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Research Article

The Iterative Method of Generalized u_0-Concave Operators

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We define the concept of the generalized u_0-concave operators, which generalize the definition of the u_0-concave operators. By using the iterative method and the partial ordering method, we prove the existence and uniqueness of fixed points of this class of the operators. As an example of the application of our results, we show the existence and uniqueness of solutions to a class of the Hammerstein integral equations.

1. Introduction and Preliminary

In [1, 2], Collatz divided the typical problems in computation mathematics into five classes, and the first class is how to solve the operator equation

$$Ax = x \tag{1.1}$$

by the iterative method, that is, construct successively the sequence

$$x_{n+1} = Ax_n \tag{1.2}$$

for some initial x_0 to solve (1.1).

Let P be a cone in real Banach space E and the partial ordering $\leq$ defined by P, that is, $x \leq y$ if and only if $y - x \in P$. The concept and properties of the cone can be found in [3–5]. People studied how to solve (1.1) by using the iterative method and the partial ordering method (see [1–11]).

In [7], Krasnosel'skiĭ gave the concept of u_0-concave operators and studied the existence and uniqueness of the fixed point for the operator by the iterative method. The concept of u_0-concave operators was defined by Krasnosel'skiĭ as follows.

Let operator $A : P \mapsto P$ and $u_0 > \theta$. Suppose that

(i) for any $x > \theta$, there exist $\alpha = \alpha(x) > 0$ and $\beta = \beta(x) > 0$, such that

$$\alpha u_0 \le Ax \le \beta u_0; \tag{1.3}$$

(ii) for any $x \in P$ satisfying $\alpha_1 u_0 \le x \le \beta_1 u_0$ ($\alpha_1 = \alpha_1(x) > 0$, $\beta_1 = \beta_1(x) > 0$) and any $0 < t < 1$, there exists $\eta = \eta(x, t) > 0$, such that

$$A(tx) \ge (1+\eta)tAx. \tag{1.4}$$

Then A is called an u_0-concave operator.

In many papers, the authors studied u_0-concave operators and obtained some results (see [3–5, 8–15]). In this paper, we generalize the concept of u_0-concave operators, give a concept of the generalized u_0-concave operators, and study the existence and uniqueness of fixed points for this class of operators by the iterative method. Our results generalize the results in [3, 4, 7, 15].

2. Main Result

In this paper, we always let P be a cone in real Banach space E and the partial ordering $\le$ defined by P. Given $w_0 \in E$, let $P(w_0) = \{x \in E \mid x \ge w_0\}$.

Definition 2.1. Let operator $A : P(w_0) \mapsto P(w_0)$ and $u_0 > \theta$. Suppose that

(i) for any $x > w_0$, there exist $\alpha = \alpha(x) > 0$ and $\beta = \beta(x) > 0$, such that

$$\alpha u_0 + w_0 \le Ax \le \beta u_0 + w_0; \tag{2.1}$$

(ii) for any $x \in P(w_0)$ satisfying $\alpha_1 u_0 + w_0 \le x \le \beta_1 u_0 + w_0$ ($\alpha_1 = \alpha_1(x) > 0$, $\beta_1 = \beta_1(x) > 0$) and any $0 < t < 1$, there exists $\eta = \eta(x, t) > 0$, such that

$$A[tx + (1-t)w_0] \ge (1+\eta)tAx + [1-(1+\eta)t]w_0. \tag{2.2}$$

Then A is called a generalized u_0-concave operator.

Remark 2.2. In Definition 2.1, let $w_0 = \theta$, we get the definition of the u_0-concave operator.

Theorem 2.3. *Let operator $A : P(w_0) \mapsto P(w_0)$ be generalized u_0-concave and increasing (i.e., $x \le y \Rightarrow Ax \le Ay$), then A has at most one fixed point in $P(w_0) \setminus \{w_0\}$.*

Proof. Let $x^{(1)} > w_0, x^{(2)} > w_0$ be two different fixed points of A, that is, $Ax^{(1)} = x^{(1)}$, $Ax^{(2)} = x^{(2)}$, and $x^{(1)} \neq x^{(2)}$. By Definition 2.1, there exist real numbers $\alpha_1 = \alpha_1(x^{(1)}) > 0$, $\beta_1 = \beta_1(x^{(1)}) > 0$, $\alpha_2 = \alpha_2(x^{(2)}) > 0$, $\beta_2 = \beta_2(x^{(2)}) > 0$, such that

$$\alpha_1 u_0 + w_0 \leq x^{(1)} \leq \beta_1 u_0 + w_0, \qquad \alpha_2 u_0 + w_0 \leq x^{(2)} \leq \beta_2 u_0 + w_0. \tag{2.3}$$

Hence $\alpha_1/\beta_2(x^{(2)} - w_0) + w_0 \leq \alpha_1 u_0 + w_0 \leq x^{(1)} \leq \beta_1 u_0 + w_0 \leq \beta_1/\alpha_2(x^{(2)} - w_0) + w_0$.

Let $\alpha = \alpha_1/\beta_2$, $\beta = \beta_1/\alpha_2$, we get that $\alpha(x^{(2)} - w_0) + w_0 \leq x^{(1)} \leq \beta(x^{(2)} - w_0) + w_0$, that is, $\alpha x^{(2)} + (1-\alpha)w_0 \leq x^{(1)} \leq \beta x^{(2)} + (1-\beta)w_0$. Let

$$t_0 = \sup\left\{t > 0 \mid tx^{(2)} + (1-t)w_0 \leq x^{(1)} \leq t^{-1}x^{(2)} + \left(1 - t^{-1}\right)w_0\right\}, \tag{2.4}$$

hence $0 < t \leq t^{-1}$, that is, $0 < t \leq 1$, then $t_0 \in (0, 1]$.

Next we will show that $t_0 = 1$. Assume that $t_0 < 1$; by (2.2) and (2.4), there exists $\eta_1 = \eta_1(x^{(2)}, t_0) > 0$, such that

$$\begin{aligned} x^{(1)} = Ax^{(1)} &\geq A\left[t_0 x^{(2)} + (1-t_0)w_0\right] \\ &\geq (1+\eta_1)t_0 Ax^{(2)} + \left[1 - (1+\eta_1)t_0\right]w_0 \\ &= (1+\eta_1)t_0 x^{(2)} + \left[1 - (1+\eta_1)t_0\right]w_0. \end{aligned} \tag{2.5}$$

By (2.2), there exists $\eta_2 = \eta_2(x^{(2)}, t_0) > 0$, such that

$$\begin{aligned} x^{(2)} = Ax^{(2)} &= A\left\{t_0\left[t_0^{-1}x^{(2)} + \left(1 - t_0^{-1}\right)w_0\right] + (1-t_0)w_0\right\} \\ &\geq (1+\eta_2)t_0 A\left[t_0^{-1}x^{(2)} + \left(1 - t_0^{-1}\right)w_0\right] + \left[1 - (1+\eta_2)t_0\right]w_0, \end{aligned} \tag{2.6}$$

hence,

$$A\left[t_0^{-1}x^{(2)} + \left(1 - t_0^{-1}\right)w_0\right] \leq (1+\eta_2)^{-1}t_0^{-1}Ax^{(2)} + \left[1 - (1+\eta_2)^{-1}t_0^{-1}\right]w_0. \tag{2.7}$$

Therefore,

$$\begin{aligned} x^{(1)} = Ax^{(1)} &\leq A\left[t_0^{-1}x^{(2)} + \left(1 - t_0^{-1}\right)w_0\right] \\ &\leq (1+\eta_2)^{-1}t_0^{-1}Ax^{(2)} + \left[1 - (1+\eta_2)^{-1}t_0^{-1}\right]w_0 \\ &\leq (1+\eta_2)^{-1}t_0^{-1}x^{(2)} + \left[1 - (1+\eta_2)^{-1}t_0^{-1}\right]w_0. \end{aligned} \tag{2.8}$$

Obviously, by (2.5) and (2.8), we get

$$(1+\eta_1)t_0 x^{(2)} + \left[1 - (1+\eta_1)t_0\right]w_0 \leq x^{(1)} \leq (1+\eta_2)^{-1}t_0^{-1}x^{(2)} + \left[1 - (1+\eta_2)^{-1}t_0^{-1}\right]w_0. \tag{2.9}$$

Let $\eta = \min\{\eta_1, \eta_2\}$, we have

$$(1+\eta)t_0x^{(2)} + [1-(1+\eta)t_0]w_0 \le x^{(1)} \le (1+\eta)^{-1}t_0^{-1}x^{(2)} + \left[1-(1+\eta)^{-1}t_0^{-1}\right]w_0, \tag{2.10}$$

in contradiction to the definition of t_0. Therefore, $t_0 = 1$.

By (2.4), $x^{(1)} = x^{(2)}$. The proof is completed. □

To prove the following Theorem 2.4, we will use the definition of the u_0-norm as follows.

Given $u_0 > \theta$, set

$$\begin{gathered} E_{u_0} = \{x \in E \mid \text{there exists a real number } \lambda > 0, \text{ such that} -\lambda u_0 \le x \le \lambda u_0\}, \\ \|x\|_{u_0} = \inf\{\lambda > 0 \mid -\lambda u_0 \le x \le \lambda u_0\}, \quad \forall x \in E_{u_0}. \end{gathered} \tag{2.11}$$

It is easy to see that E_{u_0} becomes a normed linear space under the norm $\|\cdot\|_{u_0}$. $\|x\|_{u_0}$ is called the u_0- norm of the element $x \in E_{u_0}$ (see [3, 4]).

Theorem 2.4. *Let operator $A : P(w_0) \mapsto P(w_0)$ be increasing and generalized u_0-concave. Suppose that A has a fixed point x^* in $P(w_0) \setminus \{w_0\}$, then, constructing successively the sequence $x_{n+1} = Ax_n$ $(n = 0,1,2,\ldots)$ for any initial $x_0 \in P(w_0) \setminus \{w_0\}$, we have $\|x_n - x^*\|_{u_0} \to 0$ $(n \to \infty)$.*

Proof. Suppose that $\{x_n\}$ is generated from $x_{n+1} = Ax_n$ $(n = 0,1,2,\ldots)$. Take $0 < \varepsilon_0 < 1$, such that $\varepsilon_0x^* + (1-\varepsilon_0)w_0 \le x_1 \le \varepsilon_0^{-1}x^* + (1-\varepsilon_0^{-1})w_0$. Let $y_0 = \varepsilon_0x^* + (1-\varepsilon_0)w_0$, $z_0 = \varepsilon_0^{-1}x^* + (1-\varepsilon_0^{-1})w_0$, and constructing successively the sequences $y_{n+1} = Ay_n$, $z_{n+1} = Az_n$ $(n = 0,1,2,\ldots)$. Since A is a generalized u_0-concave operator, we know that there exists $\eta_1 = \eta_1(x^*, \varepsilon_0) > 0$, such that

$$\begin{aligned} x^* = Ax^* &= A\left\{\varepsilon_0\left[\varepsilon_0^{-1}x^* + \left(1-\varepsilon_0^{-1}\right)w_0\right] + (1-\varepsilon_0)w_0\right\} \\ &\ge (1+\eta_1)\varepsilon_0A\left[\varepsilon_0^{-1}x^* + \left(1-\varepsilon_0^{-1}\right)w_0\right] + [1-(1+\eta_1)\varepsilon_0]w_0, \end{aligned} \tag{2.12}$$

hence, $A[\varepsilon_0^{-1}x^* + (1-\varepsilon_0^{-1})w_0] \le (1+\eta_1)^{-1}\varepsilon_0^{-1}Ax^* + [1-(1+\eta_1)^{-1}\varepsilon_0^{-1}]w_0$, then

$$\begin{aligned} z_1 = A(z_0) &= A\left[\varepsilon_0^{-1}x^* + \left(1-\varepsilon_0^{-1}\right)w_0\right] \le (1+\eta_1)^{-1}\varepsilon_0^{-1}Ax^* + \left[1-(1+\eta_1)^{-1}\varepsilon_0^{-1}\right]w_0 \\ &= (1+\eta_1)^{-1}\varepsilon_0^{-1}(Ax^*-w_0) + w_0 < \varepsilon_0^{-1}(Ax^*-w_0) + w_0 = \varepsilon_0^{-1}Ax^* + \left(1-\varepsilon_0^{-1}\right)w_0 \\ &= \varepsilon_0^{-1}x^* + \left(1-\varepsilon_0^{-1}\right)w_0 = z_0. \end{aligned} \tag{2.13}$$

By (2.2), we can easily get $y_1 > y_0$. So it is easy to show that

$$y_0 \le y_1 \le \cdots \le y_n \le \cdots \le x^* \le \cdots \le z_n \le \cdots \le z_1 \le z_0. \tag{2.14}$$

Let

$$t_n = \sup\left\{t > 0 \mid tx^* + (1-t)w_0 \le y_n,\ z_n \le t^{-1}x^* + \left(1-t^{-1}\right)w_0\right\} \quad (n = 0,1,2,\ldots), \tag{2.15}$$

then,

$$0 \le t_0 \le t_1 \le \cdots \le t_n \le \cdots \le 1, \tag{2.16}$$

which implies that the limit of $\{t_n\}$ exists. Let $\lim_{n\to\infty} t_n = t^*$, then $0 < t_n \le t^* \le 1$.

Next we will show that $t^* = 1$. Suppose that $0 < t^* < 1$. Since A is a generalized u_0-concave operator, then there exists $\eta_2 = \eta_2(x^*, t^*) > 0$, such that

$$A[t^*x^* + (1-t^*)w_0] \ge (1+\eta_2)t^*Ax^* + [1-(1+\eta_2)t^*]w_0 = (1+\eta_2)t^*x^* + [1-(1+\eta_2)t^*]w_0. \tag{2.17}$$

Moreover,

$$\begin{aligned} x^* = Ax^* &= A\left\{t^*\left[(t^*)^{-1}x^* + \left(1-(t^*)^{-1}\right)w_0\right] + (1-t^*)w_0\right\} \\ &\ge (1+\eta_2)t^*A\left[(t^*)^{-1}x^* + \left(1-(t^*)^{-1}\right)w_0\right] + [1-(1+\eta_2)t^*]w_0. \end{aligned} \tag{2.18}$$

Therefore,

$$A\left[(t^*)^{-1}x^* + \left(1-(t^*)^{-1}\right)w_0\right] \le (1+\eta_2)^{-1}(t^*)^{-1}x^* + \left[1-(1+\eta_2)^{-1}(t^*)^{-1}\right]w_0. \tag{2.19}$$

By (2.17) and (2.19), for any $0 < t \le t^*$, there exists $\eta_3 = \eta_3(x^*, t) > 0$, such that

$$\begin{aligned} A[tx^* + (1-t)w_0] &\ge (1+\eta_3)tx^* + [1-(1+\eta_3)t]w_0, \\ A\left[t^{-1}x^* + \left(1-t^{-1}\right)w_0\right] &\le (1+\eta_3)^{-1}t^{-1}x^* + \left[1-(1+\eta_3)^{-1}t^{-1}\right]w_0. \end{aligned} \tag{2.20}$$

Particularly, for any $0 < t_n \le t^*$ $(n = 0,1,2,\ldots)$, we have

$$\begin{aligned} A[t_nx^* + (1-t_n)w_0] &\ge (1+\eta)t_nx^* + [1-(1+\eta)t_n]w_0, \\ A\left[t_n^{-1}x^* + \left(1-t_n^{-1}\right)w_0\right] &\le (1+\eta)^{-1}t_n^{-1}x^* + \left[1-(1+\eta)^{-1}t_n^{-1}\right]w_0, \end{aligned} \tag{2.21}$$

where $\eta = \eta(t_n, x^*) > 0$.

Hence,

$$\begin{aligned} y_{n+1} &= Ay_n \ge A[t_nx^* + (1-t_n)w_0] \ge (1+\eta)t_nx^* + [1-(1+\eta)t_n]w_0, \\ z_{n+1} &= Az_n \le A\left[t_n^{-1}x^* + \left(1-t_n^{-1}\right)w_0\right] \le (1+\eta)^{-1}t_n^{-1}x^* + \left[1-(1+\eta)^{-1}t_n^{-1}\right]w_0. \end{aligned} \tag{2.22}$$

By (2.15), and (2.22), we get $t_{n+1} \geq (1+\eta)t_n$ $(n = 0,1,2,\ldots)$ therefore, $t_{n+1} \geq (1+\eta)^{n+1}t_0$ $(n = 0,1,2,\ldots)$, in contradiction to $0 < t_n \leq 1$ $(n = 1,2,\ldots)$. Hence,

$$t^* = 1. \tag{2.23}$$

Since A is a generalized u_0-concave operator, thus there exist real numbers $\alpha = \alpha(x^*) > 0$, $\beta = \beta(x^*) > 0$, such that $\alpha u_0 + w_0 \leq x^* \leq \beta u_0 + w_0$, and $t_n x^* + (1-t_n)w_0 \leq y_n \leq x_{n+1} \leq z_n \leq t_n^{-1}x^* + (1-t_n^{-1})w_0$ $(n = 0,1,2,\ldots)$, we have

$$(t_n - 1)x^* + (1-t_n)w_0 \leq x_{n+1} - x^* \leq \left(t_n^{-1} - 1\right)x^* + \left(1 - t_n^{-1}\right)w_0. \tag{2.24}$$

Moreover

$$\begin{gathered}(t_n - 1)x^* + (1-t_n)w_0 \geq (t_n - 1)\left(\beta u_0 + w_0\right) + (1-t_n)w_0 = (t_n - 1)\beta u_0,\\ \left(t_n^{-1} - 1\right)x^* + \left(1 - t_n^{-1}\right)w_0 \leq \left(t_n^{-1} - 1\right)\left(\beta u_0 + w_0\right) + \left(1 - t_n^{-1}\right)w_0 = \left(t_n^{-1} - 1\right)\beta u_0.\end{gathered} \tag{2.25}$$

Hence,

$$\left(1 - t_n^{-1}\right)\beta u_0 \leq (t_n - 1)\beta u_0 \leq x_{n+1} - x^* \leq \left(t_n^{-1} - 1\right)\beta u_0 \quad (n = 0,1,2,\ldots). \tag{2.26}$$

Consequently, by (2.23), we get $\|x_n - x^*\|_{u_0} \to 0$ $(n \to \infty)$.

To prove the following Theorem 2.5, we will use the definition of the normal cone as follows.

Let P be a cone in E. Suppose that there exist constants $N > 0$, such that

$$\theta \leq x \leq y \Rightarrow \|x\| \leq N\|y\|, \tag{2.27}$$

then P is said to be normal, and the smallest N is called the normal constant of P (see [3–5]). □

Theorem 2.5. *v Let P be a normal cone of E. If operator $A : P(w_0) \longmapsto P(w_0)$ is increasing and generalized u_0-concave, and $\eta = \eta(t,x)$ is irrelevant to x in (2.2), then A has the only one fixed point $x^* \in P(w_0) \setminus \{w_0\}$. Moreover, constructing successively the sequence $x_{n+1} = Ax_n$ $(n = 0,1,2,\ldots)$ for any initial $x_0 > w_0$, we have $\|x_n - x^*\| \to 0$ $(n \to \infty)$.*

Proof. Since A is a generalized u_0-concave operator, hence there exist real numbers $\beta > \alpha > 0$, such that $\alpha u_0 + w_0 \leq A(u_0 + w_0) \leq \beta u_0 + w_0$. Take $t_0 \in (0,1)$ small enough, then $t_0 u_0 + w_0 \leq A(u_0 + w_0) \leq (1/t_0)u_0 + w_0$.

Therefore, $t_{n+1} \geq t_n$, that is, $\{t_n\}$ is an increasing sequence and $0 < t_n \leq 1$, hence, the limit of $\{t_n\}$ exists. Set $\lim_{n\to\infty} t_n = t^*$, then $0 < t^* \leq 1$.

Let $\gamma(t) = (1+\eta(t))t$, where $\eta(t)$ which is irrelevant to x is $\eta(t,x)$ in (2.2), and A is increasing, so $t < \gamma(t) \le 1, A(tx+(1-t)w_0) \ge \gamma(t)Ax + (1-\gamma(t))w_0,$ for all $t \in (0,1)$. By $\gamma(t_0)/t_0 > 1$, we can choose a natural number $k > 0$ big enough, such that

$$\left(\frac{\gamma(t_0)}{t_0}\right)^k > \frac{1}{t_0}. \tag{2.28}$$

Let

$$y_0 = t_0^k u_0 + w_0, \quad z_0 = \frac{1}{t_0^k}u_0 + w_0; \quad y_n = Ay_{n-1}, \quad z_n = Az_{n-1} \quad (n = 1,2,\ldots). \tag{2.29}$$

Obviously, $y_0, z_0 \in P(w_0), y_0 < z_0$. Since A is increasing, we have

$$\begin{aligned}
y_1 = Ay_0 &= A\left(t_0^k u_0 + w_0\right) = A\left[t_0\left(t_0^{k-1}u_0 + w_0\right) + (1-t_0)w_0\right] \\
&\ge \gamma(t_0)A\left(t_0^{k-1}u_0 + w_0\right) + (1-\gamma(t_0))w_0 \\
&= \gamma(t_0)A\left[t_0\left(t_0^{k-2}u_0 + w_0\right) + (1-t_0)w_0\right] + (1-\gamma(t_0))w_0 \\
&\ge \gamma(t_0)\left[\gamma(t_0)A\left(t_0^{k-2}u_0 + w_0\right) + (1-\gamma(t_0))w_0\right] + (1-\gamma(t_0))w_0 \\
&= \gamma^2(t_0)A\left(t_0^{k-2}u_0 + w_0\right) + \left(1-\gamma^2(t_0)\right)w_0 \ge \cdots \ge \gamma^k(t_0)A(u_0+w_0) + \left(1-\gamma^k(t_0)\right)w_0 \\
&> t_0^{k-1}(t_0u_0 + w_0) + \left(1-t_0^{k-1}\right)w_0 = t_0^k u_0 + w_0 = y_0.
\end{aligned} \tag{2.30}$$

Since $Ax = A\{t_0[t_0^{-1}x + (1-t_0^{-1})w_0] + (1-t_0)w_0\} \ge \gamma(t_0)A[t_0^{-1}x + (1-t_0^{-1})w_0] + (1-\gamma(t_0))\mathrm{w}_0$, we get $A[t_0^{-1}x + (1-t_0^{-1})w_0] \le 1/\gamma(t_0)Ax + (1-1/\gamma(t_0))w_0$. Hence

$$\begin{aligned}
z_1 &= A\left(\frac{1}{t_0^k}u_0 + w_0\right) = A\left[\frac{1}{t_0}\left(\frac{1}{t_0^{k-1}}u_0 + w_0\right) + \left(1-\frac{1}{t_0}\right)w_0\right] \\
&\le \frac{1}{\gamma(t_0)}A\left(\frac{1}{t_0^{k-1}}u_0 + w_0\right) + \left(1-\frac{1}{\gamma(t_0)}\right)w_0 \\
&\le \cdots \le \frac{1}{\gamma^k(t_0)}A(u_0+w_0) + \left(1-\frac{1}{\gamma^k(t_0)}\right)w_0 \le \frac{1}{t_0\gamma^k(t_0)}u_0 + w_0 < \frac{1}{t_0^k}u_0 + w_0 = z_0,
\end{aligned} \tag{2.31}$$

then $y_0 \le y_1 \le z_1 \le z_0$. It is easy to see

$$y_0 \le y_1 \le \cdots \le y_n \le \cdots \le z_n \le \cdots \le z_1 \le z_0. \tag{2.32}$$

Let

$$t_n = \sup\{t > 0 \mid y_n \geq t z_n + (1-t) w_0\}. \tag{2.33}$$

Obviously, $y_n \geq t_n z_n + (1-t_n)w_0$. So $y_{n+1} \geq y_n \geq t_n z_n + (1-t_n)w_0 \geq t_n z_{n+1} + (1-t_n)w_0$.

Therefore, $t_{n+1} \geq t_n$, that is, $\{t_n\}$ is an increasing sequence and $0 < t_n \leq 1$, hence, the limit of $\{t_n\}$ exists. Set $\lim_{n\to\infty} t_n = t^*$, then $0 < t^* \leq 1$.

Next we will show that $t^* = 1$. Suppose that $0 < t^* < 1$, we have the following.

(i) If for any natural number n, $t_n < t^* < 1$, then

$$\begin{aligned}
y_{n+1} &= Ay_n \geq A[t_n z_n + (1-t_n)w_0] = A\left\{\frac{t_n}{t^*}[t^* z_n + (1-t^*)w_0] + \left(1 - \frac{t_n}{t^*}\right)w_0\right\} \\
&\geq \gamma\left(\frac{t_n}{t^*}\right)A[t^* z_n + (1-t^*)w_0] + \left(1 - \gamma\left(\frac{t_n}{t^*}\right)\right)w_0 \\
&\geq \gamma\left(\frac{t_n}{t^*}\right)\left[\gamma(t^*)Az_n + (1-\gamma(t^*))w_0\right] + \left(1 - \gamma\left(\frac{t_n}{t^*}\right)\right)w_0 \\
&= \gamma\left(\frac{t_n}{t^*}\right)\gamma(t^*)Az_n + \left(1 - \gamma\left(\frac{t_n}{t^*}\right)\gamma(t^*)\right)w_0 = \gamma\left(\frac{t_n}{t^*}\right)\gamma(t^*)z_{n+1} + \left(1 - \gamma\left(\frac{t_n}{t^*}\right)\gamma(t^*)\right)w_0,
\end{aligned} \tag{2.34}$$

hence,

$$t_{n+1} \geq \gamma\left(\frac{t_n}{t^*}\right)\gamma(t^*) = \left(1 + \eta\left(\frac{t_n}{t^*}\right)\right)\frac{t_n}{t^*}(1+\eta(t^*))t^* \geq t_n(1+\eta(t^*)). \tag{2.35}$$

Taking limits, we have $t^* \geq t^*(1+\eta(t^*)) > t^*$, a contradiction.

(ii) Suppose that there exists a natural number $N > 0$, such that $t_n = t^*(n > N)$.

When $n > N$, so we have

$$\begin{aligned}
y_{n+1} = Ay_n &\geq A[t_n z_n + (1-t_n)w_0] = A[t^* z_n + (1-t^*)w_0] \\
&\geq \gamma(t^*)Az_n + (1-\gamma(t^*))w_0 = \gamma(t^*)z_{n+1} + (1-\gamma(t^*))w_0,
\end{aligned} \tag{2.36}$$

then $t^* = t_{n+1} \geq \gamma(t^*) = (1+\eta(t^*))t^* > t^*$, a contradiction.

Therefore, $t^* = 1$.

For any natural numbers n, p, we have

$$\theta \leq y_{n+p} - y_n \leq z_{n+p} - y_n \leq z_n - y_n \leq z_n - [t_n z_n + (1-t_n)w_0] = (1-t_n)(z_n - w_0). \tag{2.37}$$

Similarly, $\theta \leq z_n - z_{n+p} \leq z_n - y_n \leq (1-t_n)(z_n - w_0)$. By the normality of P and $\lim_{n\to\infty} t_n = 1$, we get

$$\begin{aligned}
\left\|(y_{n+p} - w_0) - (y_n - w_0)\right\| &= \left\|y_{n+p} - y_n\right\| \leq N(1-t_n)\|z_n - w_0\| \to 0 \quad (n \to \infty), \\
\left\|(z_{n+p} - w_0) - (z_n - w_0)\right\| &= \left\|z_n - z_{n+p}\right\| \leq N(1-t_n)\|z_n - w_0\| \to 0 \quad (n \to \infty),
\end{aligned} \tag{2.38}$$

where N is the normal constant of P. Hence the limits of $\{y_n\}$ and $\{z_n\}$ exist. Let $\lim_{n\to\infty} y_n = y^*$, and let $\lim_{n\to\infty} z_n = z^*$, then $y_n \le y^* \le z^* \le z_n$ $(n = 0,1,2,\ldots)$, hence,

$$\theta \le z^* - y^* \le z_n - y_n \le (1-t_n)(z_n - w_0) \to \theta \quad (n \to \infty). \tag{2.39}$$

That is, $y^* = z^*$. Let $x^* = y^* = z^*$, then $y_{n+1} = Ay_n \le Ax^* \le Az_n = z_{n+1}$.

Taking limits, we get $x^* \le Ax^* \le x^*$. Hence $Ax^* = x^*$, that is, $x^* \in P(w_0)\backslash\{w_0\}$ is a fixed point of A. By Theorem 2.4, the conclusions of Theorem 2.5 hold. The proof is completed. □

3. Examples

Example 3.1. Let $I = [0,1]$, let $C(I) = \{x(t) : I \mapsto R \mid x(t)$ is continuous$\}$, let $\|x\| = \sup\{|x(t)| | t \in I\}$, let $P = \{x \in C(I) \mid x(t) \ge 0, \forall t \in I\}$, then $C(I)$ is a real Banach space and P is a normal and solid cone in $C(I)$ (P is called solid if it contains interior points, i.e., $\overset{\circ}{P} \ne \emptyset$). Take $a < 0$, let $w_0 = w_0(t) \equiv a$, $P(w_0) = \{x \in C(I) \mid x(t) \ge w_0, \forall t \in I\}$, and $\overset{\circ}{P}(w_0) = \{x + w_0 \in P(w_0) \mid x \in \overset{\circ}{P}\}$.

Considering the Hammerstein integral equation

$$x(t) = \int_0^1 k(t,s) f(s, x(s))ds, \quad t \in [0,1], \tag{3.1}$$

where $k(t,s) : I \times I \mapsto [0,+\infty)$ is continuous, $f(s,u) : I \times [a,+\infty) \mapsto R$ is increasing for u. Suppose that

(1) there exist real numbers $0 \le m \le M \le 1$, such that $m \le k(t,s) \le M$, for all $(t,s) \in I \times I$, and $f(s,u) \ge a/M$, for all$(s,u) \in I \times [a,+\infty)$,

(2) for any $\lambda \in (0,1)$ and $u \in (a,+\infty)$, there exists $\eta = \eta(\lambda) > 0$, such that

$$mf[s, \lambda u + (1-\lambda)a] \ge (1+\eta)\lambda m f(s,u) + [1-(1+\eta)\lambda]a. \tag{3.2}$$

Then (3.1) has the only one solution $x^* \in P(w_0) \setminus \{w_0\}$. Moreover, constructing successively the sequence:

$$x_n(t) = \int_0^1 k(t,s) f(s, x_{n-1}(s))ds, \quad \forall t \in I, \ n = 1,2,\ldots \tag{3.3}$$

for any initial $x_0 \in P(w_0) \setminus \{w_0\}$, we have $\|x_n - x^*\| \to 0$ $(n \to \infty)$.

Proof. Considering the operator

$$Ax(t) = \int_0^1 k(t,s) f(s, x(s))ds, \quad t \in I. \tag{3.4}$$

Obviously, $A : P(w_0) \setminus \{w_0\} \mapsto \overset{\circ}{P}(w_0)$ is increasing. Therefore, (i) of Definition 2.1 is satisfied. For any $x \in \overset{\circ}{P}(w_0)$, by (3.2), we have

$$\begin{aligned} A[\lambda x(t) + (1-\lambda)w_0] &= \int_0^1 k(t,s) f(s, \lambda x(s) + (1-\lambda)w_0) ds \\ &= \int_0^1 \frac{1}{m} k(t,s) m f(s, \lambda x(s) + (1-\lambda)w_0) ds \\ &\geq (1+\eta)\lambda \int_0^1 \frac{1}{m} k(t,s) m f(s, x(s)) ds + [1-(1+\eta)\lambda] w_0 \int_0^1 \frac{1}{m} k(t,s) ds \\ &\geq (1+\eta)\lambda A x(t) + [1-(1+\eta)\lambda] w_0. \end{aligned} \tag{3.5}$$

Therefore, (ii) of Definition 2.1 is satisfied. Hence the operator $A : P(w_0) \mapsto P(w_0)$ is generalized u_0-concave. Consequently, operator A satisfies all conditions of Theorem 2.5, thus the conclusion of Example 3.1 holds. □

Example 3.2. Let R be a real numbers set, and let $P = \{x \mid x \geq 0, x \in R\}$, then R is a real Banach space and P is a normal and solid cone in R. Let $Ax = (x+2)^{1/2} - 2$. Considering the equation: $x = Ax$. Obviously, A is a generalized u_0-concave operator and satisfies all the conditions of Theorem 2.5. Hence A has the only one fixed point $x^* \in P(-2) \setminus \{-2\} = (-2, +\infty)$. Moreover, we know $x^* = -1$ by computing.

In Example 3.2, we know that operator $A : [-2, +\infty) \mapsto [-2, +\infty)$ doesn't satisfy the definition of u_0-concave operators. Therefore, we can't obtain the fixed point of A by the fixed point theorem of u_0-concave operators. The u_0-concave operators' fixed points are all positive, but here A's fixed point is negative.

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