_n - \mu_2 B_2 x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S P_C(y_n - \mu_1 B_1 y_n), \end{gathered} \tag{1.6}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ such that

(i) $\alpha_n + \beta_n + \gamma_n = 1$, *for all $n \geq 1$;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma} u$ and $(\overline{x}, \overline{y})$ is a solution of problem (1.3), *where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

It is clear that the above result unifies and extends some corresponding results in the literature.

Based on the relaxed extragradient method and viscosity approximation method, Yao et al. [30] proposed and analyzed an iterative algorithm for finding a common element of the solution set of the general system (1.3) of variational inequalities and the fixed-point set of a strictly pseudocontractive mapping in a real Hilbert space H.

Theorem YLK (see [30, Theorem 3.2]). *Let C be a nonempty bounded closed convex subset of a real Hilbert space H. Let the mapping $B_i : C \to H$ be μ_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. *Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated iteratively by*

$$\begin{gathered} z_n = P_C(x_n - \mu_2 B_2 x_n), \\ y_n = \alpha_n Q x_n + (1 - \alpha_n) P_C(z_n - \mu_1 B_1 z_n), \\ x_{n+1} = \beta_n x_n + \gamma_n P_C(z_n - \mu_1 B_1 z_n) + \delta_n S y_n, \quad \forall n \geq 0, \end{gathered} \tag{1.7}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ are four sequences in $[0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$ *for all $n \geq 0$;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty} (\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$.

Then the sequence $\{x_n\}$ generated by (1.7) *converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma} \cdot Q\overline{x}$ and $(\overline{x}, \overline{y})$ is a solution of the general system* (1.3) *of variational inequalities, where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Motivated by the above work, in this paper, we introduce an iterative algorithm for finding a common element of the solution set of the variational inequality (1.1), the solution set of the general system (1.3) and the fixed-point set of the strictly pseudocontractive mapping $S : C \to C$ via a hybrid extragradient method based on the well-known extragradient method, viscosity approximation method, and Mann-type iteration method, that is,

$$\begin{aligned} z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n Q x_n + (1-\alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{1.8}$$

where $\{\lambda_n\} \subset (0, \infty)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0,1]$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Moreover, we prove that the studied iterative algorithm converges strongly to an element of $\operatorname{Fix}(S) \cap \Gamma \cap \operatorname{VI}(A, C)$ under some mild conditions imposed on algorithm parameters. Our method improves and extends Yao et al. [30, Theorem 3.2] in the following aspects:

(i) the problem of finding an element of $\operatorname{Fix}(S) \cap \Gamma$ in [30, Theorem 3.2] is extended to the the problem of finding an element of $\operatorname{Fix}(S) \cap \Gamma \cap \operatorname{VI}(A, C)$;

(ii) the requirement of boundedness of C in [30, Theorem 3.2] is removed;

(iii) the condition $(\gamma_n + \delta_n)k \leq \gamma_n < (1 - 2\rho)\delta_n$, for all $n \geq 0$ in [30, Theorem 3.2] is replaced by the one $(\gamma_n + \delta_n)k \leq \gamma_n$, for all $n \geq 0$;

(iv) the argument of Step 5 in the proof of [30, Theorem 3.2] is simplified under the lack of the condition $\gamma_n < (1 - 2\rho)\delta_n$, for all $n \geq 0$;

(v) our iterative algorithm is similar to but different from the one of [30, Theorem 3.2] because the problem of finding an element of $\operatorname{Fix}(S) \cap \Gamma \cap \operatorname{VI}(A, C)$ is more challenging than the problem of finding an element of $\operatorname{Fix}(S) \cap \Gamma$ in [30, Theorem 3.2].

2. Preliminaries

In this section, we collect some notations and lemmas. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C. \tag{2.1}$$

A mapping $A : C \to H$ is called Lipschitz continuous if there exists a real number $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C. \tag{2.2}$$

Recall that a mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.3}$$

It is clear that every inverse strongly monotone mapping is a monotone and Lipschitz continuous mapping. Also, recall that a mapping $S : C \to C$ is said to be k-strictly pseudocontractive if there exists a constant $0 \le k < 1$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.4}$$

For such a case, we also say that S is a k-strict pseudo-contraction [31]. It is clear that, in a real Hilbert space H, inequality (2.4) is equivalent to the following:

$$\langle Sx - Sy, x - y\rangle \le \|x - y\|^2 - \frac{1-k}{2}\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C. \tag{2.5}$$

This immediately implies that if S is a k-strictly pseudocontractive mapping, then $I - S$ is $(1 - k)/2$-inverse strongly monotone; see [32] for more details. We use $\mathrm{Fix}(S)$ to denote the set of fixed points of S. It is well known that the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings which are mappings $S : C \to C$ such that $\|Sx - Sy\| \le \|x - y\|$, for all $x, y \in C$. A mapping $Q : C \to C$ is called a contraction if there exists a constant $\rho \in [0, 1)$ such that $\|Qx - Qy\| \le \rho\|x - y\|$ for all $x, y \in C$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{2.6}$$

The mapping P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.7}$$

It is known that $P_C x$ is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad \forall x \in H,\ y \in C. \tag{2.8}$$

In order to prove the main result in this paper, we will need the following lemmas in the sequel.

Lemma 2.1 (see [33]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in* $[0, 1]$ *with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.2 (see [34, Proposition 2.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $S : C \to C$ be a self-mapping of C.*

(i) *If S is a k-strict pseudocontractive mapping, then S satisfies the Lipschitz condition*

$$\|Sx - Sy\| \le \frac{1+k}{1-k}\|x - y\|, \quad \forall x, y \in C. \tag{2.9}$$

(ii) *If S is a k-strict pseudocontractive mapping, then the mapping $I - S$ is demiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \to \tilde{x}$ weakly and $(I - S)x_n \to 0$ strongly, then $(I - S)\tilde{x} = 0$.*

(iii) *If S is k-(quasi-)strict pseudo-contraction, then the fixed-point set* $\mathrm{Fix}(S)$ *of S is closed and convex so that the projection $P_{\mathrm{Fix}(S)}$ is well defined.*

Lemma 2.3 (see [9, Lemma 2.1]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$s_{n+1} \le (1 - \alpha_n)s_n + \alpha_n\beta_n, \quad \forall n \ge 0, \tag{2.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$, *or equivalently,*

$$\prod_{n=0}^{\infty}(1 - \alpha_n) := \lim_{n\to\infty}\prod_{k=1}^{n}(1 - \alpha_k) = 0; \tag{2.11}$$

(ii) $\limsup_{n\to\infty}\beta_n \le 0$; *or*

(ii)′ $\sum_{n=0}^{\infty} \alpha_n\beta_n$ *is convergent.*

Then, $\lim_{n\to\infty}s_n = 0$.

Lemma 2.4 (see [30]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a k-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers. Assume $(\gamma + \delta)k \le \gamma$. Then*

$$\|\gamma(x - y) + \delta(Sx - Sy)\| \le (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C. \tag{2.12}$$

The following lemma is an immediate consequence of an inner product.

Lemma 2.5. *In a real Hilbert space H, there holds the inequality*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.13}$$

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.8) *implies that*

$$u \in VI(A, C) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \forall\lambda > 0. \tag{2.14}$$

It is also known that a set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply that $\langle x - y, f - g\rangle \ge 0$. A monotone set-valued mapping $T : H \to 2^H$ is maximal if its graph $Gph(T)$ is not properly contained in the graph of any other monotone set-valued mapping. It is known that a monotone set-valued mapping $T : H \to 2^H$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in Gph(T)$ implies that $f \in Tx$. Let A be a *monotone and Lipschitz continuous mapping of C into H. Let $N_C v$ be the normal cone to C at $v \in C$, that is,*

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall \in C\}. \tag{2.15}$$

Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.16}$$

It is known that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(A, C)$; see [35] for more details.

3. Main Results

The main idea for showing strong convergence of the sequence $\{x_n\}$ generated by (1.8) to an element of $\mathrm{VI}(A, C)$ is first to transform the variational inequality problem (1.1) into the zero point problem of a maximal monotone mapping T and then to derive the strong convergence of $\{x_n\}$ to a zero of T by using the technique in [10]. We are now in a position to state and prove the main result in this paper.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n Q x_n + (1 - \alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{3.1}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\gamma_{n+1}/(1 - \beta_{n+1}) - \gamma_n/(1 - \beta_n)) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ generated by (3.1) *converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma\cap VI(A,C)} \cdot Q\overline{x}$ and $(\overline{x}, \overline{y})$ is a solution of the general system* (1.3) *of variational inequalities, where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Proof. We divide the proof into several steps.

Step 1. $\{x_n\}$ is bounded.

Indeed, take $x^* \in \mathrm{Fix}(S) \cap \Gamma \cap \mathrm{VI}(A, C)$ arbitrarily. Then $Sx^* = x^*$, $x^* = P_C(x^* - \lambda_n Ax^*)$ and

$$x^* = P_C\left[P_C\left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right]. \tag{3.2}$$

Since $A : C \to H$ be α-inverse strongly monotone and $0 < \lambda_n \leq 2\alpha$, we have for all $n \geq 0$,

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 \\
&= \|(x_n - x^*) - \lambda_n(Ax_n - Ax^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.3}$$

For simplicity, we write $y^* = P_C(x^* - \mu_2 B_2 x^*)$ and $u_n = P_C(z_n - \mu_2 B_2 z_n)$ for all $n \geq 0$. Since $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$ and $0 < \mu_i < 2\beta_i$ for $i = 1, 2$, we know that for all $n \geq 0$,

$$\begin{aligned}
&\left\|P_C\left[P_C\left(z_n - \mu_2 B_2 z_n\right) - \mu_1 B_1 P_C\left(z_n - \mu_2 B_2 z_n\right)\right] - x^*\right\|^2 \\
&\quad= \left\|P_C\left[P_C\left(z_n - \mu_2 B_2 z_n\right) - \mu_1 B_1 P_C\left(z_n - \mu_2 B_2 z_n\right)\right]\right. \\
&\qquad \left.-P_C\left[P_C\left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right]\right\|^2 \\
&\quad\leq \left\|\left[P_C\left(z_n - \mu_2 B_2 z_n\right) - \mu_1 B_1 P_C\left(z_n - \mu_2 B_2 z_n\right)\right]\right. \\
&\qquad \left.-\left[P_C\left(x^* - \mu_2 B_2 x^*\right) - \mu_1 B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right]\right\|^2 \\
&\quad= \left\|\left[P_C\left(z_n - \mu_2 B_2 z_n\right) - P_C\left(x^* - \mu_2 B_2 x^*\right)\right]\right. \\
&\qquad \left.-\mu_1\left[B_1 P_C\left(z_n - \mu_2 B_2 z_n\right) - B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right]\right\|^2 \\
&\quad\leq \left\|P_C\left(z_n - \mu_2 B_2 z_n\right) - P_C\left(x^* - \mu_2 B_2 x^*\right)\right\|^2 \\
&\qquad - \mu_1\left(2\beta_1 - \mu_1\right)\left\|B_1 P_C\left(z_n - \mu_2 B_2 z_n\right) - B_1 P_C\left(x^* - \mu_2 B_2 x^*\right)\right\|^2 \\
&\quad\leq \left\|\left(z_n - \mu_2 B_2 z_n\right) - \left(x^* - \mu_2 B_2 x^*\right)\right\|^2 - \mu_1\left(2\beta_1 - \mu_1\right)\left\|B_1 u_n - B_1 y^*\right\|^2 \\
&\quad= \left\|\left(z_n - x^*\right) - \mu_2\left(B_2 z_n - B_2 x^*\right)\right\|^2 - \mu_1\left(2\beta_1 - \mu_1\right)\left\|B_1 u_n - B_1 y^*\right\|^2 \\
&\quad\leq \|z_n - x^*\|^2 - \mu_2\left(2\beta_2 - \mu_2\right)\|B_2 z_n - B_2 x^*\|^2 - \mu_1\left(2\beta_1 - \mu_1\right)\left\|B_1 u_n - B_1 y^*\right\|^2 \\
&\quad\leq \|x_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \\
&\qquad - \mu_2\left(2\beta_2 - \mu_2\right)\|B_2 z_n - B_2 x^*\|^2 - \mu_1\left(2\beta_1 - \mu_1\right)\left\|B_1 u_n - B_1 y^*\right\|^2 \\
&\quad\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.4}$$

Hence we get

$$\begin{aligned}
\|y_n - x^*\| &= \|\alpha_n(Qx_n - x^*) + (1-\alpha_n)(P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] - x^*)\| \\
&\le \alpha_n\|Qx_n - x^*\| + (1-\alpha_n)\|P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] - x^*\| \\
&\le \alpha_n(\rho\|x_n - x^*\| + \|Qx^* - x^*\|) + (1-\alpha_n)\|x_n - x^*\| \\
&= (1-(1-\rho)\alpha_n)\|x_n - x^*\| + (1-\rho)\alpha_n \frac{\|Qx^* - x^*\|}{1-\rho} \\
&\le \max\left\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\right\}.
\end{aligned} \tag{3.5}$$

Since $(\gamma_n + \delta_n)k \le \gamma_n$ for all $n \ge 0$, utilizing Lemma 2.4 we obtain from (3.5)

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\| \\
&\le \beta_n\|x_n - x^*\| + \|\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\| \\
&\le \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|y_n - x^*\| \\
&\le \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\max\left\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\right\} \\
&\le \max\left\{\|x_n - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\right\}.
\end{aligned} \tag{3.6}$$

By induction, we obtain that for all $n \ge 0$

$$\|x_n - x^*\| \le \max\left\{\|x_0 - x^*\|, \frac{\|Qx^* - x^*\|}{1-\rho}\right\}. \tag{3.7}$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce immediately that $\{z_n\}, \{y_n\}, \{Sy_n\}$, and $\{u_n\}$ are bounded, where $u_n = P_C(z_n - \mu_2 B_2 z_n)$ for all $n \ge 0$.

Now, put

$$t_n := P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \quad \forall n \ge 0. \tag{3.8}$$

Then it is easy to see that $\{t_n\}$ is bounded because P_C, B_1, and B_2 are Lipschitz continuous and $\{z_n\}$ is bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Indeed, define $x_{n+1} = \beta_n x_n + (1-\beta_n) w_n$ for all $n \geq 0$. It follows that

$$\begin{aligned} w_{n+1} - w_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1} y_{n+1} + \delta_{n+1} S y_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n y_n + \delta_n S y_n}{1-\beta_n} \\ &= \frac{\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(S y_{n+1} - S y_n)}{1-\beta_{n+1}} + \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) y_n \\ &\quad + \left(\frac{\delta_{n+1}}{1-\beta_{n+1}} - \frac{\delta_n}{1-\beta_n}\right) S y_n. \end{aligned} \tag{3.9}$$

Since $(\gamma_n + \delta_n)k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.4 we have

$$\left\|\gamma_{n+1}(y_{n+1} - y_n) + \delta_{n+1}(S y_{n+1} - S y_n)\right\| \leq (\gamma_{n+1} + \delta_{n+1}) \left\|y_{n+1} - y_n\right\|. \tag{3.10}$$

Next, we estimate $\|y_{n+1} - y_n\|$. Observe that

$$\begin{aligned} \|z_{n+1} - z_n\| &= \|P_C(x_{n+1} - \lambda_{n+1} A x_{n+1}) - P_C(x_n - \lambda_n A x_n)\| \\ &\leq \|(x_{n+1} - \lambda_{n+1} A x_{n+1}) - (x_n - \lambda_n A x_n)\| \\ &= \|(x_{n+1} - x_n) - \lambda_{n+1}(A x_{n+1} - A x_n) + (\lambda_n - \lambda_{n+1}) A x_n\| \\ &\leq \|(x_{n+1} - x_n) - \lambda_{n+1}(A x_{n+1} - A x_n)\| + |\lambda_{n+1} - \lambda_n| \|A x_n\| \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n| \|A x_n\|, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \|t_{n+1} - t_n\|^2 &= \big\|P_C\left[P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - \mu_1 B_1 P_C(z_{n+1} - \mu_2 B_2 z_{n+1})\right] \\ &\quad - P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right]\big\|^2 \\ &\leq \big\|\left[P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - \mu_1 B_1 P_C(z_{n+1} - \mu_2 B_2 z_{n+1})\right] \\ &\quad - \left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right]\big\|^2 \\ &= \big\|\left[P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - P_C(z_n - \mu_2 B_2 z_n)\right] \\ &\quad - \mu_1\left[B_1 P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - B_1 P_C(z_n - \mu_2 B_2 z_n)\right]\big\|^2 \\ &\leq \left\|P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - P_C(z_n - \mu_2 B_2 z_n)\right\|^2 \\ &\quad - \mu_1(2\beta_1 - \mu_1)\left\|B_1 P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - B_1 P_C(z_n - \mu_2 B_2 z_n)\right\|^2 \\ &\leq \left\|P_C(z_{n+1} - \mu_2 B_2 z_{n+1}) - P_C(z_n - \mu_2 B_2 z_n)\right\|^2 \\ &\leq \left\|(z_{n+1} - \mu_2 B_2 z_{n+1}) - (z_n - \mu_2 B_2 z_n)\right\|^2 \\ &= \left\|(z_{n+1} - z_n) - \mu_2(B_2 z_{n+1} - B_2 z_n)\right\|^2 \\ &\leq \|z_{n+1} - z_n\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 z_{n+1} - B_2 z_n\|^2 \\ &\leq \|z_{n+1} - z_n\|^2. \end{aligned} \tag{3.12}$$

Combining (3.11) with (3.12), we get

$$\begin{aligned}\|t_{n+1}-t_n\| &= \big\|P_C\left[P_C\left(z_{n+1}-\mu_2B_2z_{n+1}\right)-\mu_1B_1P_C\left(z_{n+1}-\mu_2B_2z_{n+1}\right)\right]\\&\qquad -P_C\left[P_C\left(z_n-\mu_2B_2z_n\right)-\mu_1B_1P_C\left(z_n-\mu_2B_2z_n\right)\right]\big\|\\&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.\end{aligned} \tag{3.13}$$

This together with (3.13) implies that

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|t_{n+1}+\alpha_{n+1}(Qx_{n+1}-t_{n+1})-t_n-\alpha_n(Qx_n-t_n)\|\\&\leq \|t_{n+1}-t_n\|+\alpha_{n+1}\|Qx_{n+1}-t_{n+1}\|+\alpha_n\|Qx_n-t_n\|\\&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|+\alpha_{n+1}\|Qx_{n+1}-t_{n+1}\|+\alpha_n\|Qx_n-t_n\|.\end{aligned} \tag{3.14}$$

Hence it follows from (3.9), (3.10), and (3.14) that

$$\begin{aligned}\|w_{n+1}-w_n\| &\leq \frac{\|\gamma_{n+1}(y_{n+1}-y_n)+\delta_{n+1}(Sy_{n+1}-Sy_n)\|}{1-\beta_{n+1}}\\&\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|\|y_n\|+\left|\frac{\delta_{n+1}}{1-\beta_{n+1}}-\frac{\delta_n}{1-\beta_n}\right|\|Sy_n\|\\&\leq \frac{\gamma_{n+1}+\delta_{n+1}}{1-\beta_{n+1}}\|y_{n+1}-y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Sy_n\|)\\&= \|y_{n+1}-y_n\|+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Sy_n\|)\\&\leq \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|+\alpha_{n+1}\|Qx_{n+1}-t_{n+1}\|+\alpha_n\|Qx_n-t_n\|\\&\quad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Sy_n\|).\end{aligned} \tag{3.15}$$

Since $\{x_n\}$, $\{y_n\}$, and $\{t_n\}$ are bounded, it follows from conditions (ii), (iv), (v) that

$$\begin{aligned}&\limsup_{n\to\infty}(\|w_{n+1}-w_n\|-\|x_{n+1}-x_n\|)\\&\quad\leq \limsup_{n\to\infty}\Bigg\{|\lambda_{n+1}-\lambda_n|\|Ax_n\|+\alpha_{n+1}\|Qx_{n+1}-t_{n+1}\|+\alpha_n\|Qx_n-t_n\|\\&\qquad+\left|\frac{\gamma_{n+1}}{1-\beta_{n+1}}-\frac{\gamma_n}{1-\beta_n}\right|(\|y_n\|+\|Sy_n\|)\Bigg\}=0.\end{aligned} \tag{3.16}$$

Hence by Lemma 2.1 we get $\lim_{n\to\infty}\|w_n-x_n\|=0$. Thus,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|w_n-x_n\|=0. \tag{3.17}$$

Step 3. $\lim_{n\to\infty}\|B_2z_n - B_2x^*\| = 0$, $\lim_{n\to\infty}\|B_1u_n - B_1y^*\| = 0$ and $\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$, where $y^* = P_C(x^* - \mu_2 B_2 x^*)$.

Indeed, utilizing Lemma 2.4 and the convexity of $\|\cdot\|^2$, we get from (3.1) and (3.4)

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\right\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}\left[\gamma_n(y_n - x^*) + \delta_n(Sy_n - x^*)\right]\right\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\left\|y_n - x^*\right\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (\gamma_n + \delta_n)\left[\alpha_n\|Qx_n - x^*\|^2 + (1 - \alpha_n)\|t_n - x^*\|^2\right] \\
&\le \beta_n\|x_n - x^*\|^2 + \alpha_n\|Qx_n - x^*\|^2 + (\gamma_n + \delta_n)\|t_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + \alpha_n\|Qx_n - x^*\|^2 + (\gamma_n + \delta_n) \\
&\quad \times \Big[\|x_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\
&\qquad -\mu_1(2\beta_1 - \mu_1)\left\|B_1u_n - B_1y^*\right\|^2\Big] \\
&= \|x_n - x^*\|^2 + \alpha_n\|Qx_n - x^*\|^2 - (\gamma_n + \delta_n) \\
&\quad \times \Big[\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 \\
&\qquad -\mu_2(2\beta_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 - \mu_1(2\beta_1 - \mu_1)\left\|B_1u_n - B_1y^*\right\|^2\Big].
\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}
(\gamma_n + \delta_n)&\Big[\lambda_n(2\alpha - \lambda_n)\|Ax_n - Ax^*\|^2 + \mu_2(2\beta_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\
&\quad +\mu_1(2\beta_1 - \mu_1)\left\|B_1u_n - B_1y^*\right\|^2\Big] \\
&\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n\|Qx_n - x^*\|^2 \\
&\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + \alpha_n\|Qx_n - x^*\|^2.
\end{aligned} \tag{3.19}$$

Since $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$, $\liminf_{n\to\infty}(\gamma_n + \delta_n) > 0$ and $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$, we have

$$\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0, \qquad \lim_{n\to\infty}\left\|B_1u_n - B_1y^*\right\| = 0, \qquad \lim_{n\to\infty}\|B_2z_n - B_2x^*\| = 0. \tag{3.20}$$

Step 4. $\lim_{n\to\infty}\|Sy_n - y_n\| = 0$.

Indeed, noticing the firm nonexpansivity of P_C we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\
&\le \langle (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), z_n - x^* \rangle \\
&= \frac{1}{2}\Big[\|x_n - x^* - \lambda_n(Ax_n - Ax^*)\|^2 + \|z_n - x^*\|^2 \\
&\qquad - \|(x_n - x^*) - \lambda_n(Ax_n - Ax^*) - (z_n - x^*)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|(x_n - z_n) - \lambda_n(Ax_n - Ax^*)\|^2\Big] \\
&= \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\
&\qquad + 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \lambda_n^2\|Ax_n - Ax^*\|^2\Big] \\
&\le \frac{1}{2}\Big[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|\Big],
\end{aligned} \tag{3.21}$$

that is,

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|. \tag{3.22}$$

Similarly to the above argument, we obtain

$$\begin{aligned}
\|u_n - y^*\|^2 &= \|P_C(z_n - \mu_2 B_2 z_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\le \langle (z_n - \mu_2 B_2 z_n) - (x^* - \mu_2 B_2 x^*), u_n - y^* \rangle \\
&= \frac{1}{2}\Big[\|z_n - x^* - \mu_2(B_2 z_n - B_2 x^*)\|^2 + \|u_n - y^*\|^2 \\
&\qquad - \|(z_n - x^*) - \mu_2(B_2 z_n - B_2 x^*) - (u_n - y^*)\|^2\Big] \\
&\le \frac{1}{2}\Big[\|z_n - x^*\|^2 + \|u_n - y^*\|^2 - \|(z_n - u_n) - \mu_2(B_2 z_n - B_2 x^*) - (x^* - y^*)\|^2\Big] \\
&= \frac{1}{2}\Big[\|z_n - x^*\|^2 + \|u_n - y^*\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 \\
&\qquad + 2\mu_2\langle z_n - u_n - (x^* - y^*), B_2 z_n - B_2 x^*\rangle - \mu_2^2\|B_2 z_n - B_2 x^*\|^2\Big],
\end{aligned} \tag{3.23}$$

that is,

$$\|u_n - y^*\|^2 \le \|z_n - x^*\|^2 - \|z_n - u_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - u_n - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|. \tag{3.24}$$

Substituting (3.22) in (3.24), we have

$$\begin{aligned}\left\|u_n - y^*\right\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| \\ - \left\|z_n - u_n - \left(x^* - y^*\right)\right\|^2 + 2\mu_2\left\|z_n - u_n - \left(x^* - y^*\right)\right\|\|B_2 z_n - B_2 x^*\|.\end{aligned} \tag{3.25}$$

Further, similarly to the above argument, we derive

$$\begin{aligned}\|t_n - x^*\|^2 &= \left\|P_C\left(u_n - \mu_1 B_1 u_n\right) - P_C\left(y^* - \mu_1 B_1 y^*\right)\right\|^2 \\ &\leq \left\langle\left(u_n - \mu_1 B_1 u_n\right) - \left(y^* - \mu_1 B_1 y^*\right), t_n - x^*\right\rangle \\ &= \frac{1}{2}\Big[\left\|u_n - y^* - \mu_1\left(B_1 u_n - B_1 y^*\right)\right\|^2 + \|t_n - x^*\|^2 \\ &\quad -\left\|\left(u_n - y^*\right) - \mu_1\left(B_1 u_n - B_1 y^*\right) - \left(t_n - x^*\right)\right\|^2\Big] \\ &\leq \frac{1}{2}\Big[\left\|u_n - y^*\right\|^2 + \|t_n - x^*\|^2 - \left\|\left(u_n - t_n\right) - \mu_1\left(B_1 u_n - B_1 y^*\right) + \left(x^* - y^*\right)\right\|^2\Big] \\ &= \frac{1}{2}\Big[\left\|u_n - y^*\right\|^2 + \|t_n - x^*\|^2 - \left\|u_n - t_n + \left(x^* - y^*\right)\right\|^2 \\ &\quad + 2\mu_1\left\langle u_n - t_n + \left(x^* - y^*\right), B_1 u_n - B_1 y^*\right\rangle - \mu_1^2\left\|B_1 u_n - B_1 y^*\right\|^2\Big],\end{aligned} \tag{3.26}$$

that is,

$$\|t_n - x^*\|^2 \leq \left\|u_n - y^*\right\|^2 - \left\|u_n - t_n + \left(x^* - y^*\right)\right\|^2 + 2\mu_1\left\|u_n - t_n + \left(x^* - y^*\right)\right\|\left\|B_1 u_n - B_1 y^*\right\|. \tag{3.27}$$

Substituting (3.25) in (3.27), we have

$$\begin{aligned}\|t_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| \\ - \left\|z_n - u_n - \left(x^* - y^*\right)\right\|^2 + 2\mu_2\left\|z_n - u_n - \left(x^* - y^*\right)\right\|\|B_2 z_n - B_2 x^*\| \\ - \left\|u_n - t_n + \left(x^* - y^*\right)\right\|^2 + 2\mu_1\left\|u_n - t_n + \left(x^* - y^*\right)\right\|\left\|B_1 u_n - B_1 y^*\right\|.\end{aligned} \tag{3.28}$$

Thus from (3.1) and (3.28), it follows that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \left\|\beta_n(x_n-x^*)+\gamma_n(y_n-x^*)+\delta_n(Sy_n-x^*)\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(\gamma_n+\delta_n)\left\|y_n-x^*\right\|^2\\
&= \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left\|y_n-x^*\right\|^2\\
&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\left[\alpha_n\|Qx_n-x^*\|^2+(1-\alpha_n)\|t_n-x^*\|^2\right]\\
&\le \beta_n\|x_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\|t_n-x^*\|^2\\
&\le \beta_n\|x_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\\
&\quad\times\Big[\|x_n-x^*\|^2-\|x_n-z_n\|^2+2\lambda_n\|x_n-z_n\|\|Ax_n-Ax^*\|\\
&\qquad-\left\|z_n-u_n-(x^*-y^*)\right\|^2+2\mu_2\left\|z_n-u_n-(x^*-y^*)\right\|\|B_2z_n-B_2x^*\|\\
&\qquad-\left\|u_n-t_n+(x^*-y^*)\right\|^2+2\mu_1\left\|u_n-t_n+(x^*-y^*)\right\|\left\|B_1u_n-B_1y^*\right\|\Big]\\
&= \|x_n-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\\
&\quad\times\Big[2\lambda_n\|x_n-z_n\|\|Ax_n-Ax^*\|+2\mu_2\left\|z_n-u_n-(x^*-y^*)\right\|\|B_2z_n-B_2x^*\|\\
&\qquad+2\mu_1\left\|u_n-t_n+(x^*-y^*)\right\|\left\|B_1u_n-B_1y^*\right\|\Big]\\
&\qquad-(1-\beta_n)\left[\|x_n-z_n\|^2+\left\|z_n-u_n-(x^*-y^*)\right\|^2+\left\|u_n-t_n+(x^*-y^*)\right\|^2\right],
\end{aligned}\tag{3.29}$$

which hence implies that

$$\begin{aligned}
&(1-\beta_n)\left[\|x_n-z_n\|^2+\left\|z_n-u_n-(x^*-y^*)\right\|^2+\left\|u_n-t_n+(x^*-y^*)\right\|^2\right]\\
&\quad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\\
&\qquad\times\Big[2\lambda_n\|x_n-z_n\|\|Ax_n-Ax^*\|+2\mu_2\left\|z_n-u_n-(x^*-y^*)\right\|\|B_2z_n-B_2x^*\|\\
&\qquad+2\mu_1\left\|u_n-t_n+(x^*-y^*)\right\|\left\|B_1u_n-B_1y^*\right\|\Big]\\
&\quad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+\alpha_n\|Qx_n-x^*\|^2+(1-\beta_n)\\
&\qquad\times\Big[2\lambda_n\|x_n-z_n\|\|Ax_n-Ax^*\|+2\mu_2\left\|z_n-u_n-(x^*-y^*)\right\|\|B_2z_n-B_2x^*\|\\
&\qquad+2\mu_1\left\|u_n-t_n+(x^*-y^*)\right\|\left\|B_1u_n-B_1y^*\right\|\Big].
\end{aligned}\tag{3.30}$$

Since $\limsup_{n\to\infty}\beta_n < 1$, $0 < \lambda_n \le 2\alpha$, $\alpha_n \to 0$, $\|Ax_n - Ax^*\| \to 0$, $\|B_2z_n - B_2x^*\| \to 0$, $\|B_1u_n - B_1y^*\| \to 0$ and $\|x_{n+1} - x_n\| \to 0$, it follows from the boundedness of $\{x_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{t_n\}$ that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0, \qquad \lim_{n\to\infty}\left\|z_n - u_n - (x^* - y^*)\right\| = 0, \qquad \lim_{n\to\infty}\left\|u_n - t_n + (x^* - y^*)\right\| = 0. \tag{3.31}$$

Consequently, it immediately follows that

$$\lim_{n\to\infty}\|z_n - t_n\| = 0, \qquad \lim_{n\to\infty}\|x_n - t_n\| = 0. \tag{3.32}$$

This together with $\|y_n - t_n\| \le \alpha_n\|Qx_n - t_n\| \to 0$ implies that

$$\lim_{n\to\infty}\left\|x_n - y_n\right\| = 0. \tag{3.33}$$

Since

$$\left\|\delta_n(Sy_n - x_n)\right\| \le \|x_{n+1} - x_n\| + \gamma_n\left\|y_n - x_n\right\|, \tag{3.34}$$

it follows that

$$\lim_{n\to\infty}\left\|Sy_n - x_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|Sy_n - y_n\right\| = 0. \tag{3.35}$$

Step 5. $\limsup_{n\to\infty}\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle \le 0$, where $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma\cap\mathrm{VI}(A,C)} \cdot Q\overline{x}$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle = \lim_{i\to\infty}\langle Q\overline{x} - \overline{x}, x_{n_i} - \overline{x}\rangle. \tag{3.36}$$

Also, since H is reflexive and $\{y_n\}$ is bounded, without loss of generality we may assume that $y_{n_i} \to p$ weakly for some $p \in C$. First, it is clear from Lemma 2.2 that $p \in \mathrm{Fix}(S)$. Now let us show that $p \in \Gamma$. We note that

$$\begin{aligned}
&\left\|y_n - G(y_n)\right\| \\
&\le \alpha_n\left\|Qx_n - G(y_n)\right\| + (1-\alpha_n)\left\|P_C\left[P_C(z_n - \mu_2B_2z_n) - \mu_1B_1P_C(z_n - \mu_2B_2z_n)\right] - G(y_n)\right\| \\
&= \alpha_n\left\|Qx_n - G(y_n)\right\| + (1-\alpha_n)\left\|G(z_n) - G(y_n)\right\| \\
&\le \alpha_n\left\|Qx_n - G(y_n)\right\| + (1-\alpha_n)\left\|x_n - y_n\right\| \\
&\longrightarrow 0.
\end{aligned} \tag{3.37}$$

According to Lemma 2.2 we obtain $p \in \Gamma$. Further, let us show that $p \in \mathrm{VI}(A, C)$. As a matter of fact, since $\|x_n - z_n\| \to 0$ and $\|x_n - y_n\| \to 0$, we deduce that $x_{n_i} \to p$ weakly and $z_{n_i} \to p$ weakly. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases} \tag{3.38}$$

where $N_C v$ is the normal cone to C at $v \in C$. In this case, the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \mathrm{VI}(A, C)$; see [10] for more details. Let $\mathrm{Gph}(T)$ be the graph of T and let $(v, w) \in \mathrm{Gph}(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av \rangle \geq 0$ for all $t \in C$. On the other hand, from $z_n = P_C(x_n - \lambda_n A x_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n A x_n - z_n, z_n - v \rangle \geq 0 \tag{3.39}$$

and hence

$$\left\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + A x_n \right\rangle \geq 0. \tag{3.40}$$

From $\langle v - t, w - Av \rangle \geq 0$ for all $t \in C$ and $z_{n_i} \in C$, we have

$$\begin{aligned} \langle v - z_{n_i}, w \rangle &\geq \langle v - z_{n_i}, Av \rangle \\ &\geq \langle v - z_{n_i}, Av \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + A x_{n_i} \right\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i} \rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle \\ &\geq \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i} \rangle - \left\langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle, \end{aligned} \tag{3.41}$$

Hence, we obtain $\langle v - p, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $p \in T^{-1}0$ and hence $p \in \mathrm{VI}(A, C)$. Therefore, $p \in \mathrm{Fix}(S) \cap \Gamma \cap \mathrm{VI}(A, C)$. Hence it follows from (2.8) and (3.36) that

$$\begin{aligned} \limsup_{n \to \infty} \langle Q\overline{x} - \overline{x}, x_n - \overline{x} \rangle &= \lim_{i \to \infty} \langle Q\overline{x} - \overline{x}, x_{n_i} - \overline{x} \rangle \\ &= \langle Q\overline{x} - \overline{x}, p - \overline{x} \rangle \\ &\leq 0. \end{aligned} \tag{3.42}$$

Step 6. $\lim_{n \to \infty} x_n = \overline{x}$.

Indeed, since $G : C \to C$ is nonexpansive, we have

$$\|t_n - \overline{x}\| = \|G(z_n) - G(\overline{x})\| \leq \|x_n - \overline{x}\|. \tag{3.43}$$

Note that

$$\begin{aligned}\langle Qx_n - \overline{x}, y_n - \overline{x}\rangle &= \langle Qx_n - \overline{x}, x_n - \overline{x}\rangle + \langle Qx_n - \overline{x}, y_n - x_n\rangle \\ &= \langle Qx_n - Q\overline{x}, x_n - \overline{x}\rangle + \langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \langle Qx_n - \overline{x}, y_n - x_n\rangle \\ &\le \rho\|x_n - \overline{x}\|^2 + \langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \|Qx_n - \overline{x}\|\|y_n - x_n\|.\end{aligned} \tag{3.44}$$

Utilizing Lemmas 2.4 and 2.5, we obtain from (3.4) and the convexity of $\|\cdot\|^2$

$$\begin{aligned}\|x_{n+1} - \overline{x}\|^2 &= \|\beta_n(x_n - \overline{x}) + \gamma_n(y_n - \overline{x}) + \delta_n(Sy_n - \overline{x})\|^2 \\ &\le \beta_n\|x_n - \overline{x}\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}[\gamma_n(y_n - \overline{x}) + \delta_n(Sy_n - \overline{x})]\right\|^2 \\ &\le \beta_n\|x_n - \overline{x}\|^2 + (\gamma_n + \delta_n)\|y_n - \overline{x}\|^2 \\ &\le \beta_n\|x_n - \overline{x}\|^2 + (\gamma_n + \delta_n)\left[(1 - \alpha_n)^2\|t_n - \overline{x}\|^2 + 2\alpha_n\langle Qx_n - \overline{x}, y_n - \overline{x}\rangle\right] \\ &\le \beta_n\|x_n - \overline{x}\|^2 + (\gamma_n + \delta_n)\left[(1 - \alpha_n)\|x_n - \overline{x}\|^2 + 2\alpha_n\langle Qx_n - \overline{x}, y_n - \overline{x}\rangle\right] \\ &= (1 - (\gamma_n + \delta_n)\alpha_n)\|x_n - \overline{x}\|^2 + (\gamma_n + \delta_n)2\alpha_n\langle Qx_n - \overline{x}, y_n - \overline{x}\rangle \\ &\le (1 - (\gamma_n + \delta_n)\alpha_n)\|x_n - \overline{x}\|^2 \\ &\quad + (\gamma_n + \delta_n)2\alpha_n\left[\rho\|x_n - \overline{x}\|^2 + \langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \|Qx_n - \overline{x}\|\|y_n - x_n\|\right] \\ &\le [1 - (1 - 2\rho)(\gamma_n + \delta_n)\alpha_n]\|x_n - \overline{x}\|^2 \\ &\quad + (\gamma_n + \delta_n)2\alpha_n[\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \|Qx_n - \overline{x}\|\|y_n - x_n\|] \\ &= [1 - (1 - 2\rho)(\gamma_n + \delta_n)\alpha_n]\|x_n - \overline{x}\|^2 \\ &\quad + (1 - 2\rho)(\gamma_n + \delta_n)\alpha_n\frac{2[\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \|Qx_n - \overline{x}\|\|y_n - x_n\|]}{1 - 2\rho}.\end{aligned} \tag{3.45}$$

Note that $\liminf_{n\to\infty}(1 - 2\rho)(\gamma_n + \delta_n) > 0$. It follows that $\sum_{n=0}^{\infty}(1 - 2\rho)(\gamma_n + \delta_n)\alpha_n = \infty$. It is clear that

$$\limsup_{n\to\infty}\frac{2[\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle + \|Qx_n - \overline{x}\|\|y_n - x_n\|]}{1 - 2\rho} \le 0 \tag{3.46}$$

because $\limsup_{n\to\infty}\langle Q\overline{x} - \overline{x}, x_n - \overline{x}\rangle \le 0$ and $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore, all conditions of Lemma 2.3 are satisfied. Consequently, we immediately deduce that $x_n \to \overline{x}$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let*

$S : C \to C$ be a k-strictly pseudocontractive mapping such that $\mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. *For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated iteratively by*

$$\begin{gathered} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n u + (1-\alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{gathered} \tag{3.47}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma\cap VI(A,C)} \cdot Q\overline{x}$ and $(\overline{x}, \overline{y})$ is a solution of the general system (1.3) *of variational inequalities, where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Corollary 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let $S : C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. *Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1/2)$. For given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{gathered} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = \alpha_n Q x_n + (1-\alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{gathered} \tag{3.48}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$;

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma\cap VI(A,C)} \cdot Q\overline{x}$ and $(\overline{x}, \overline{y})$ is a solution of the general system (1.3) *of variational inequalities, where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

Corollary 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be α-inverse strongly monotone and $B_i : C \to H$ be β_i-inverse strongly monotone for $i = 1, 2$. Let*

$S : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \cap VI(A, C) \neq \emptyset$. *For fixed $u \in C$ and given $x_0 \in C$ arbitrarily, let the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be generated iteratively by*

$$\begin{aligned} z_n &= P_C(x_n - \lambda_n A x_n), \\ y_n &= \alpha_n u + (1-\alpha_n) P_C\left[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{3.49}$$

where $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$, $\{\lambda_n\} \subset (0, 2\alpha]$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset [0, 1]$ such that

(i) $\beta_n + \gamma_n + \delta_n = 1$ *for all* $n \geq 0$*;*

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$*;*

(iv) $\lim_{n\to\infty}(\gamma_{n+1}/(1-\beta_{n+1}) - \gamma_n/(1-\beta_n)) = 0$*;*

(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathrm{Fix}(S)\cap\Gamma\cap VI(A,C)} u$ and $(\overline{x}, \overline{y})$ is a solution of the general system (1.3) *of variational inequalities, where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$.*

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