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Research Article

An Implicit Extragradient Method for Hierarchical Variational Inequalities

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As a well-known numerical method, the extragradient method solves numerically the variational inequality $\mathrm{VI}(C, A)$ of finding $u \in C$ such that $\langle Au, v-u\rangle \geq 0$, for all $v \in C$. In this paper, we devote to solve the following hierarchical variational inequality $\mathrm{HVI}(C, A, f)$ Find $\tilde{x} \in \mathrm{VI}(C, A)$ such that $\langle (I-f)\tilde{x}, x-\tilde{x}\rangle \geq 0$, for all $x \in \mathrm{VI}(C, A)$. We first suggest and analyze an implicit extragradient method for solving the hierarchical variational inequality $\mathrm{HVI}(C, A, f)$. It is shown that the net defined by the suggested implicit extragradient method converges strongly to the unique solution of $\mathrm{HVI}(C, A, f)$ in Hilbert spaces. As a special case, we obtain the minimum norm solution of the variational inequality $\mathrm{VI}(C, A)$.

1. Introduction

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$. It is well known that the variational inequality theory has emerged as an important tool in studying a wide class of obstacle, unilateral, and equilibrium problems; which arise in several branches of pure and applied sciences in a unified and general framework. Several numerical methods have been developed for solving variational inequalities and related optimization problems, see [1–24] and the references therein. In particular, Korpelevich's extragradient method which was introduced by Korpelevič [4] in 1976 generates a sequence $\{x_n\}$ via the recursion

$$y_n = P_C[x_n - \lambda A x_n],$$
$$x_{n+1} = P_C[x_n - \lambda A y_n], \quad n \geq 0, \tag{1.2}$$

where P_C is the metric projection from R^n onto C, $A : C \to H$ is a monotone operator, and λ is a constant. Korpelevich [4] proved that the sequence $\{x_n\}$ converges strongly to a solution of $\mathrm{VI}(C, A)$. Note that the setting of the space is Euclid space R^n.

Recently, hierarchical fixed point problems and hierarchical minimization problems have attracted many authors' attention due to their link with some convex programming problems. See [25–32]. Motivated and inspired by these results in the literature, in this paper we are devoted to solve the following hierarchical variational inequality $\mathrm{HVI}(C, A, f)$:

$$\text{Find } z \in \mathrm{VI}(C, A) \text{ such that } \langle (I - f)z, x^* - z \rangle \geq 0, \quad \forall x^* \in \mathrm{VI}(C, A). \tag{1.3}$$

For this purpose, in this paper, we first suggest and analyze an implicit extragradient method. It is shown that the net defined by this implicit extragradient method converges strongly to the unique solution of $\mathrm{HVI}(C, A, f)$ in Hilbert spaces. As a special case, we obtain the minimum norm solution of the variational inequality $\mathrm{VI}(C, A)$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let C be a closed convex subset of H. Recall that a mapping $A : C \to H$ is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2, \quad \forall u, v \in C. \tag{2.1}$$

A mapping $f : C \to H$ is said to be ρ-contraction if there exists a constant $\rho \in [0, 1)$ such that

$$\|f(x) - f(y)\| \leq \rho \|x - y\| \quad \forall x, y \in C. \tag{2.2}$$

It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}. \tag{2.3}$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

(i) $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;

(ii) $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$;

(iii) $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$, $y \in C$;

(iv) $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ for all $x \in H$, $y \in C$.

Such properties of P_C will be crucial in the proof of our main results. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from property (iii) that

$$x^* \in \mathrm{VI}(C,A) \Longleftrightarrow x^* = P_C(x^* - \mu A x^*), \quad \forall \mu > 0. \tag{2.4}$$

We need the following lemmas for proving our main result.

Lemma 2.1 (see [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone, and let $\lambda > 0$ be a constant. Then, one has*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^2 \le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{2.5}$$

In particular, if $0 \le \lambda \le 2\alpha$, then $I - \lambda A$ is nonexpansive.

Lemma 2.2 (see [32]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the mapping $F : C \to H$ is monotone and weakly continuous along segments, that is, $F(x + ty) \to F(x)$ weakly as $t \to 0$. Then, the variational inequality*

$$x^* \in C, \quad \langle F x^*, x - x^* \rangle \ge 0, \quad \forall x \in C \tag{2.6}$$

is equivalent to the dual variational inequality

$$x^* \in C, \quad \langle F x, x - x^* \rangle \ge 0, \quad \forall x \in C. \tag{2.7}$$

3. Main Result

In this section, we will introduce our implicit extragradient algorithm and show its strong convergence to the unique solution of $\mathrm{HVI}(C, A, f)$.

Algorithm 1. Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Let $f : C \to H$ be a (nonself) contraction with coefficient $\rho \in [0,1)$. For any $t \in (0,1)$, define a net $\{x_t\}$ as follows:

$$\begin{aligned} y_t &= P_C[t f(x_t) + (1-t)(x_t - \lambda A x_t)], \\ x_t &= P_C[y_t - \lambda A y_t], \quad t \in (0,1), \end{aligned} \tag{3.1}$$

where $\lambda \in [a,b] \subset (0, 2\alpha)$ is a constant.

Note the fact that f is a possible nonself mapping. Hence, if we take $f = 0$, then (3.1) reduces to

$$\begin{aligned} y_t &= P_C[(1-t)(x_t - \lambda A x_t)], \\ x_t &= P_C[y_t - \lambda A y_t], \quad t \in (0,1), \end{aligned} \tag{3.2}$$

Remark 3.1. We notice that the net $\{x_t\}$ defined by (3.1) is well defined. In fact, we can define a self-mapping $W_t : C \to C$ as follows:

$$W_t x := P_C[I - \lambda A] P_C\left[t f(x) + (1-t)(I - \lambda A)x\right], \quad \forall x \in C. \tag{3.3}$$

From Lemma 2.1, we know that if $\lambda \in (0, 2\alpha)$, the mapping $I - \lambda A$ is nonexpansive.

For any $x, y \in C$, we have

$$\begin{aligned} \|W_t x - W_t y\| &= \big\| P_C[I - \lambda A] P_C\left[t f(x) + (1-t)(I - \lambda A)x\right] \\ &\quad - P_C[I - \lambda A] P_C\left[t f(y) + (1-t)(I - \lambda A)y\right] \big\| \\ &\leq \|t f(x) - t f(y) + (1-t)(I - \lambda A)x - (1-t)(I - \lambda A)y\| \\ &\leq t\|f(x) - f(y)\| + (1-t)\|(I - \lambda A)x - (I - \lambda A)y\| \\ &\leq t\rho\|x - y\| + (1-t)\|x - y\| \\ &= \left[1 - (1-\rho)t\right]\|x - y\|. \end{aligned} \tag{3.4}$$

This shows that the mapping W_t is a contraction. By Banach contractive mapping principle, we immediately deduce that the net (3.1) is well defined.

Theorem 3.2. *Suppose the solution set Ω of* HVI(C, A, f) *is nonempty. Then the net $\{x_t\}$ generated by the implicit extragradient method* (3.1) *converges in norm, as $t \to 0^+$, to the unique solution z of the hierarchical variational inequality* HVI(C, A, f). *In particular, if one takes that $f = 0$, then the net $\{x_t\}$ defined by* (3.2) *converges in norm, as $t \to 0^+$, to the minimum-norm solution of the variational inequality* VI(C, A).

Proof. Take that $x^* \in \Omega$. Since $x^* \in \mathrm{VI}(C, A)$, using the relation (2.4), we have $x^* = P_C[x^* - \mu A x^*]$, for all $\mu > 0$. In particular, if we take $\mu = \lambda(1-t)$, we obtain

$$x^* = P_C[x^* - \lambda(1-t)Ax^*] = P_C[tx^* + (1-t)(x^* - \lambda A x^*)], \quad \forall t \in (0,1). \tag{3.5}$$

From (3.1), we have

$$\begin{aligned} \|y_t - x^*\| &= \left\| P_C\left[t f(x_t) + (1-t)(x_t - \lambda A x_t)\right] - P_C[tx^* + (1-t)(x^* - \lambda A x^*)] \right\| \\ &\leq \left\| t\left(f(x_t) - x^*\right) + (1-t)[(x_t - \lambda A x_t) - (x^* - \lambda A x^*)] \right\| \\ &\leq t\|f(x_t) - f(x^*)\| + t\|f(x^*) - x^*\| + (1-t)\|(I - \lambda A)x_t - (I - \lambda A)x^*\| \\ &\leq t\rho\|x_t - x^*\| + t\|f(x^*) - x^*\| + (1-t)\|x_t - x^*\| \\ &= \left[1 - (1-\rho)t\right]\|x_t - x^*\| + t\|f(x^*) - x^*\|. \end{aligned} \tag{3.6}$$

Noting that $I - \lambda A$ is nonexpansive, thus,

$$\begin{aligned}\|x_t - x^*\| &= \left\|P_C\left[y_t - \lambda A y_t\right] - P_C[x^* - \lambda A x^*]\right\| \\ &\le \left\|\left(y_t - \lambda A y_t\right) - (x^* - \lambda A x^*)\right\| \\ &\le \left\|y_t - x^*\right\| \\ &\le \left[1 - (1-\rho)t\right]\|x_t - x^*\| + t\left\|f(x^*) - x^*\right\|.\end{aligned} \tag{3.7}$$

That is,

$$\|x_t - x^*\| \le \frac{1}{1-\rho}\left\|f(x^*) - x^*\right\|. \tag{3.8}$$

Therefore, $\{x_t\}$ is bounded and so are $\{f(x_t)\}$, $\{y_t\}$. Since A is α-inverse strongly monotone, it is $1/\alpha$-Lipschitz continuous. Consequently, $\{Ax_t\}$ and $\{Ay_t\}$ are also bounded.

From (3.6),(2.5), and the convexity of the norm, we deduce

$$\begin{aligned}\|x_t - x^*\|^2 &\le \left\|t\left(f(x_t) - x^*\right) + (1-t)[(x_t - \lambda A x_t) - (x^* - \lambda A x^*)]\right\|^2 \\ &\le t\left\|f(x_t) - x^*\right\|^2 + (1-t)\|(I - \lambda A)x_t - (I - \lambda A)x^*\|^2 \\ &\le t\left\|f(x_t) - x^*\right\|^2 + (1-t)\left[\|x_t - x^*\|^2 + \lambda(\lambda - 2\alpha)\|Ax_t - Ax^*\|^2\right] \\ &\le t\left\|f(x_t) - x^*\right\|^2 + \|x_t - x^*\|^2 + (1-t)a(b - 2\alpha)\|Ax_t - Ax^*\|^2.\end{aligned} \tag{3.9}$$

Therefore, we have

$$(1-t)a(2\alpha - b)\|Ax_t - Ax^*\|^2 \le t\left\|f(x_t) - x^*\right\|^2. \tag{3.10}$$

Hence

$$\lim_{t \to 0}\|Ax_t - Ax^*\| = 0. \tag{3.11}$$

By the property (ii) of the metric projection P_C, we have

$$\begin{aligned}
\|y_t - x^*\|^2 &= \|P_C[tf(x_t) + (1-t)(x_t - \lambda Ax_t)] - P_C(x^* - \lambda Ax^*)\|^2 \\
&\le \langle tf(x_t) + (1-t)(x_t - \lambda Ax_t) - (x^* - \lambda Ax^*), y_t - x^* \rangle \\
&= \frac{1}{2}\Big\{ \|(x_t - \lambda Ax_t) - (x^* - \lambda Ax^*) - t(I - \lambda A - f)x_t\|^2 + \|y_t - x^*\|^2 \\
&\qquad - \|(x_t - \lambda Ax_t) - (x^* - \lambda Ax^*) - (y_t - x^*) - t(I - \lambda A - f)x_t\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|(x_t - \lambda Ax_t) - (x^* - \lambda Ax^*)\|^2 + tM + \|y_t - x^*\|^2 \\
&\qquad - \|(x_t - y_t) - \lambda(Ax_t - Ax^*) - t(I - \lambda A - f)x_t\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_t - x^*\|^2 + tM + \|y_t - x^*\|^2 - \|x_t - y_t\|^2 \\
&\qquad + 2\lambda\langle x_t - y_t, Ax_t - Ax^* \rangle + 2t\langle (I - \lambda A - f)x_t, x_t - y_t \rangle \\
&\qquad - \|\lambda(Ax_t - Ax^*) + t(I - \lambda A - f)x_t\|^2 \Big\} \\
&\le \frac{1}{2}\Big\{ \|x_t - x^*\|^2 + tM + \|y_t - x^*\|^2 - \|x_t - y_t\|^2 \\
&\qquad + 2\lambda\|x_t - y_t\|\|Ax_t - Ax^*\| + 2t\|(I - \lambda A - f)x_t\|\|x_t - y_t\| \Big\},
\end{aligned} \tag{3.12}$$

where $M > 0$ is some appropriate constant. It follows that

$$\begin{aligned}
\|y_t - x^*\|^2 \le \|x_t - x^*\|^2 + tM - \|x_t - y_t\|^2 + 2\lambda\|x_t - y_t\|\|Ax_t - Ax^*\| \\
+ 2t\|(I - \lambda A - f)x_t\|\|x_t - y_t\|,
\end{aligned} \tag{3.13}$$

and hence (by (3.7))

$$\begin{aligned}
\|x_t - x^*\|^2 &\le \|y_t - x^*\|^2 \\
&\le \|x_t - x^*\|^2 + tM - \|x_t - y_t\|^2 + 2\lambda\|x_t - y_t\|\|Ax_t - Ax^*\| \\
&\quad + 2t\|(I - \lambda A - f)x_t\|\|x_t - y_t\|,
\end{aligned} \tag{3.14}$$

which implies that

$$\|x_t - y_t\|^2 \le tM + 2\lambda\|x_t - y_t\|\|Ax_t - Ax^*\| + 2t\|(I - \lambda A - f)x_t\|\|x_t - y_t\|. \tag{3.15}$$

Since $\|Ax_t - Ax^*\| \to 0$, we derive

$$\lim_{t \to 0^+} \|x_t - y_t\| = 0. \tag{3.16}$$

Next, we show that the net $\{x_t\}$ is relatively norm-compact as $t \to 0^+$. Assume that $\{t_n\} \subset (0,1)$ is such that $t_n \to 0^+$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $y_n := y_{t_n}$.

By the property (ii) of metric projection P_C, we have

$$\begin{aligned}
\|y_t - x^*\|^2 &= \|P_C[tf(x_t) + (1-t)(x_t - \lambda A x_t)] - P_C[tx^* + (1-t)(x^* - \lambda A x^*)]\|^2 \\
&\le \langle t(f(x_t) - x^*) + (1-t)[(x_t - \lambda A x_t) - (x^* - \lambda A x^*)], y_t - x^* \rangle \\
&\le t\langle x^* - f(x_t), x^* - y_t\rangle + (1-t)\|(x_t - \lambda A x_t) - (x^* - \lambda A x^*)\| \|y_t - z_0\| \\
&\le t\langle x^* - f(x^*), x^* - y_t\rangle + t\langle f(x^*) - f(x_t), x^* - y_t\rangle + (1-t)\|x_t - x^*\| \|y_t - x^*\| \\
&\le t\langle x^* - f(x^*), x^* - y_t\rangle + t\|f(x^*) - f(x_t)\| \|x^* - y_t\| + (1-t)\|y_t - x^*\|^2 \\
&\le t\langle x^* - f(x^*), x^* - y_t\rangle + t\rho\|x^* - y_t\|^2 + (1-t)\|y_t - x^*\|^2 \\
&= [1 - (1-\rho)t]\|y_t - x^*\|^2 + t\langle x^* - f(x^*), x^* - y_t\rangle.
\end{aligned} \tag{3.17}$$

Hence

$$\|y_t - x^*\|^2 \le \frac{1}{1-\rho}\langle x^* - f(x^*), x^* - y_t\rangle. \tag{3.18}$$

Therefore,

$$\|x_t - x^*\|^2 \le \|y_t - x^*\|^2 \le \frac{1}{1-\rho}\langle x^* - f(x^*), x^* - y_t\rangle. \tag{3.19}$$

In particular,

$$\|x_n - x^*\|^2 \le \frac{1}{1-\rho}\langle x^* - f(x^*), x^* - y_n\rangle. \tag{3.20}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $z \in C$. Since $\|x_t - y_t\| \to 0$, we have $\|x_n - y_n\| \to 0$. Hence, $\{y_n\}$ also converges weakly to the same point z.

Next we show that $z \in \mathrm{VI}(C, A)$. We define a mapping T by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases} \tag{3.21}$$

Then T is maximal monotone (see [33]). Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $y_n \in C$, we have $\langle v - y_n, w - Av\rangle \ge 0$. On the other hand, from $y_n = P_C[t_n f(x_n) + (1-t_n)(x_n - \lambda A x_n)]$, we have

$$\langle v - y_n, y_n - [t_n f(x_n) + (1-t_n)(x_n - \lambda A x_n)]\rangle \ge 0, \tag{3.22}$$

that is,

$$\left\langle v - y_n, \frac{y_n - x_n}{\lambda} + Ax_n + \frac{t_n}{\lambda}(I - \lambda A - f)x_n \right\rangle \geq 0. \tag{3.23}$$

Therefore, we have

$$\begin{aligned}
\langle v - y_{n_i}, w \rangle &\geq \langle v - y_{n_i}, Av \rangle \\
&\geq \langle v - y_{n_i}, Av \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + Ax_{n_i} + \frac{t_{n_i}}{\lambda}(I - \lambda A - f)x_{n_i} \right\rangle \\
&= \left\langle v - y_{n_i}, Av - Ax_{n_i} - \frac{y_{n_i} - x_{n_i}}{\lambda} - \frac{t_{n_i}}{\lambda}(I - \lambda A - f)x_{n_i} \right\rangle \\
&= \langle v - y_{n_i}, Av - Ay_{n_i} \rangle + \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle \\
&\quad - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{t_{n_i}}{\lambda}(I - \lambda A - f)x_{n_i} \right\rangle \\
&\geq \langle v - y_{n_i}, Ay_{n_i} - Ax_{n_i} \rangle - \left\langle v - y_{n_i}, \frac{y_{n_i} - x_{n_i}}{\lambda} + \frac{t_{n_i}}{\lambda}(I - \lambda A - f)x_{n_i} \right\rangle.
\end{aligned} \tag{3.24}$$

Noting that $t_{n_i} \to 0$, $\|y_{n_i} - x_{n_i}\| \to 0$, and A is Lipschitz continuous, we obtain $\langle v - z, w \rangle \geq 0$. Since T is maximal monotone, we have $z \in T^{-1}(0)$ and hence $z \in \mathrm{VI}(C, A)$.

Therefore we can substitute x^* for z in (3.20) to get

$$\|x_n - z\|^2 \leq \frac{1}{1 - \rho} \langle z - f(z), z - y_n \rangle. \tag{3.25}$$

Consequently, the weak convergence of $\{x_n\}$ and $\{y_n\}$ to z actually implies that $x_n \to z$ strongly. This has proved the relative norm-compactness of the net $\{x_t\}$ as $t \to 0^+$.

Now we return to (3.20) and take the limit as $n \to \infty$ to get

$$\|z - x^*\|^2 \leq \frac{1}{1 - \rho} \langle x^* - f(x^*), x^* - z \rangle, \quad x^* \in \mathrm{VI}(C, A). \tag{3.26}$$

In particular, z solves the following VI

$$z \in \mathrm{VI}(C, A), \quad \langle (I - f)x^*, x^* - z \rangle \geq 0, \quad x^* \in \mathrm{VI}(C, A), \tag{3.27}$$

or the equivalent dual VI (see Lemma 2.2)

$$z \in \mathrm{VI}(C, A), \quad \langle (I - f)z, x^* - z \rangle \geq 0, \quad x^* \in \mathrm{VI}(C, A). \tag{3.28}$$

Therefore, $z = (P_{\mathrm{VI}(C,A)} f)z$. That is, z is the unique solution in $\mathrm{VI}(C, A)$ of the contraction $P_{\mathrm{VI}(C,A)} f$. Clearly this is sufficient to conclude that the entire net $\{x_t\}$ converges in norm to z as $t \to 0^+$.

Finally, if we take that $f = 0$, then VI (3.28) is reduced to

$$z \in \text{VI}(C, A), \quad \langle (z, x^* - z) \rangle \geq 0, \quad x^* \in \text{VI}(C, A). \tag{3.29}$$

Equivalently,

$$\|z\|^2 \leq \langle z, x^* \rangle, \quad x^* \in \text{VI}(C, A). \tag{3.30}$$

This clearly implies that

$$\|z\| \leq \|x^*\|, \quad x^* \in \text{VI}(C, A). \tag{3.31}$$

Therefore, z is the minimum-norm solution of VI(C, A). This completes the proof. □

Remark 3.3. (1) Note that our Implicit Extragradient Algorithms (3.1) and (3.2) have strong convergence in an infinite dimensional Hilbert space.

(2) In many problems, it is needed to find a solution with minimum norm; see [34–38]. Our Algorithm (3.2) solves the minimum norm solution of VI(C, A).

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References

[1] G. Stampacchia, "Formes bilinéaires coercitives sur les ensembles convexes," *Comptes rendus de I'Academie des. Sciences*, vol. 258, pp. 4413–4416, 1964.

[2] J.-L. Lions and G. Stampacchia, "Variational inequalities," *Communications on Pure and Applied Mathematics*, vol. 20, pp. 493–519, 1967.

[3] F. Liu and M. Z. Nashed, "Regularization of nonlinear ill-posed variational inequalities and convergence rates," *Set-Valued Analysis*, vol. 6, no. 4, pp. 313–344, 1998.

[4] G. M. Korpelevič, "An extragradient method for finding saddle points and for other problems," *Èkonomika i Matematicheskie Metody*, vol. 12, no. 4, pp. 747–756, 1976.

[5] R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer Series in Computational Physics, Springer, New York, NY, USA, 1984.

[6] A. N. Iusem and B. F. Svaiter, "A variant of Korpelevich's method for variational inequalities with a new search strategy," *Optimization*, vol. 42, no. 4, pp. 309–321, 1997.

[7] E. N. Khobotov, "Modification of the extra-gradient method for solving variational inequalities and certain optimization problems," *USSR Computational Mathematics and Mathematical Physics*, vol. 27, no. 5, pp. 120–127, 1989.

[8] M. V. Solodov and B. F. Svaiter, "A new projection method for variational inequality problems," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 765–776, 1999.

[9] M. Aslam Noor, "Some developments in general variational inequalities," *Applied Mathematics and Computation*, vol. 152, no. 1, pp. 199–277, 2004.

[10] Y. Yao and J.-C. Yao, "On modified iterative method for nonexpansive mappings and monotone mappings," *Applied Mathematics and Computation*, vol. 186, no. 2, pp. 1551–1558, 2007.

[11] Y. Censor, A. Gibali, and S. Reich, "Two extensions of Korpelevich's extragradient method for solving the variational inequality problem in Euclidean space," Tech. Rep., 2010.
[12] Y. Censor, A. Gibali, and S. Reich, "The subgradient extragradient method for solving variational inequalities in Hilbert space," *Journal of Optimization Theory and Applications*. In press.
[13] W. Takahashi and M. Toyoda, "Weak convergence theorems for nonexpansive mappings and monotone mappings," *Journal of Optimization Theory and Applications*, vol. 118, no. 2, pp. 417–428, 2003.
[14] Y. Yao, Y. C. Liou, and G. Marino, "Strong convergence of two iterative algorithms for nonexpansive mappings in Hilbert spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 279058, 7 pages, 2009.
[15] Y. Yao, Y.-C. Liou, and Y.-J. Wu, "An extragradient method for mixed equilibrium problems and fixed point problems," *Fixed Point Theory and Applications*, vol. 2009, Article ID 632819, 15 pages, 2009.
[16] Y. Yao, Y.-C. Liou, and J.-C. Yao, "A new hybrid iterative algorithm for fixed-point problems, variational inequality problems, and mixed equilibrium problems," *Fixed Point Theory and Applications*, vol. 2008, Article ID 417089, 15 pages, 2008.
[17] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.
[18] Y. Yao, Y.-C. Liou, and J.-C. Yao, "An extragradient method for fixed point problems and variational inequality problems," *Journal of Inequalities and Applications*, vol. 2007, Article ID 38752, 12 pages, 2007.
[19] S. Wang, G. Marino, and F. Wang, "Strong convergence theorems for a generalized equilibrium problem with a relaxed monotone mapping and a countable family of nonexpansive mappings in a Hilbert space," *Fixed Point Theory and Applications*, vol. 2010, Article ID 230304, 22 pages, 2010.
[20] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, "A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem," *Fixed Point Theory and Applications*, vol. 2010, Article ID 230304, 19 pages, 2010.
[21] F. Cianciaruso, G. Marino, L. Muglia, and Y. Yao, "On a two-step algorithm for hierarchical fixed point problems and variational inequalities," *Journal of Inequalities and Applications*, vol. 2009, Article ID 208692, 13 pages, 2009.
[22] J.-W. Peng, S.-Y. Wu, and J.-C. Yao, "A new iterative method for finding common solutions of a system of equilibrium problems, fixed-point problems, and variational inequalities," *Abstract and Applied Analysis*, vol. 2010, Article ID 428293, 27 pages, 2010.
[23] L.-C. Zeng, Q. H. Ansari, D. S. Shyu, and J.-C. Yao, "Strong and weak convergence theorems for common solutions of generalized equilibrium problems and zeros of maximal monotone operators," *Fixed Point Theory and Applications*, vol. 2010, Article ID 590278, 33 pages, 2010.
[24] L.-C. Zeng, S. Kum, and J.-C. Yao, "Algorithm for solving a generalized mixed equilibrium problem with perturbation in a Banach space," *Fixed Point Theory and Applications*, vol. 2010, Article ID 794503, p. 22, 2010.
[25] A. Moudafi and P.-E. Maingé, "Towards viscosity approximations of hierarchical fixed-point problems," *Fixed Point Theory and Applications*, vol. 2006, Article ID 95453, p. 10, 2006.
[26] H.-K. Xu, "Viscosity method for hierarchical fixed point approach to variational inequalities," *Taiwanese Journal of Mathematics*, vol. 14, no. 2, pp. 463–478, 2010.
[27] F. Cianciaruso, V. Colao, L. Muglia, and H.-K. Xu, "On an implicit hierarchical fixed point approach to variational inequalities," *Bulletin of the Australian Mathematical Society*, vol. 80, no. 1, pp. 117–124, 2009.
[28] P.-E. Maingé and A. Moudafi, "Strong convergence of an iterative method for hierarchical fixed-point problems," *Pacific Journal of Optimization*, vol. 3, no. 3, pp. 529–538, 2007.
[29] A. Moudafi, "Krasnoselski-Mann iteration for hierarchical fixed-point problems," *Inverse Problems*, vol. 23, no. 4, pp. 1635–1640, 2007.
[30] Y. Yao and Y.-C. Liou, "Weak and strong convergence of Krasnoselski-Mann iteration for hierarchical fixed point problems," *Inverse Problems*, vol. 24, no. 1, Article ID 015015, p. 8, 2008.
[31] G. Marino, V. Colao, L. Muglia, and Y. Yao, "Krasnoselski-Mann iteration for hierarchical fixed points and equilibrium problem," *Bulletin of the Australian Mathematical Society*, vol. 79, no. 2, pp. 187–200, 2009.
[32] X. Lu, H.-K. Xu, and X. Yin, "Hybrid methods for a class of monotone variational inequalities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 1032–1041, 2009.
[33] R. T. Rockafellar, "Monotone operators and the proximal point algorithm," *SIAM Journal on Control and Optimization*, vol. 14, no. 5, pp. 877–898, 1976.

[34] A. Sabharwal and L. C. Potter, "Convexly constrained linear inverse problems: iterative least-squares and regularization," *IEEE Transactions on Signal Processing*, vol. 46, no. 9, pp. 2345–2352, 1998.
[35] Y.-L. Cui and X. Liu, "Notes on Browder's and Halpern's methods for nonexpansive mappings," *Fixed Point Theory*, vol. 10, no. 1, pp. 89–98, 2009.
[36] X. Liu and Y. Cui, "The common minimal-norm fixed point of a finite family of nonexpansive mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 1, pp. 76–83, 2010.
[37] Y. Yao, R. Chen, and H.-K. Xu, "Schemes for finding minimum-norm solutions of variational inequalities," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 7-8, pp. 3447–3456, 2010.
[38] Y. Yao and H. K. Xu, "Iterative methods for finding minimum-norm fixed points of nonexpansive mappings with applications," *Optimization*. In press.