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Research Article

Fixed Points and Random Stability of a Generalized Apollonius Type Quadratic Functional Equation

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Using the fixed-point method, we prove the generalized Hyers-Ulam stability of a generalized Apollonius type quadratic functional equation in random Banach spaces.

1. Introduction

The stability problem of functional equations was originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias [4] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* of functional equations. A generalization of the Th. M. Rassias theorem was obtained by Găvruţa [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Th. M. Rassias' approach.

On the other hand, in 1982–1998, J. M. Rassias generalized the Hyers' stability result by presenting a weaker condition controlled by a product of different powers of norms.

Theorem 1.1 (see [6–12]). *Assume that there exist constants $\Theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : E \to E'$ is a mapping from a normed space E into a Banach space E', such that the inequality*

$$\| f(x+y) - f(x) - f(y) \| \leq \epsilon \|x\|^{p_1} \|y\|^{p_2}, \tag{1.1}$$

for all $x, y \in E$, then there exists a unique additive mapping $T : E \longrightarrow E'$, such that

$$\|f(x) - L(x)\| \leq \frac{\Theta}{2 - 2^p}\|x\|^p, \tag{1.2}$$

for all $x \in E$.

The control function $\|x\|^p \cdot \|y\|^q + \|x\|^{p+q} + \|y\|^{p+q}$ was introduced by Ravi et al. [13] and was used in several papers (see [14–19]).

The functional equation $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The generalized Hyers-Ulam stability of the quadratic functional equation was proved by Skof [20] for mappings $f : X \longrightarrow Y$, where X is a normed space and Y is a Banach space. Cholewa [21] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [22] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [23–44]).

In [45], Park and Th. M. Rassias defined and investigated the following generalized Apollonius type quadratic functional equation:

$$\begin{aligned} &Q\left(\left(\sum_{i=1}^{n} z_i\right) - \left(\sum_{i=1}^{n} x_i\right)\right) + Q\left(\left(\sum_{i=1}^{n} z_i\right) - \left(\sum_{i=1}^{n} y_i\right)\right) \\ &\quad = \frac{1}{2}Q\left(\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} y_i\right)\right) + 2Q\left(\left(\sum_{i=1}^{n} z_i\right) - \frac{\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} y_i\right)}{2}\right) \end{aligned} \tag{1.3}$$

in Banach spaces.

2. Preliminaries

We define the notion of a random normed space, which goes back to Šerstnev et al. (see, e.g., [46, 47]).

In the sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [47, 48]. Throughout this paper, let Δ^+ be the space of distribution functions, that is,

$$\begin{aligned} \Delta^+ := \{F : \mathbb{R} \cup \{-\infty, \infty\} \longrightarrow [0,1] : F \text{ is leftcontinuous,} \\ \text{nondecreasing on } \mathbb{R},\ F(0) = 0 \text{ and } F(+\infty) = 1\}, \end{aligned} \tag{2.1}$$

and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, that is, $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0 & \text{if } t \le 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{2.2}$$

Definition 2.1 (see [48]). A function $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a t-norm) if T satisfies the following conditions:

(TN_1) T is commutative and associative;

(TN_2) T is continuous;

(TN_3) $T(a,1) = a$ for all $a \in [0,1]$;

(TN_4) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Łukasiewicz t-norm). Recall (see [46, 49]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, $T_{i=1}^n x_i$ is defined recurrently by

$$T_{i=1}^n x_i = \begin{cases} x_1 & \text{if } n = 1, \\ T\left(T_{i=1}^{n-1} x_i, x_n\right) & \text{if } n \ge 2. \end{cases} \tag{2.3}$$

$T_{i=n}^\infty x_i$ is defined as $T_{i=1}^\infty x_{n+i}$.

Definition 2.2 (see [47]). A *random normed space* (briefly, RN-space) is a triple (X, Λ, T), where X is a vector space, T is a continuous t-norm, and Λ is a mapping from X into D^+, such that the following conditions hold:

(RN_1) $\Lambda_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN_2) $\Lambda_{\alpha x}(t) = \Lambda_x(t/(|\alpha|))$ for all $x \in X$, $\alpha \neq 0$;

(RN_3) $\Lambda_{x+y}(t+s) \ge T(\Lambda_x(t), \Lambda_y(s))$ for all $x, y \in X$ and all $t, s \ge 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, Λ, T_M), where $\Lambda_u(t) = t/(t + \|u\|)$ for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 2.3. Let (X, Λ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N, such that $\Lambda_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \ge N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N, such that $\Lambda_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \ge m \ge N$.

(3) An RN-space (X, Λ, T) is said to be *complete* if every Cauchy sequence in X is convergent to a point in X. A complete RN-space is said to be a *random Banach space*.

Theorem 2.4 (see [48]). *If* (X, Λ, T) *is an RN-space and* $\{x_n\}$ *is a sequence, such that* $x_n \to x$, *then* $\lim_{n\to\infty}\Lambda_{x_n}(t) = \Lambda_x(t)$ *almost everywhere.*

Starting with the paper [50], the stability of some functional equations in the framework of fuzzy normed spaces or random normed spaces has been investigated in [51–57].

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let (X, d) be a generalized metric space. An operator $T : X \to X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L \geq 0$ such that $d(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz constant L is less than 1, then the operator T is called a strictly contractive operator. Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. We recall the following theorem by Diaz and Margolis.

Theorem 2.5 (see [58, 59]). *Let* (X, d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$, *then for each given element* $x \in X$, *either*

$$d\left(J^n x, J^{n+1} x\right) = \infty, \tag{2.4}$$

for all nonnegative integers n, *or there exists a positive integer* n_0, *such that*

(1) $d(J^n x, J^{n+1} x) < \infty$, *for all* $n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed-point* y^* *of* J;

(3) y^* *is the unique fixed-point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq (1/(1 - L))d(y, Jy)$ *for all* $y \in Y$.

In 1996, Isac and Th. M Rassias [60] were the first to provide applications of stability theory of functional equations for the proof of new fixed-point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [61–67]).

In this paper, we prove the generalized Hyers-Ulam stability of the generalized Apollonius type quadratic functional equation (1.3) in random Banach space by using the fixed point method.

Throughout this paper, assume that X is a vector spaces and (Y, μ, T) is a complete RN-space.

3. Generalized Hyers-Ulam Stability of the Quadratic Functional Equation (1.3) in RN-Spaces

Let $x = \sum_i^n x_i$, $y = \sum_i^n y_i$, $z = \sum_i^n z_i$, and for a given mapping $Q : X \to Y$, consider the mapping $DQ : X^3 \to Y$, defined by

$$DQ(z,x,y) = Q(z-x) + Q((z-y)) - \frac{1}{2}Q(x-y) + 2Q\left(z - \frac{x+y}{2}\right), \tag{3.1}$$

for all $x, y, z \in X$.

Using the fixed-point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $DQ(z,x,y) = 0$ in complete RN-spaces.

Theorem 3.1. *Let $\rho : X^3 \to D^+$ be a mapping ($\rho(z_1,\dots,z_n,x_1,\dots,x_n,y_1,\dots,y_n)$ is denoted by $\rho_{z,x,y}$) such that, for some $0 < \alpha < 4$,*

$$\rho_{2z,2x,2y}(t) \geq \rho_{z,x,y}(t), \tag{3.2}$$

for all $x, y, z \in X$ and all $t > 0$. Suppose that an even mapping $Q : X \to Y$ with $Q(0) = 0$ satisfies the inequality

$$\mu_{DQ(z,x,y)}(t) \geq \rho_{z,x,y}(t), \tag{3.3}$$

for all $x, y, z \in X$ and all $t > 0$, then there exists a unique quadratic mapping $R : X \to Y$, such that

$$\mu_{Q(x)-R(x)}\left(\frac{t}{4-\alpha}\right) \geq \rho_{x,x,-x}(t), \tag{3.4}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $z = x$ and $y = -x$ in (3.3), we get

$$\mu_{Q(2x)-4Q(x)}(t) \geq \rho_{x,x,-x}(t), \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Therefore,

$$\mu_{Q(2x)/4-Q(x)}(t) \geq \rho_{x,x,-x}(4t), \tag{3.6}$$

for all $x \in X$ and all $t > 0$.

Let S be the set of all even mappings $h : X \to Y$ with $h(0) = 0$ and introduce a generalized metric on S as follows:

$$d(h,k) = \inf\left\{u \in \mathbb{R}^+ : \mu_{h(x)-k(x)}(ut) \geq \rho_{x,x,-x}(t),\ \forall x \in X,\ \forall t > 0\right\}, \tag{3.7}$$

where, as usual, $\inf \emptyset = +\infty$. It is easy to show that (S,d) is a generalized complete metric space (see [68, Lemma 2.1]).

Now, we define the mapping $J : S \to S$

$$Jh(x) := \frac{h(2x)}{4}, \tag{3.8}$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$ such that $d(f,g) < \varepsilon$. Therefore,

$$\begin{aligned}\mu_{Jg(x)-Jf(x)}\left(\frac{\alpha\varepsilon}{4}t\right) &= \mu_{g(2x)/4-f(2x)/4}\left(\frac{\alpha\varepsilon}{4}t\right) = \mu_{g(2x)-f(2x)}(\alpha\varepsilon t)\\ &\geq \rho_{2x,2x,-2x}(\alpha t) \geq \rho_{x,x,-x}(t),\end{aligned} \tag{3.9}$$

that is, if $d(f,g) < \varepsilon$, we have $d(Jf, Jg) < (\alpha/4)\varepsilon$. Hence,

$$d(Jf, Jg) \leq \frac{\alpha}{4} d(f,g), \tag{3.10}$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $L = \alpha/4 < 1$.

It follows from (3.6) that

$$\mu_{JQ(x)-Q(x)}\left(\frac{1}{4}t\right) \geq \rho_{x,x,-x}(t), \tag{3.11}$$

for all $x \in X$ and all $t > 0$, which means that $d(JQ, Q) \leq 1/4$.

By Theorem 2.5, there exists a unique mapping $R : X \to Y$, such that R is a fixed point of J, that is, $R(2x) = 4R(x)$ for all $x \in X$.

Also, $d(J^m Q, R) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} \frac{Q(2^m x)}{2^{2m}} = R(x), \tag{3.12}$$

for all $x \in X$.

It follows from (3.2) and (3.3) that

$$\begin{aligned}\mu_{DQ(2^m z, 2^m x, 2^m y)/2^{2m}}(t) &\geq \rho_{2^m z, 2^m x, 2^m y}\left(2^{2m} t\right) = \rho_{2^m z, 2^m x, 2^m y}\left(\alpha^m \left(\frac{4}{\alpha}\right)^m t\right)\\ &\geq \rho_{z,x,y}\left(\left(\frac{4}{\alpha}\right)^m t\right),\end{aligned} \tag{3.13}$$

for all $x, y, z \in X$ and all $t > 0$. Letting $m \to \infty$ in (3.13), we find that $\mu_{DR(z,x,y)}(t) = 1$ for all $t > 0$, which implies $DR(z,x,y) = 0$. By [45, Lemma 2.1], the mapping, $R : X \to Y$ is quadratic.

Since R is the unique fixed point of J in the set $\Omega = \{g \in S : d(f,g) < \infty\}$, R is the unique mapping such that

$$\mu_{Q(x)-R(x)}(ut) \geq \rho_{x,x,-x}(t), \tag{3.14}$$

for all $x, y, z \in X$ and all $t > 0$. Using the fixed-point alternative, we obtain that

$$d(Q,R) \le \frac{1}{1-L} d(Q,JQ) \le \frac{1}{4(1-L)} = \frac{1}{4-\alpha}, \tag{3.15}$$

which implies the inequality

$$\mu_{Q(x)-R(x)}\left(\frac{1}{4-\alpha}t\right) \ge \rho_{x,x,-x}(t), \tag{3.16}$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{Q(x)-R(x)}(t) \ge \rho_{x,x,-x}((4-\alpha)t), \tag{3.17}$$

for all $x \in X$ and all $t > 0$. This completes the proof. □

Theorem 3.2. *Let $\rho : X^3 \to D^+$ be a mapping ($\rho(z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n)$ is denoted by $\rho_{z,x,y}$) such that, for some $\alpha > 4$,*

$$\rho_{z/2,x/2,y/2}(t) \ge \rho_{z,x,y}(\alpha t), \tag{3.18}$$

for all $x, y, z \in X$ and all $t > 0$. Suppose that an even mapping $Q : X \to Y$ satisfying $Q(0) = 0$ and (3.3)*, then there exists a unique quadratic mapping $R : X \to Y$, such that*

$$\mu_{Q(x)-R(x)}(t) \ge \rho_{x,x,-x}((\alpha-4)t), \tag{3.19}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 3.1.

We consider the mapping $J : S \to S$ defined by

$$Jh(x) := 4h\left(\frac{x}{2}\right), \tag{3.20}$$

for all $h \in S$ and $x \in X$. Let $f, g \in S$, such that $d(f, g) < \varepsilon$, then

$$\begin{aligned} \mu_{Jg(x)-Jf(x)}\left(\frac{4\varepsilon}{\alpha}t\right) &= \mu_{4g(x/2)-4f(x/2)}\left(\frac{4\varepsilon}{\alpha}t\right) = \mu_{g(x/2)-f(x/2)}\left(\frac{\varepsilon}{\alpha}t\right) \\ &\ge \rho_{x/2,x/2,-x/2}\left(\frac{t}{\alpha}\right) \ge \rho_{x,x,-x}(t), \end{aligned} \tag{3.21}$$

that is, if $d(f, g) < \varepsilon$, we have $d(Jf, Jg) < (4/\alpha)\varepsilon$. This means that

$$d(Jf, Jg) \le \frac{4}{\alpha} d(f, g), \tag{3.22}$$

for all $f, g \in S$, that is, J is a strictly contractive self-mapping on S with the Lipschitz constant $L = 4/\alpha < 1$.

By Theorem 2.5, there exists a unique mapping $R : X \to Y$, such that R is a fixed point of J, that is, $R(x/2) = (1/4)R(x)$ for all $x \in X$.

Also, $d(J^m Q, R) \to 0$ as $m \to \infty$, which implies the equality

$$\lim_{m\to\infty} 2^{2m} Q\left(\frac{x}{2^m}\right) = R(x), \tag{3.23}$$

for all $x \in X$.

It follows from (3.5) that

$$\mu_{JQ(x)-Q(x)}\left(\frac{1}{\alpha}t\right) \geq \rho_{x/2,x/2,-x/2}\left(\frac{t}{\alpha}\right) \geq \rho_{x,x,-x}(t), \tag{3.24}$$

for all $x \in X$ and all $t > 0$, which implies that $d(JQ, Q) \leq 1/\alpha$.

Since R is the unique fixed point of J in the set $\Omega = \{g \in S : d(f, g) < \infty\}$, and R is the unique mapping, such that

$$\mu_{Q(x)-R(x)}(ut) \geq \rho_{x,x,-x}(t), \tag{3.25}$$

for all $x \in X$ and all $t > 0$. Using the fixed point alternative, we obtain that

$$d(Q, R) \leq \frac{1}{1-L} d(Q, JQ) \leq \frac{1}{\alpha(1-L)} = \frac{1}{\alpha(1-4/\alpha)}, \tag{3.26}$$

which implies the inequality

$$\mu_{Q(x)-R(x)}\left(\frac{1}{\alpha-4}t\right) \geq \rho_{x,x,-x}(t), \tag{3.27}$$

for all $x \in X$ and all $t > 0$. So

$$\mu_{Q(x)-R(x)}(t) \geq \rho_{x,x,-x}((\alpha-4)t), \tag{3.28}$$

for all $x \in X$ and all $t > 0$.

The rest of the proof is similar to the proof of Theorem 3.1. □

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