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Research Article

Strong Convergence Theorems of the Ishikawa Process with Errors for Strictly Pseudocontractive Mapping of Browder-Petryshyn Type in Banach Spaces

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We prove several strong convergence theorems for the Ishikawa iterative sequence with errors to a fixed point of strictly pseudocontractive mapping of Browder-Petryshyn type in Banach spaces and give sufficient and necessary conditions for the convergence of the scheme to a fixed point of the mapping. The results presented in this work give an affirmative answer to the open question raised by Zeng et al. 2006, and generalize the corresponding result of Zeng et al. 2006, Osilike and Udomene 2001, and others.

1. Introduction and Preliminaries

Let E be a real Banach space and E^* its dual. $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by the following:

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}, \quad \forall x \in E. \tag{1.1}$$

It is well known that if E is smooth, then J is single-valued. In this paper, we denote a single-valued selection of the normalized duality mapping by j. I denotes the identity operator. $F(T)$ is the fixed point set of T, that is, $F(T) = \{x : Tx = x\}$.

Definition 1.1 (see [1]). A mapping $T : D(T) \subset E \to E$ is said to be strictly pseudocontractive if there exists $\lambda > 0$ and $j(x-y) \in J(x-y)$, such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x-y\|^2 - \lambda\|x-y-(Tx-Ty)\|^2, \quad \forall x, y \in D(T). \tag{1.2}$$

Remark 1.2. (i) Without loss of generality, we may assume $\lambda \in (0,1)$. Inequality (1.2) can be written in the form

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \ge \lambda\|(I-T)x-(I-T)y\|^2. \tag{1.3}$$

(ii) If E is a Hilbert space, then inequality (1.2) is equivalent to the following inequality:

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2, \quad k = 1-2\lambda < 1. \tag{1.4}$$

(iii) T is a Lipschitz continuous mapping, that is, $\exists L > 0$, s.t, $\|Tx-Ty\| \le L\|x-y\|$. In fact, by (1.3), we have

$$\|x-y\| \ge \lambda\|x-y-(Tx-Ty)\| \ge \lambda\|Tx-Ty\| - \lambda\|x-y\|, \tag{1.5}$$

so that,

$$\|Tx-T\| \le L\|x-y\|, \quad \forall x, y \in D(T), \tag{1.6}$$

where $L = (\lambda+1)/\lambda$.

Definition 1.3. A mapping $T : D(T) \subset E \to E$ is said to be

(i) compact, if for any bounded sequence $\{x_n\}$ in $D(T)$, there exists a strongly convergent subsequence of $\{Tx_n\}$, or

(ii) demicompact, if for any bounded sequence $\{x_n\}$ in $D(T)$, whenever $\{x_n - Tx_n\}$ is strongly convergent, there exists a strongly convergent subsequence of $\{x_n\}$.

Let us recall some important iterative processes.

Definition 1.4 (Ishikawa iterative process with errors in the sense of Liu [2]). Let K be a nonempty convex subset of E with $K + K \subseteq K$. For any $x_1 \in K$, the sequence $\{x_n\}$ is defined as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n + u_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n + v_n, \quad n \ge 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in $[0,1]$, and $\{u_n\}$, $\{v_n\}$, are appropriate sequences in K.

If $\beta_n = v_n = 0$ for all n, then (1.7) reduces to Mann iterative process with errors as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n. \tag{1.8}$$

Definition 1.5 (Ishikawa iterative process with errors in the sense of Xu [3]). Let K be a nonempty convex subset of E. For any $x_1 \in K$, the sequence $\{x_n\}$ is defined as follows:

$$\begin{aligned} x_{n+1} &= (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n, \\ y_n &= (1-\beta_n-\delta_n)x_n + \beta_n T x_n + \delta_n v_n, \quad n \geq 1, \end{aligned} \tag{1.9}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, and $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\beta_n\}$, $\{\delta_n\}$ are real sequences in $[0,1]$ satisfying $\alpha_n + \gamma_n \leq 1$, $\beta_n + \delta_n \leq 1$, for all $n \geq 1$.

If $\beta_n = \delta_n = 0$ for all n, then (1.9) reduces to Mann iterative process with errors as follows:

$$x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T x_n + \gamma_n u_n. \tag{1.10}$$

Remark 1.6. (i) If $u_n = v_n = 0$ in (1.7) or $\gamma_n = \delta_n = 0$ in (1.9), then (1.7) and (1.9) reduce to Ishikawa iterative process [4],

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1. \end{aligned} \tag{1.11}$$

(ii) If $u_n = 0$ in (1.8) or $\gamma_n = 0$ in (1.10), then (1.8) and (1.10) reduce to Mann iterative process [5],

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1. \tag{1.12}$$

In 1974, Rhoades [6] proved strong convergence theorem by the Mann iterative process to a fixed point of strictly pseudocontractive mapping defined on a nonempty compact convex subset of a Hilbert space. In 2001, Osilike and Udomene [7] proved weak and strong convergence theorems for strictly pseudocontractive mapping in a real q-uniformly smooth Banach space E which is also uniform convex.

In 2006, Zeng et al. [8] established the sufficient and necessary conditions on the strong convergence to a fixed point of strictly pseudocontractive mapping in a real q-uniformly smooth Banach space. They got the following main results.

Theorem 1.7. *Let $q > 1$ and E be a real q-uniformly smooth Banach space, let K be a nonempty closed convex subset of E with $K + K \subseteq K$, and let $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{u_n\}$ be a bounded sequence in K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in $[0,1]$*

satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} \|u_n\| < \infty$;

(ii) $\alpha_n \leq \lambda(q/c_q)^{1/(q-1)}$, *and* $\sum_{n=1}^{\infty} \beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q-1)\}$ *and* c_q *is a constant depending on* q.

From an arbitrary $x_1 \in K$, *let* $\{x_n\}$ *be defined by the following:*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1. \tag{1.13}$$

Then $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\{x_n\}$ *is bounded and* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, *where* $d(x, F(T)) = \inf_{p\in F(T)} \|x - p\|$.

In the end of Zeng et al. [8], they raised an open question.

Open Question 1. Can the Ishikawa iterative process with errors (1.7) be extended to Theorem 1.7?

At the same year, Zeng et al. [9] proved the following strong convergence theorem for strictly pseudocontractive mappings.

Theorem 1.8. *Let* $q > 1$ *and* E *be a real* q*-uniformly smooth Banach space. Let* K *be a nonempty closed convex subset of* E, *and let* $T : K \to K$ *be compact or demicompact, and strictly pseudocontractive with* $F(T) \neq \emptyset$. *Let* $\{u_n\}$ *be a bounded sequence in* K. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \leq 1$, *for all* $n \geq 1$;

(ii) $\overline{\lim}_{n\to\infty} \alpha_n < \lambda(q/c_q)^{1/(q-1)}$, $\overline{\lim}_{n\to\infty} \beta_n < 1/L$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n^{\tau} < \infty$, *where* $\tau = \min\{1, (q-1)\}$.

From an arbitrary $x_1 \in K$, *let* $\{x_n\}$ *be defined by the following:*

$$x_{n+1} = (1-\alpha_n-\gamma_n)x_n + \alpha_n T y_n + \gamma_n u_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1. \tag{1.14}$$

If $\{x_n\}$ *is the bounded sequence, then* $\{x_n\}$ *converges strongly to a fixed point of* T.

They raised another open question.

Open Question 2. Can the Ishikawa iterative process with errors (1.9) be extended to Theorem 1.8?

We have answered the Open Question 1 in [10]. The purpose of this paper is to answer the Open Question 2, and we prove some strong convergence theorems for strictly pseudocontractive mapping in Banach spaces, which improve Theorem 1.8 in the following:

(i) q-uniformly smooth Banach spaces can be replaced by general Banach spaces.

(ii) Remove the boundedness assumption of $\{x_n\}$.

(iii) Iterative process (1.14) can be replaced by Ishikawa iterative process with errors (1.9).

Respectively, our results improve and generalize the corresponding results of Zeng el al. [8], Osilike and Udomene [7], and others.

In the sequel, we will need the following lemmas.

Lemma 1.9 (see [11]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + c_n)a_n + b_n, \quad n \ge 1. \tag{1.15}$$

If $\sum_{n=1}^{\infty} c_n < +\infty$, $\sum_{n=1}^{\infty} b_n < +\infty$, we have (i) $\lim_{n\to\infty} a_n$ exists. (ii) In particular, if $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.10 (see [12]). *Let E be a Banach space and $J : E \to 2^{E^*}$ be the normalized duality mapping, then for any $x, y \in E$, the following conclusions hold*

(i) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle$, *for all* $j(x + y) \in J(x + y)$;

(ii) $\|x + y\|^2 \ge \|x\|^2 + 2\langle y, j(x)\rangle$, *for all* $j(x) \in J(x)$.

2. Main Results

In the rest of paper, we denote by L the Lipschitz constant.

Lemma 2.1. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $x_1 \in K$; $\{x_n\}$ is defined by* (1.9) *and satisfying the following conditions:*

(i) $\beta_n \le \alpha_n$, $\delta_n \le \gamma_n$, $\sum_{n=1}^{\infty} \gamma_n < +\infty$;

(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < +\infty$, $\sum_{n=1}^{\infty} \alpha_n = +\infty$.

Then

(1) there exist two sequences $\{r_n\}$, $\{s_n\}$ in $[0, +\infty)$, such that $\sum_{n=1}^{\infty} r_n < +\infty$, $\sum_{n=1}^{\infty} s_n < +\infty$, and

$$\|x_{n+1} - q\| \le (1 + r_n)\|x_n - q\| + s_n, \quad \forall q \in F(T), \ n \ge 1. \tag{2.1}$$

Furthermore, $\lim_{n\to\infty} \|x_n - q\|$ exists.

(2) For any integer $n, m \geq 1$, there exists a constant $M_1 > 0$, such that

$$\|x_{n+m} - q\| \leq M_1 \|x_n - q\| + M_1 \sum_{k=n}^{n+m-1} s_k, \quad \forall q \in F(T). \tag{2.2}$$

Proof. (1) Let $q \in F(T)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K, we have

$$0 < M := \max\left\{\sup_{n\geq 1}\|u_n - q\|, \sup_{n\geq 1}\|v_n - q\|\right\} < +\infty. \tag{2.3}$$

Since T is a strictly pseudocontractive mapping, by Remark 1.2(i),

$$\langle (I-T)x - (I-T)y, j(x-y)\rangle \geq \lambda \left\|(I-T)x - (I-T)y\right\|^2 \geq 0. \tag{2.4}$$

By Kato [13], the above inequality is equivalent to

$$\|x - y\| \leq \|x - y + \gamma[(I-T)x - (I-T)y]\|, \quad \forall x, y \in K, \ \gamma > 0. \tag{2.5}$$

Let $a_n = \alpha_n + \gamma_n$ and from (1.9), we have

$$x_{n+1} = (1 - a_n)x_n + a_n T y_n + \gamma_n (u_n - T y_n). \tag{2.6}$$

It follows that

$$\begin{aligned} x_n = {} & (1 + a_n)x_{n+1} + a_n (I - T)x_{n+1} - a_n x_n + 2a_n^2 (x_n - T y_n) \\ & + a_n (T x_{n+1} - T y_n) + \gamma_n (1 + 2a_n)(T y_n - u_n). \end{aligned} \tag{2.7}$$

Observe that

$$q = (1 + a_n)q + a_n (I - T)q - a_n q. \tag{2.8}$$

From (2.7) and (2.8), we have

$$\begin{aligned} x_n - q = {} & (1 + a_n)(x_{n+1} - q) + a_n [(I - T)x_{n+1} - (I - T)q] - a_n (x_n - q) \\ & + 2a_n^2 (x_n - T y_n) + a_n (T x_{n+1} - T y_n) + \gamma_n (1 + 2a_n)(T y_n - u_n). \end{aligned} \tag{2.9}$$

By inequality (2.5), we get

$$\begin{aligned}\|x_n-q\| &\geq (1+a_n)\|x_{n+1}-q+\frac{a_n}{1+a_n}\left[(I-T)x_{n+1}-(I-T)q\right]\| \\ &\quad - a_n\|x_n-q\| - 2a_n^2\|x_n-Ty_n\| - a_n\|Tx_{n+1}-Ty_n\| \\ &\quad - \gamma_n(1+2a_n)\|Ty_n-u_n\| \\ &\geq (1+a_n)\|x_{n+1}-q\| - a_n\|x_n-q\| - 2a_n^2\|x_n-Ty_n\| \\ &\quad - a_n\|Tx_{n+1}-Ty_n\| - \gamma_n(1+2a_n)\|Ty_n-u_n\|.\end{aligned} \tag{2.10}$$

So

$$\begin{aligned}\|x_{n+1}-q\| \leq \|x_n-q\| + 2a_n^2\|x_n-Ty_n\| + a_n\|Tx_{n+1}-Ty_n\| \\ + \gamma_n(1+2a_n)\|Ty_n-u_n\|.\end{aligned} \tag{2.11}$$

Furthermore, set $b_n = \beta_n + \delta_n \leq 1$, then

$$y_n = (1-b_n)x_n + b_nTx_n + \delta_n(v_n - Tx_n). \tag{2.12}$$

We make the following estimations.

$$\begin{aligned}\|y_n-q\| &= \|(1-b_n)(x_n-q) + b_n(Tx_n-q) + \delta_n(v_n-Tx_n)\| \\ &\leq [1+(L-1)b_n]\|x_n-q\| + \delta_n\|v_n-Tx_n\| \\ &\leq L\|x_n-q\| + \delta_nL\|x_n-q\| + \delta_nM \\ &= L(1+\delta_n)\|x_n-q\| + \delta_nM,\end{aligned} \tag{2.13}$$

$$\begin{aligned}\|x_n-Ty_n\| &\leq \|x_n-q\| + \|q-Ty_n\| \\ &\leq \|x_n-q\| + L\|y_n-q\| \\ &\leq \|x_n-q\| + L^2(1+\delta_n)\|x_n-q\| + L\delta_nM \\ &= \left[1+L^2(1+\delta_n)\right]\|x_n-q\| + L\delta_nM,\end{aligned} \tag{2.14}$$

$$\begin{aligned}\|Ty_n-u_n\| &\leq L\|y_n-q\| + \|u_n-q\| \\ &\leq L^2(1+\delta_n)\|x_n-q\| + (1+L\delta_n)M,\end{aligned} \tag{2.15}$$

$$\begin{aligned}
\|Tx_{n+1} - Ty_n\| &\leq L\|x_{n+1} - y_n\| \\
&= L\|x_n - y_n + a_n(Ty_n - x_n) + \gamma_n(u_n - Ty_n)\| \\
&\leq L\|x_n - y_n\| + La_n\|Ty_n - x_n\| + L\gamma_n\|Ty_n - u_n\| \\
&= L\|b_n(x_n - Tx_n) + \delta_n(Tx_n - v_n)\| + La_n\|Ty_n - x_n\| + L\gamma_n\|Ty_n - u_n\| \\
&\leq Lb_n\|x_n - Tx_n\| + L\delta_n\|Tx_n - v_n\| + La_n\|Ty_n - x_n\| + L\gamma_n\|Ty_n - u_n\| \\
&\leq L(1+L)b_n\|x_n - q\| + L^2\delta_n\|x_n - q\| + La_n\left[1 + L^2(1+\delta_n)\right]\|x_n - q\| \\
&\quad + L^3\gamma_n(1+\delta_n)\|x_n - q\| + L\delta_n M + L^2 a_n\delta_n M + L\gamma_n(1 + L\delta_n)M.
\end{aligned} \tag{2.16}$$

Substituting (2.14), (2.15), and (2.16) in (2.11), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| \leq\ & \|x_n - q\| + 2a_n^2\left[1 + L^2(1+\delta_n)\right]\|x_n - q\| + L(1+L)a_n b_n\|x_n - q\| \\
&+ a_n L^2\delta_n\|x_n - q\| + La_n^2\left[1 + L^2(1+\delta_n)\right]\|x_n - q\| \\
&+ a_n L^3\gamma_n(1+\delta_n)\|x_n - q\| + 2a_n^2 L\delta_n M + a_n L\delta_n M + L^2 a_n^2\delta_n M \\
&+ La_n\gamma_n(1 + L\delta_n)M \\
=\ & (1 + r_n)\|x_n - q\| + s_n,
\end{aligned} \tag{2.17}$$

where

$$\begin{aligned}
r_n &= 2a_n^2\left[1 + L^2(1+\delta_n)\right] + L(1+L)a_n b_n + a_n L^2\delta_n \\
&\quad + La_n^2\left[1 + L^2(1+\delta_n)\right] + a_n L^3\gamma_n(1+\delta_n), \\
s_n &= 2a_n^2 L\delta_n M + a_n L\delta_n M + L^2 a_n^2\delta_n M + La_n\gamma_n(1 + L\delta_n)M.
\end{aligned} \tag{2.18}$$

By conditions (i) and (ii), we have $\sum_{n=1}^{\infty} r_n < +\infty$, $\sum_{n=1}^{\infty} s_n < +\infty$. It follows from Lemma 1.9 that $\lim_{n\to\infty}\|x_n - q\|$ exists. This completes the proof of part (1).

(2) If $x \geq 0$, then $1 + x \leq e^x$. For any integer $n, m \geq 1$ and from part (1), we have

$$\begin{aligned}
\|x_{n+m} - q\| &\leq (1 + r_{n+m-1})\|x_{n+m-1} - q\| + s_{n+m-1} \\
&\leq e^{r_{n+m-1}} e^{r_{n+m-2}}\|x_{n+m-2} - q\| + e^{r_{n+m-1}} s_{n+m-2} + s_{n+m-1} \\
&\cdots \\
&\leq e^{\sum_{k=n}^{n+m-1} r_k}\|x_n - q\| + e^{\sum_{k=n}^{n+m-1} r_k}\sum_{k=n}^{n+m-1} s_k \\
&\leq M_1\|x_n - q\| + M_1\sum_{k=n}^{n+m-1} s_k,
\end{aligned} \tag{2.19}$$

where $M_1 = e^{\sum_{k=1}^{\infty} r_k}$. This completes the proof of part (2). □

Lemma 2.2. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined as in Lemma 2.1. Then there exists a subsequence x_{n_j} of $\{x_n\}$, such that*

$$\lim_{j \to \infty} \|x_{n_j} - Tx_{n_j}\| = 0. \tag{2.20}$$

Proof. Let $q \in F(T)$. It follows from (1.3), (1.9), and Lemma 1.10(i) that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \|x_n - q + \alpha_n(Ty_n - x_n) + \gamma_n(u_n - x_n)\|^2 \\
&\quad \le \|x_n - q\|^2 + 2\langle \alpha_n(Ty_n - x_n) + \gamma_n(u_n - x_n), j(x_{n+1} - q)\rangle \\
&\quad = \|x_n - q\|^2 - 2\alpha_n\langle x_{n+1} - Tx_{n+1}, j(x_{n+1} - q)\rangle + 2\alpha_n\langle x_{n+1} - x_n, j(x_{n+1} - q)\rangle \\
&\qquad + 2\alpha_n\langle Ty_n - Tx_{n+1}, j(x_{n+1} - q)\rangle + 2\gamma_n\langle u_n - x_n, j(x_{n+1} - q)\rangle \\
&\quad \le \|x_n - q\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 + 2\alpha_n^2\langle Ty_n - x_n, j(x_{n+1} - q)\rangle \\
&\qquad + 2\alpha_n\langle Ty_n - Tx_{n+1}, j(x_{n+1} - q)\rangle + (2\alpha_n\gamma_n + 2\gamma_n)\langle u_n - x_n, j(x_{n+1} - q)\rangle \\
&\quad \le \|x_n - q\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 + 2\alpha_n^2\|Ty_n - x_n\|\|x_{n+1} - q\| \\
&\qquad + 2\alpha_n L\|y_n - x_{n+1}\|\|x_{n+1} - q\| + (2\alpha_n\gamma_n + 2\gamma_n)\|u_n - x_n\|\|x_{n+1} - q\|.
\end{aligned} \tag{2.21}$$

Let

$$\begin{aligned}
k_n &= 2\alpha_n^2\|Ty_n - x_n\|\|x_{n+1} - q\| + 2\alpha_n L\|y_n - x_{n+1}\|\|x_{n+1} - q\| \\
&\quad + (2\alpha_n\gamma_n + 2\gamma_n)\|u_n - x_n\|\|x_{n+1} - q\|.
\end{aligned} \tag{2.22}$$

Then (2.21) becomes

$$\|x_{n+1} - q\|^2 \le \|x_n - q\|^2 - 2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 + k_n. \tag{2.23}$$

From Lemma 2.1(1), $\lim_{n\to\infty}\|x_n - q\|$ exists. So $\{\|x_n - q\|\}$ is bounded. By inequalities (2.14), (2.15), and (2.16), the sequences $\{\|Ty_n - x_n\|\}$, $\{\|y_n - x_{n+1}\|\}$, $\{\|u_n - x_n\|\}$ are all bounded. Notice the conditions of $\sum_{n=1}^{\infty}\alpha_n^2 < +\infty$ and $\sum_{n=1}^{\infty}\gamma_n < +\infty$, then $\sum_{n=1}^{\infty}k_n < +\infty$. It follows from (2.23) that

$$2\alpha_n\lambda\|x_{n+1} - Tx_{n+1}\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + k_n, \tag{2.24}$$

so

$$2\lambda\sum_{i=1}^{n}\alpha_i\|x_{i+1} - Tx_{i+1}\|^2 \le \|x_1 - q\|^2 + \sum_{i=1}^{n}k_i. \tag{2.25}$$

Hence, $\sum_{n=1}^{\infty}\alpha_n\|x_{n+1} - Tx_{n+1}\|^2 < \infty$. Since $\sum_{n=1}^{\infty}\alpha_n = \infty$, so we have $\underline{\lim}_{n\to\infty}\|x_{n+1} - Tx_{n+1}\| = 0$.

By virtue of Lemma 1.10(ii), we obtain

$$\begin{aligned}
&\|x_{n+1} - Tx_{n+1}\|^2 \\
&\quad = \left\|(1-\alpha_n-\gamma_n)x_n + \alpha_n Ty_n + \gamma_n u_n - Tx_{n+1}\right\|^2 \\
&\quad = \left\|(x_n - Tx_n) + (Tx_n - Tx_{n+1}) + \alpha_n(Ty_n - x_n) + \gamma_n(u_n - x_n)\right\|^2 \\
&\quad \ge \|x_n - Tx_n\|^2 + 2\langle (Tx_n - Tx_{n+1}) + \alpha_n(Ty_n - x_n) + \gamma_n(u_n - x_n), j(x_n - \mathrm{T}x_n)\rangle,
\end{aligned} \tag{2.26}$$

therefore,

$$\begin{aligned}
\|x_n - Tx_n\|^2 \le{}& \|x_{n+1} - Tx_{n+1}\|^2 + 2\langle Tx_{n+1} - Tx_n, j(x_n - Tx_n)\rangle \\
&+ 2\alpha_n\langle x_n - Ty_n, j(x_n - Tx_n)\rangle + 2\gamma_n\langle x_n - u_n, j(x_n - Tx_n)\rangle \\
\le{}& \|x_{n+1} - Tx_{n+1}\|^2 + 2\|Tx_{n+1} - Tx_n\|\|x_n - Tx_n\| \\
&+ 2\alpha_n\|x_n - Ty_n\|\|x_n - Tx_n\| + 2\gamma_n\|x_n - u_n\|\|x_n - Tx_n\|.
\end{aligned} \tag{2.27}$$

Observe the right side of the above inequality, since

$$\begin{aligned}
\|Tx_{n+1} - Tx_n\| &\le L\|x_{n+1} - x_n\| \\
&\le \alpha_n\|Ty_n - x_n\| + \gamma_n\|u_n - x_n\| \longrightarrow 0 \quad (n \longrightarrow \infty),
\end{aligned} \tag{2.28}$$

and $\{\|x_n - Tx_n\|\}$, $\{\|x_n - Ty_n\|\}$, $\{\|x_n - u_n\|\}$ are all bounded. Together with $\underline{\lim}_{n\to\infty}\|x_{n+1} - Tx_{n+1}\| = 0$, then $\underline{\lim}_{n\to\infty}\|x_n - Tx_n\| = 0$, that is, there exists a subsequence x_{n_j} of $\{x_n\}$, such that

$$\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0. \tag{2.29}$$

□

Theorem 2.3. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \longrightarrow K$ be a strictly pseudocontractive mapping with $F(T) \ne \emptyset$. Let $\{x_n\}$ be defined as in Lemma 2.1, then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf_{p\in F(T)}\|x - p\|$.*

Proof. The necessity is obvious. So, we will prove the sufficiency. From Lemma 2.1(1), we have

$$\|x_{n+1} - q\| \le (1 + r_n)\|x_n - q\| + s_n, \quad \forall q \in F(T), n \ge 1. \tag{2.30}$$

Therefore,

$$d(x_{n+1}, F(T)) \le (1 + r_n)d(x_n, F(T)) + s_n. \tag{2.31}$$

Note that $\sum_{n=1}^{\infty} r_n < +\infty$, $\sum_{n=1}^{\infty} s_n < +\infty$. By Lemma 1.9 and $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, we get $\lim_{n\to\infty} d(x_n, F(T)) = 0$.

Next, we prove $\{x_n\}$ is a cauchy sequence. For each $\varepsilon > 0$, there exists a natural number n_1, such that

$$d(x_n, F(T)) \leq \frac{\varepsilon}{12M_1}, \quad \forall n \geq n_1, \tag{2.32}$$

where M_1 is the constant in Lemma 2.1 (2). Hence, there exists $p_1 \in F(T)$ and a natural number $n_2 > n_1$, such that

$$\|x_{n_2} - p_1\| \leq \frac{\varepsilon}{4M_1}, \qquad \sum_{k=n_2}^{\infty} s_k < \frac{\varepsilon}{4M_1}. \tag{2.33}$$

From Lemma 2.1(2) and (2.33), for all $n \geq n_2$, we have

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|p_1 - x_n\| \\ &\leq 2M_1\|x_{n_2} - p_1\| + M_1 \sum_{k=n_2}^{n+m-1} s_k + M_1 \sum_{k=n_2}^{n-1} s_k \\ &\leq 2M_1 \frac{\varepsilon}{4M_1} + 2M_1 \frac{\varepsilon}{4M_1} = \varepsilon. \end{aligned} \tag{2.34}$$

Hence, $\{x_n\}$ is a cauchy sequence. Since K is a closed subset of E, so $\{x_n\}$ converges strongly to a $p \in K$.

Finally, we prove $p \in F(T)$. In fact, since $d(p, F(T)) = 0$. So, for any $\varepsilon_1 > 0$, there exists $p' \in F(T)$, such that $\|p' - p\| < \varepsilon_1$. Then we have

$$\begin{aligned} \|Tp - p\| &\leq \|Tp - p'\| + \|p' - p\| \\ &\leq (1 + L)\varepsilon_1. \end{aligned} \tag{2.35}$$

By the arbitrary of ε_1, we know that $\|Tp - p\| = 0$. Therefore, $p \in F(T)$. □

A mapping $T : K \to K$ is said to satisfy Condition(A) [14], if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in [0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$.

Theorem 2.4. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a strictly pseudocontractive mapping with $F(T) \neq \emptyset$, and satisfy Condition(A). Let $\{x_n\}$ be defined as in Lemma 2.1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 2.2, there exists a subsequence x_{n_j} of $\{x_n\}$, such that

$$\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0. \tag{2.36}$$

By Condition(A), $\lim_{j\to\infty} f(d(x_{n_j}, F(T))) = 0$. Since f is a nondecreasing function and $f(0) = 0$, therefore $\lim_{j\to\infty} d(x_{n_j}, F(T)) = 0$. The rest of the proof is the same to Theorem 2.3. □

Theorem 2.5. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a compact and strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined as in Lemma 2.1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 2.1 (1), it follows that $\lim_{n\to\infty}\|x_n - q\|$ exists, for any $q \in F(T)$. By Lemma 2.2, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0$. Since $\{x_{n_j}\}$ is bounded and T is compact, $\{Tx_{n_j}\}$ has a strongly convergent subsequence. Without loss of generality, we may assume that $\{Tx_{n_j}\}$ converges strongly to $p \in K$. Next, we prove $p \in F(T)$.

$$\|x_{n_j} - p\| \le \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - p\| \longrightarrow 0 \quad (j \longrightarrow \infty), \tag{2.37}$$

that is, $\lim_{j\to\infty}\|x_{n_j} - p\| = 0$. By the Lipschitz continuity of T, it follows that

$$\|p - Tp\| \le \|p - x_{n_j}\| + \|x_{n_j} - Tp\| \longrightarrow 0 \quad (j \longrightarrow \infty). \tag{2.38}$$

This means that $p \in F(T)$. By Lemmas 1.9(ii) and 2.1(i), the sequence $\{x_n\}$ converges strongly to $p \in F(T)$. □

Theorem 2.6. *Let K be a nonempty closed convex subset of a real Banach space E. Let $T : K \to K$ be a demicompact and strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be defined as in Lemma 2.1. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 2.1(1), it follows that $\lim_{n\to\infty}\|x_n - q\|$ exists, for any $q \in F(T)$. By Lemma 2.2, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0$. Since $\{x_{n_j}\}$ is bounded together with T being demicompact, there exists a subsequence of $\{x_{n_j}\}$ which converges strongly to some $p \in K$. Taking into account that $\lim_{j\to\infty}\|x_{n_j} - Tx_{n_j}\| = 0$ and the Lipschitz continuity of T, we have $p \in F(T)$. By Lemma 1.9, $\{x_n\}$ converges strongly to $p \in F(T)$. □

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