

Research Article

General Viscosity Approximation Methods for Common Fixed Points of Nonexpansive Semigroups in Hilbert Spaces

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This paper is devoted to the strong convergence of two kinds of general viscosity iteration processes for approximating common fixed points of a nonexpansive semigroup in Hilbert spaces. The results presented in this paper improve and generalize some corresponding results in (X. Li et al., 2009, S. Li et al., 2009, and Marino and Xu, 2006).

1. Introduction

Let H be a real Hilbert space and A be a linear bounded operator on H . Throughout this paper, we always assume that A is strongly positive; that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x \rangle \geq \bar{\gamma} \|x\|^2, \quad \forall x \in H. \quad (1.1)$$

We recall that a mapping $T : H \rightarrow H$ is said to be contractive if there exists a constant $\alpha \in [0, 1)$ such that $\|Tx - Ty\| \leq \alpha \|x - y\|$ for all $x, y \in H$. $T : H \rightarrow H$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \quad (1.2)$$

(ii) L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in H; \quad (1.3)$$

(iii) pseudocontractive if

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2, \quad \forall x, y \in H; \quad (1.4)$$

(iv) ϕ -strongly pseudocontractive if there exists a strictly increasing function $\phi : [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|, \quad \forall x, y \in H. \quad (1.5)$$

It is obvious that pseudocontractive mapping is more general than ϕ -strongly pseudocontractive mapping. If $\phi(r) = \alpha r$ with $0 < \alpha \leq 1$, then ϕ -strongly pseudocontractive mapping reduces to β -strongly pseudocontractive mapping with $1 - \alpha = \beta \in [0, 1)$, which is more general than contractive mapping.

A nonexpansive semigroup is a family

$$\Gamma := \{T(s) : s \geq 0\} \quad (1.6)$$

of self-mappings on H such that

- (1) $T(0) = I$, where I is the identity mapping on H ;
- (2) $T(s + t)x = T(s)T(t)x$ for all $x \in H$ and $s, t \geq 0$;
- (3) $T(s)$ is nonexpansive for each $s \geq 0$;
- (4) for each $x \in H$, the mapping $T(\cdot)x$ from R^+ into H is continuous.

We denote by $F(\Gamma)$ the common fixed points set of nonexpansive semigroup Γ , that is,

$$F(\Gamma) = \bigcap_{s \geq 0} F(T(s)) = \{x \in H : T(s)x = x \text{ for each } s \geq 0\}. \quad (1.7)$$

In the sequel, we always assume that $F(\Gamma) \neq \emptyset$.

In recent decades, many authors studied the convergence of iterative algorithms for nonexpansive mappings, nonexpansive semigroup, and pseudocontractive semigroup in Banach spaces (see, e.g., [1–15]). Let $f : H \rightarrow H$ be a contractive mapping with coefficient $\alpha \in (0, 1)$, $T : H \rightarrow H$ be a nonexpansive mapping, and A be a strongly positive and linear bounded operator with coefficient $\bar{\gamma} > 0$. Let F denote the fixed points set of T . Recently, Marino and Xu [6] considered the general viscosity approximation process as follows:

$$x_t = (I - tA)Tx_t + t\gamma f(x_t), \quad (1.8)$$

where $t \in (0, 1)$ such that $t < \|A\|^{-1}$ and $0 < \gamma < \bar{\gamma}/\alpha$. Marino and Xu [6] proved that the sequence $\{x_t\}$ generated by (1.8) converges strongly as $t \rightarrow 0$ to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F, \quad (1.9)$$

which is the optimality condition for the minimization problem

$$\min_{x \in F} \frac{1}{2} \langle Ax, x \rangle + h(x), \quad (1.10)$$

where h is a potential function for γf , that is, $h'(x) = \gamma f(x)$ for all $x \in H$.

Let $\Gamma := \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on H and $f : H \rightarrow H$ be a contractive mapping with coefficient $\alpha \in (0, 1)$. Very recently, S. Li et al. [5] considered the following general viscosity iteration process:

$$x_n = (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x_n ds + \alpha_n \gamma f(x_n), \quad \forall n \geq 1, \quad (1.11)$$

where $\{\alpha_n\} \subset (0, 1]$ and $\{t_n\}$ are two sequences satisfying certain conditions. S. Li et al. [5] claimed that the sequence $\{x_n\}$ generated by (1.11) converges strongly as $t_n \rightarrow \infty$ to $x^* \in F(\Gamma)$ which solves the following variational inequality:

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(\Gamma). \quad (1.12)$$

More research work related to general viscosity iteration processes for nonexpansive mapping and nonexpansive semigroup can be found (see, e.g., [5, 6, 12]).

An interesting work is to extend some results involving general viscosity approximation processes for nonexpansive mapping, nonexpansive semigroup, and contractive mapping to nonexpansive semigroup and ϕ -strongly pseudocontractive mapping (pseudocontractive mapping, resp.). Motivated by the works mentioned above, in this paper, on one hand we study the convergence of general implicit viscosity iteration process (1.11) constructed from the nonexpansive semigroup $\Gamma := \{T(t) : t \geq 0\}$ and ϕ -strongly pseudocontractive mapping (pseudocontractive mapping, resp.) in Hilbert spaces. On the other hand, we consider the convergence of the following general viscosity iteration process:

$$x_n = (I - \alpha_n A)T(t_n)x_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 1, \quad (1.13)$$

where $\alpha_n \in (0, 1]$, $\gamma > 0$, $T(t_n) \in \Gamma$ and f is a ϕ -strongly pseudocontractive mapping (pseudocontractive mapping, resp.). The results presented in this paper improve and generalize some corresponding results in [4–6].

2. Preliminaries

A mapping T with domain $D(T)$ and range $R(T)$ in H is said to be demiclosed at a point $p \in H$ if $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to $x \in D(T)$ and $\{Tx_n\}$ converges strongly to p , then $Tx = p$.

For the sake of convenience, we restate the following lemmas that will be used.

Lemma 2.1 (see [6]). *Let A be a strongly positive and linear bounded operator on a real Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2 (see [16]). *Let E be a Banach space and $T : E \rightarrow E$ be a ϕ -strongly pseudocontractive and continuous mapping. Then T has a unique fixed point in E .*

Lemma 2.3 (see [9]). *Let E be a uniformly convex Banach space, K a nonempty closed convex subset of E and $T : K \rightarrow E$ a nonexpansive mapping. Then $I - T$ is demiclosed at zero.*

Lemma 2.4 (see [10]). *Let C be a nonempty bounded closed convex subset of a real Hilbert space H and $\Gamma = \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on H . Then for any $h \geq 0$,*

$$\limsup_{t \rightarrow \infty} \sup_{x \in C} \left\| \frac{1}{t} \int_0^t T(s)x ds - T(h) \left(\frac{1}{t} \int_0^t T(s)x ds \right) \right\| = 0. \quad (2.1)$$

3. Main Results

We first discuss the convergence of general implicit viscosity iteration process (1.11) constructed from a nonexpansive semigroup $\Gamma := \{T(s) : s \geq 0\}$.

Theorem 3.1. *Let $\Gamma := \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on H and $f : H \rightarrow H$ be an L_f -Lipschitzian ϕ -strongly pseudocontractive mapping with $\lim_{t \rightarrow +\infty} \phi(t) = +\infty$. Let A be a strongly positive and linear bounded operator on H with coefficient $\bar{\gamma}$. Then for any $0 < \gamma \leq \bar{\gamma}$, the sequence $\{x_n\}$ generated by (1.11) is well defined. Suppose that*

$$\lim_{t \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} t_n = \infty. \quad (3.1)$$

Then the sequence $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a common fixed point $x^ \in F(\Gamma)$ that is the unique solution in $F(\Gamma)$ to variational inequality (VI):*

$$\langle \gamma f(x^*) - Ax^*, x^* - p \rangle \geq 0, \quad \forall p \in F(\Gamma). \quad (3.2)$$

Proof. Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we may assume without loss of generality that $\alpha_n < \|A\|^{-1}$, for any $n \geq 1$. Let us define a mapping $T_n : H \rightarrow H$ provided by

$$T_n x := \alpha_n \gamma f(x) + (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} T(s)x ds, \quad \forall n \geq 1. \quad (3.3)$$

An application of Lemma 2.1 yields that

$$\begin{aligned}
\langle T_n x - T_n y, x - y \rangle &= \left\langle (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} [T(s)x - T(s)y] ds, x - y \right\rangle + \alpha_n \gamma \langle f(x) - f(y), x - y \rangle \\
&\leq \|I - \alpha_n A\| \|x - y\|^2 + \alpha_n \gamma (\|x - y\|^2 - \phi(\|x - y\|) \|x - y\|) \\
&\leq [1 - \alpha_n (\bar{\gamma} - \gamma)] \|x - y\|^2 - \alpha_n \gamma \phi(\|x - y\|) \|x - y\| \\
&\leq \|x - y\|^2 - \alpha_n \gamma \phi(\|x - y\|) \|x - y\|,
\end{aligned} \tag{3.4}$$

and thus T_n is ϕ -strongly pseudocontractive and strongly continuous. It follows from Lemma 2.2 that T_n has a unique fixed point (say) $x_n \in H$, that is, $\{x_n\}$ generated by (1.11) is well defined.

Taking $p \in F(\Gamma)$, we have

$$\begin{aligned}
\|x_n - p\|^2 &= \alpha_n \langle \gamma f(x_n) - Ap, x_n - p \rangle + \left\langle (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} [T(s)x_n - p] ds, x_n - p \right\rangle \\
&\leq \alpha_n \langle \gamma f(x_n) - \gamma f(p), x_n - p \rangle + \alpha_n \langle \gamma f(p) - Ap, x_n - p \rangle + \|I - \alpha_n A\| \|x_n - p\|^2 \\
&\leq [1 - \alpha_n (\bar{\gamma} - \gamma)] \|x_n - p\|^2 - \alpha_n \gamma \phi(\|x_n - p\|) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| \|x_n - p\|
\end{aligned} \tag{3.5}$$

and so

$$(\bar{\gamma} - \gamma) \|x_n - p\| + \gamma \phi(\|x_n - p\|) \leq \|\gamma f(p) - Ap\|. \tag{3.6}$$

This implies that $\|x_n - p\| \leq \phi^{-1}(\|\gamma f(p) - Ap\|/\gamma)$ and $\{x_n\}$ is bounded.

We denote $z_n = (1/t_n) \int_0^{t_n} T(s)x_n ds$ and have $\|z_n - p\| \leq \|x_n - p\|$, for any $p \in F(\Gamma)$. Since $\{x_n\}$ and $\{z_n\}$ are bounded, it follows from the Lipschitzian conditions of Γ and f that $\{Az_n\}$ and $\{f(x_n)\}$ are two bounded sequences. Therefore,

$$\|x_n - z_n\| = \alpha_n \|\gamma f(x_n) - Az_n\| \longrightarrow 0. \tag{3.7}$$

Let

$$C = \left\{ x \in H : \|x - p\| \leq \phi^{-1} \left(\frac{\|\gamma f(p) - Ap\|}{\gamma} \right) \right\}. \tag{3.8}$$

Since $t_n \rightarrow \infty$, C is a nonempty bounded closed convex subset and $T(s)$ -invariant (i.e., $T(s)C$ is a subset of C), it follows from Lemma 2.4 that

$$\lim_{n \rightarrow \infty} \|z_n - T(s)z_n\| = 0, \quad \forall s \geq 0. \tag{3.9}$$

For each $s \geq 0$, we know that

$$\begin{aligned} \|x_n - T(s)x_n\| &\leq \|x_n - z_n\| + \|z_n - T(s)z_n\| + \|T(s)z_n - T(s)x_n\| \\ &\leq 2\|x_n - z_n\| + \|z_n - T(s)z_n\|. \end{aligned} \quad (3.10)$$

Consequently, we have from formulas (3.7) and (3.9) that

$$\lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| = 0, \quad \forall s \geq 0. \quad (3.11)$$

Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges weakly to some x^* . It is known from Lemma 2.3 that $I - T(s)$ is demiclosed at zero for each $s \geq 0$, where I is the identity mapping on H . Thus, $x^* \in F(\Gamma)$ follows readily.

In addition, by (1.11) and Lemma 2.1, we observe

$$\begin{aligned} \|x_n - x^*\|^2 &= \alpha_n \langle \gamma f(x_n) - Ax^*, x_n - x^* \rangle + \left\langle (I - \alpha_n A) \frac{1}{t_n} \int_0^{t_n} [T(s)x_n - x^*] ds, x_n - x^* \right\rangle \\ &\leq \alpha_n \langle \gamma f(x_n) - \gamma f(x^*), x_n - x^* \rangle + \alpha_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle + \|I - \alpha_n A\| \|x_n - x^*\|^2 \\ &\leq [1 - \alpha_n(\bar{\gamma} - \gamma)] \|x_n - x^*\|^2 - \alpha_n \gamma \phi(\|x_n - x^*\|) \|x_n - x^*\| + \alpha_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle, \end{aligned} \quad (3.12)$$

which implies that

$$\gamma \phi(\|x_n - x^*\|) \|x_n - x^*\| \leq \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle. \quad (3.13)$$

This means that $\{x_{n_k}\}$ converges strongly to x^* . If there exists another subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to y^* , then from (3.11) and (3.13) we know that $\{x_{n_j}\}$ converges strongly to $y^* \in F(\Gamma)$. For any $p \in F(\Gamma)$, it follows from (1.11) that

$$\begin{aligned} \langle Az_n - \gamma f(x_n), x_n - p \rangle &= \frac{1}{\alpha_n} \langle z_n - x_n, x_n - p \rangle \\ &= \frac{1}{\alpha_n} \left(\left\langle \frac{1}{t_n} \int_0^{t_n} [T(s)x_n - p] ds, x_n - p \right\rangle - \|x_n - p\|^2 \right) \\ &\leq 0. \end{aligned} \quad (3.14)$$

The convergence of sequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ yields that

$$\begin{aligned} \langle Ax^* - \gamma f(x^*), x^* - y^* \rangle &\leq 0, \\ \langle Ay^* - \gamma f(y^*), y^* - x^* \rangle &\leq 0. \end{aligned} \quad (3.15)$$

Thus,

$$\begin{aligned}\bar{\gamma}\|x^* - y^*\|^2 &\leq \langle A(x^* - y^*), x^* - y^* \rangle \\ &\leq \gamma \langle f(x^*) - f(y^*), x^* - y^* \rangle \\ &\leq \gamma \|x^* - y^*\|^2 - \gamma \phi(\|x^* - y^*\|) \|x^* - y^*\|.\end{aligned}\tag{3.16}$$

This implies that $x^* = y^*$. Therefore, $\{x_n\}$ converges strongly to $x^* \in F(\Gamma)$. From (3.14) and the deduction above, we know that x^* is also the unique solution to VI (3.2). This completes the proof. \square

Theorem 3.2. *Let $\Gamma := \{T(s) : s \geq 0\}$ be a nonexpansive semigroup on H and $f : H \rightarrow H$ be an L_f -Lipschitzian pseudocontractive mapping. Let A be a strongly positive and linear bounded operator on H with coefficient $\bar{\gamma}$. Then for any $0 < \gamma < \bar{\gamma}$, the sequence $\{x_n\}$ generated by (1.11) is well defined. Suppose that*

$$\lim_{t \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} t_n = \infty.\tag{3.17}$$

Then the sequence $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a common fixed point $x^* \in F(\Gamma)$ that is the unique solution in $F(\Gamma)$ to VI (3.2).

Proof. Similar to the proof of Theorem 3.1, we can verify that the sequence $\{x_n\}$ generated by (1.11) is well defined,

$$\begin{aligned}\|x_n - p\| &\leq \frac{1}{\bar{\gamma} - \gamma} \|\gamma f(p) - Ap\| \quad \text{for a fixed } p \in F(\Gamma), \\ \lim_{n \rightarrow \infty} \|x_n - T(s)x_n\| &= 0, \quad \forall s \geq 0.\end{aligned}\tag{3.18}$$

Thus, $\{x_n\}$ is bounded and so there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges weakly to some x^* . It is obvious that $x^* \in F(\Gamma)$.

In addition, by (1.11) and Lemma 2.1, we can show that

$$\|x_n - x^*\|^2 \leq \frac{1}{\bar{\gamma} - \gamma} \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle.\tag{3.19}$$

This means that $\{x_{n_k}\}$ converges strongly to x^* . The rest of the proof is almost the same as Theorem 3.1. This completes the proof. \square

Remark 3.3. (1) Theorems 3.1 and 3.2 improve and generalize Theorem 3.1 of [5] from contractive mapping to ϕ -strongly pseudocontractive mapping and pseudocontractive mapping, respectively. (2) Theorems 3.1 and 3.2 also improve and generalize Theorem 3.2 of [6] from nonexpansive mapping to nonexpansive semigroup, and from contractive mapping to ϕ -strongly pseudocontractive mapping and pseudocontractive mapping, respectively.

A strong mean convergence theorem for nonexpansive mappings was first established by Baillon [17], and later generalized to that for nonlinear semigroup (see, e.g., [8]). It is clear

that Theorems 3.1 and 3.2 are valid for nonexpansive mappings. Thus, we have the following mean ergodic assertions of general viscosity iteration process for nonexpansive mappings in Hilbert spaces.

Corollary 3.4. *Let H, f, A be as in Theorem 3.1, $T : H \rightarrow H$ be a nonexpansive mapping such that the fixed points set F of T is nonempty. Let $\{\alpha_n\} \subset (0, 1]$ be a real sequence such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then for any $0 < \gamma \leq \bar{\gamma}$, there exists a unique $\{x_n\}$ such that*

$$x_n = (I - \alpha_n A) \frac{1}{n+1} \sum_{j=0}^n T^j x_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0. \quad (3.20)$$

Moreover, the sequence $\{x_n\}$ generated by (3.20) converges strongly as $n \rightarrow \infty$ to a common fixed point $x^* \in F$ that is the unique solution in F to variational inequality (VI):

$$\langle \gamma f(x^*) - Ax^*, x^* - p \rangle \geq 0, \quad \forall p \in F. \quad (3.21)$$

Corollary 3.5. *Let H, f, A be as in Theorem 3.2, $T : H \rightarrow H$ be a nonexpansive mapping such that the fixed points set F of T is nonempty. Let $\{\alpha_n\} \subset (0, 1]$ be a real sequence such that $\lim_{n \rightarrow \infty} \alpha_n = 0$. Then for any $0 < \gamma < \bar{\gamma}$, there exists a unique $\{x_n\}$ satisfying (3.20). Moreover, the sequence $\{x_n\}$ generated by (3.20) converges strongly as $n \rightarrow \infty$ to a common fixed point $x^* \in F$ that is the unique solution in F to VI (3.21).*

We now turn to discuss the convergence of general implicit viscosity iteration process (1.13) constructed from a nonexpansive semigroup $\Gamma := \{T(t) : t \geq 0\}$.

Theorem 3.6. *Let $\Gamma := \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on H and $f : H \rightarrow H$ be an L_f -Lipschitzian ϕ -strongly pseudocontractive mapping with $\lim_{t \rightarrow +\infty} \phi(t) = +\infty$. Let A be a strongly positive and linear bounded operator with coefficient $\bar{\gamma}$. Then for any $0 < \gamma \leq \bar{\gamma}$, the sequence $\{x_n\}$ generated by (1.13) is well defined. Suppose that for any bounded subset $K \subset H$,*

$$\limsup_{s \rightarrow 0} \sup_{x \in K} \|T(s)x - x\| = 0, \quad (3.22)$$

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \quad (3.23)$$

Then the sequence $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a common fixed point $x^* \in F(\Gamma)$ that is the unique solution in $F(\Gamma)$ to VI (3.2).

Proof. Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, we assume without loss of generality that $\alpha_n < \|A\|^{-1}$, for any $n \geq 1$. Let

$$T_n^f x := \alpha_n \gamma f(x) + (I - \alpha_n A)T(t_n)x, \quad \forall n \geq 1. \quad (3.24)$$

By Lemma 2.2, we know

$$\begin{aligned}
\langle T_n^f x - T_n^f y, x - y \rangle &= \langle (I - \alpha_n A)(T(t_n)x - T(t_n)y), x - y \rangle + \alpha_n \gamma \langle f(x) - f(y), x - y \rangle \\
&\leq \|I - \alpha_n A\| \|x - y\|^2 + \alpha_n \gamma (\|x - y\|^2 - \phi(\|x - y\|) \|x - y\|) \\
&\leq \|x - y\|^2 - \alpha_n \gamma \phi(\|x - y\|) \|x - y\|,
\end{aligned} \tag{3.25}$$

and thus T_n^f is ϕ -strongly pseudocontractive and strongly continuous. It follows from Lemma 2.2 that T_n^f has a unique fixed point (say) $x_n \in H$, that is, $\{x_n\}$ generated by (1.13) is well defined.

Taking $p \in F(\Gamma)$, we note

$$\begin{aligned}
\|x_n - p\|^2 &= \alpha_n \langle \gamma f(x_n) - Ap, x_n - p \rangle + \langle (I - \alpha_n A)(T(t_n)x_n - p), x_n - p \rangle \\
&\leq \alpha_n \langle \gamma f(x_n) - \gamma f(p), x_n - p \rangle + \alpha_n \langle \gamma f(p) - Ap, x_n - p \rangle + \|I - \alpha_n A\| \|x_n - p\|^2 \\
&\leq [1 - \alpha_n (\bar{\gamma} - \gamma)] \|x_n - p\|^2 - \alpha_n \gamma \phi(\|x_n - p\|) \|x_n - p\| + \alpha_n \|\gamma f(p) - Ap\| \|x_n - p\|,
\end{aligned} \tag{3.26}$$

and so $\|x_n - p\| \leq \phi^{-1}(\|\gamma f(p) - Ap\|/\gamma)$, the sequence $\{x_n\}$ is bounded. It follows from the Lipschitzian conditions of Γ and f that $\{AT(t_n)x_n\}$ and $\{f(x_n)\}$ are bounded. (1.13) implies that

$$\|x_n - T(t_n)x_n\| = \alpha_n \|\gamma f(x_n) - AT(t_n)x_n\| \longrightarrow 0. \tag{3.27}$$

For any given $t > 0$,

$$\begin{aligned}
\|x_n - T(t)x_n\| &= \sum_{k=0}^{[t/t_n]-1} \|T((k+1)t_n)x_n - T(kt_n)x_n\| + \left\| T(t)x_n - T\left(\left[\frac{t}{t_n}\right]t_n\right)x_n \right\| \\
&\leq \left[\frac{t}{t_n}\right] \|x_n - T(t_n)x_n\| + \left\| T\left(t - \left[\frac{t}{t_n}\right]t_n\right)x_n - x_n \right\| \\
&\leq t \frac{\alpha_n}{t_n} \|AT(t_n)x_n - \gamma f(x_n)\| + \max\{\|T(s)x_n - x_n\| : 0 \leq s \leq t_n\},
\end{aligned} \tag{3.28}$$

where $[t/t_n]$ is the integral part of t/t_n . Since $\lim_{n \rightarrow \infty} (\alpha_n/t_n) = 0$ and $T(\cdot)x : R^+ \rightarrow H$ is continuous for any $x \in H$, it follows from (3.22) that

$$\lim_{n \rightarrow \infty} \|x_n - T(t)x_n\| = 0 \quad \forall t \geq 0. \tag{3.29}$$

Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ which converges weakly to some x^* . By Lemma 2.3, we know that $x^* \in F(\Gamma)$.

In addition, by (1.13) and Lemma 2.1, we observe

$$\begin{aligned}
\|x_n - x^*\|^2 &= \alpha_n \langle \gamma f(x_n) - Ax^*, x_n - x^* \rangle + \langle (I - \alpha_n A)(T(t_n)x_n - x^*), x_n - x^* \rangle \\
&\leq \alpha_n \langle \gamma f(x_n) - \gamma f(x^*), x_n - x^* \rangle \\
&\quad + \alpha_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle + \|I - \alpha_n A\| \|x_n - x^*\|^2 \\
&\leq [1 - \alpha_n(\bar{\gamma} - \gamma)] \|x_n - x^*\|^2 - \alpha_n \gamma \phi(\|x_n - x^*\|) \|x_n - x^*\| \\
&\quad + \alpha_n \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle,
\end{aligned} \tag{3.30}$$

which implies that

$$\gamma \phi(\|x_n - x^*\|) \|x_n - x^*\| \leq \langle \gamma f(x^*) - Ax^*, x_n - x^* \rangle. \tag{3.31}$$

For any $p \in F(\Gamma)$, it follows from (1.13) that

$$\begin{aligned}
\langle AT(t_n)x_n - \gamma f(x_n), x_n - p \rangle &= \frac{1}{\alpha_n} \langle T(t_n)x_n - x_n, x_n - p \rangle \\
&= \frac{1}{\alpha_n} \left(\langle T(t_n)x_n - p, x_n - p \rangle - \|x_n - p\|^2 \right) \\
&\leq 0.
\end{aligned} \tag{3.32}$$

The rest of the proof is the same as Theorem 3.1. This completes the proof. \square

To illustrate Theorem 3.6, we give the following example concerned with a nonexpansive semigroup $\Gamma := \{T(t) : t \geq 0\}$ on H .

Example 3.7. Let H be a Hilbert space. For each given $t \geq 0$, let $T(t) : H \rightarrow H$ be defined by

$$T(t)x = e^{-t}x, \quad \forall x \in H. \tag{3.33}$$

Then it is easy to check that $\Gamma := \{T(t) : t \geq 0\}$ is a nonexpansive semigroup satisfying (3.22) and $F(\Gamma)$ is a singleton $\{\theta\}$, where θ is the zero point in H .

Combining the proofs of Theorems 3.2 and 3.6, we can easily conclude the following result.

Theorem 3.8. *Let $f : H \rightarrow H$ be an L_f -Lipschitzian pseudocontractive mapping and $\Gamma := \{T(t) : t \geq 0\}$ be a nonexpansive semigroup on H such that (3.22) holds. Let A be a strongly positive and linear bounded operator with coefficient $\bar{\gamma}$. Then for any $0 < \gamma < \bar{\gamma}$, the sequence $\{x_n\}$ generated by (1.13) is well defined. Suppose that*

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} \frac{\alpha_n}{t_n} = 0. \tag{3.34}$$

Then the sequence $\{x_n\}$ converges strongly as $n \rightarrow \infty$ to a common fixed point $x^* \in F(\Gamma)$ that is the unique solution in $F(\Gamma)$ to VI (3.2).

Remark 3.9. (1) Theorems 3.6 and 3.8 improve and generalize Theorem 3.2 of [6] from nonexpansive mapping to nonexpansive semigroup, and from contraction mapping to ϕ -strongly pseudocontractive mapping and pseudocontractive mapping, respectively. (2) If A is the identity mapping I , f , and Γ are restricted on a nonempty closed convex subset in H , then Theorem 3.6 of [4] follows by Theorems 3.6 and 3.8. So, Theorems 3.6 and 3.8 generalize Theorem 3.6 of [4].

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