FIXED POINTS OF MULTIMAPS WHICH ARE NOT NECESSARILY NONEXPANSIVE

NASEER SHAHZAD AND AMJAD LONE

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Let C be a nonempty closed bounded convex subset of a Banach space X whose characteristic of noncompact convexity is less than 1 and T a continuous 1-χ-contractive SL map (which is not necessarily nonexpansive) from C to $KC(X)$ satisfying an inwardness condition, where $KC(X)$ is the family of all nonempty compact convex subsets of X. It is proved that T has a fixed point. Some fixed points results for noncontinuous maps are also derived as applications. Our result contains, as a special case, a recent result of Benavides and Ramírez (2004).

1. Introduction

During the last four decades, various fixed point results for nonexpansive single-valued maps have been extended to multimaps, see, for instance, the works of Benavides and Ramírez [2], Kirk and Massa [6], Lami Dozo [7], Lim [8], Markin [10], Xu [12], and the references therein. Recently, Benavides and Ramírez [3] obtained a fixed point theorem for nonexpansive multimaps in a Banach space whose characteristic of noncompact convexity is less than 1. More precisely, they proved the following theorem.

Theorem 1.1 (see [3]). *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ and $T : C \to KC(X)$ a nonexpansive 1-χ-contractive map. If T satisfies*

$$T(x) \subset I_C(x) \quad \forall x \in C, \tag{1.1}$$

then T has a fixed point.

Benavides and Ramírez further remarked that the assumption of nonexpansiveness in the above theorem can not be avoided. In this paper, we prove a fixed point result for multimaps which are not necessarily nonexpansive. To establish this, we define a new class of multimaps which includes nonexpansive maps. To show the generality of our result, we present an example. As consequences of our main result, we also derive some fixed point theorems for $*$-nonexpansive maps.

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2. Preliminaries

Let C be a nonempty closed subset of a Banach space X. Let $CB(X)$ denote the family of all nonempty closed bounded subsets of X and $KC(X)$ the family of all nonempty compact convex subsets of X. The Kuratowski and Hausdorff measures of noncompactness of a nonempty bounded subset of X are, respectively, defined by

$$\begin{aligned}\alpha(B) &= \inf\{d > 0 : B \text{ can be covered by finitely many sets of diameter} \le d\},\\ \chi(B) &= \inf\{d > 0 : B \text{ can be covered by finitely many balls of radius} \le d\}.\end{aligned} \tag{2.1}$$

Let H be the Hausdorff metric on $CB(X)$ and $T : C \to CB(X)$ a map. Then T is called

(1) contraction if there exists a constant $k \in [0,1)$ such that

$$H(T(x), T(y)) \le k\|x - y\|, \quad \forall x, y \in C; \tag{2.2}$$

(2) nonexpansive if

$$H(T(x), T(y)) \le \|x - y\|, \quad \forall x, y \in C; \tag{2.3}$$

(3) ϕ-condensing (resp., 1-ϕ-contractive), where $\phi = \alpha(\cdot)$ or $\chi(\cdot)$ if $T(C)$ is bounded and, for each bounded subset B of C with $\phi(B) > 0$, the following holds:

$$\phi(T(B)) < \phi(B) \quad (\text{resp., } \phi(T(B)) \le \phi(B)); \tag{2.4}$$

here $T(B) = \bigcup_{x \in B} T(x)$;

(4) upper semicontinuous if $\{x \in C : T(x) \subset V\}$ is open in C whenever $V \subset X$ is open;

(5) lower semicontinuous if the set $\{x \in C : T(x) \cap V \neq \phi\}$ is open in C whenever $V \subset X$ is open;

(6) continuous (with respect to H) if $H(T(x_n), T(x)) \to 0$ whenever $x_n \to x$;

(7) $*$-nonexpansive (see [5]) if for all $x, y \in C$ and $u_x \in T(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in T(x)\}$, there exists $u_y \in T(y)$ with $d(y, u_y) = \inf\{d(y, w) : w \in T(y)\}$ such that

$$d(u_x, u_y) \le d(x, y). \tag{2.5}$$

A sequence $\{x_n\}$ is called asymptotically T-regular if $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Let $\phi = \alpha$ or χ. The modulus of noncompactness convexity associated to ϕ is defined by

$$\Delta_{X,\phi}(\epsilon) = \inf\{1 - d(0, A) : A \subset B_X \text{ is convex, with } \phi(A) \ge \epsilon\}, \tag{2.6}$$

where B_X is the unit ball of X. The characteristic of noncompact convexity of X associated with the measure of noncompactness ϕ is defined in the following way:

$$\epsilon_\phi(X) = \sup\{\epsilon \ge 0 : \Delta_{X,\phi}(\epsilon) = 0\}. \tag{2.7}$$

Note that

$$\Delta_{X,\alpha}(\epsilon) \le \Delta_{X,\chi}(\epsilon) \tag{2.8}$$

and so

$$\epsilon_\alpha(X) \ge \epsilon_\chi(X). \tag{2.9}$$

Let C be a nonempty subset of X, $\mathcal{D}$ a directed set, and $\{x_\alpha : \alpha \in \mathcal{D}\}$ a bounded net in X. We use $r(C,\{x_\alpha\})$ and $A(C,\{x_\alpha\})$ to denote the asymptotic radius and the asymptotic center of $\{x_\alpha : \alpha \in \mathcal{D}\}$ in C, that is,

$$\begin{aligned} r(x,\{x_\alpha\}) &= \limsup_\alpha \|x_\alpha - x\|, \\ r(C,\{x_\alpha\}) &= \inf\{r(x,\{x_\alpha\}) : x \in C\}, \\ A(C,\{x_\alpha\}) &= \{x \in C : r(x,\{x_\alpha\}) = r(C,\{x_\alpha\})\}. \end{aligned} \tag{2.10}$$

It is known that $A(C,\{x_\alpha\})$ is a nonempty weakly compact convex set if C is a nonempty closed convex subset of a reflexive Banach space. For details, we refer the reader to [1, 3].

Let A be a set and $B \subset A$. A net $\{x_\alpha : \alpha \in \mathcal{D}\}$ in A is eventually in B if there exists $\alpha_0 \in \mathcal{D}$ such that $x_\alpha \in B$ for all $\alpha \ge \alpha_0$. A net $\{x_\alpha : \alpha \in \mathcal{D}\}$ in a set A is called an ultranet if either $\{x_\alpha : \alpha \in \mathcal{D}\}$ is eventually in B or $\{x_\alpha : \alpha \in \mathcal{D}\}$ is eventually in $A - B$, for each subset B of A.

A Banach space X is said to satisfy the nonstrict Opial condition if, whenever a sequence $\{x_n\}$ in X converges weakly to x, then for any $y \in X$,

$$\limsup_n \|x_n - x\| \le \limsup_n \|x_n - y\|. \tag{2.11}$$

Let C be a nonempty closed convex subset of a Banach space X and $x \in X$. Then the inward set $I_C(x)$ is defined by

$$I_C(x) = \{x + \lambda(y - x) : y \in C,\ \lambda \ge 0\}. \tag{2.12}$$

Note that $C \subset I_C(x)$ and $I_C(x)$ is convex.

We need the following results in the sequel.

Lemma 2.1 (see [9]). *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to K(X)$ a contraction. If T satisfies*

$$T(x) \subset \overline{I_C(x)} \quad \forall x \in C, \tag{2.13}$$

then T has a fixed point.

Lemma 2.2 (see [4]). *Let C be a nonempty closed bounded convex subset of a Banach space X and $T : C \to KC(X)$ an upper semicontinuous ϕ-condensing, where $\phi(\cdot) = \alpha(\cdot)$ or $\chi(\cdot)$. If T satisfies*

$$T(x) \cap \overline{I_C(x)} \ne \varnothing \quad \forall x \in C, \tag{2.14}$$

then T has a fixed point.

LEMMA 2.3 (see [3]). *Let C be a nonempty closed convex subset of a reflexive Banach space X and $\{x_\beta : \beta \in D\}$ a bounded ultranet in C. Then*

$$r_C(A(C,\{x_\beta\})) \leq (1-\Delta_{X,\alpha}(1^-))r(C,\{x_\beta\}); \tag{2.15}$$

here $r_C(A(C,\{x_\beta\})) = \inf\{\sup\{\|x-y\| : y \in A(C,\{x_\beta\})\} : x \in C\}$.

3. Main results

Let C be a nonempty weakly compact convex subset of a Banach space X and $T : C \to KC(X)$ a continuous map.

Definition 3.1. The map T is called subsequentially limit-contractive (SL) if for every asymptotically T-regular sequence $\{x_n\}$ in C,

$$\limsup_n H(T(x_n),T(x)) \leq \limsup_n \|x_n - x\| \tag{3.1}$$

for all $x \in A(C,\{x_n\})$.

Note that if C is a nonempty closed convex subset of a uniformly convex Banach space and $\{x_n\}$ is bounded, then $A(C,\{x_n\})$ has a unique asymptotic center, say x_0, and so in the above definition, we have

$$\limsup_n H(T(x_n),T(x_0)) \leq \limsup_n \|x_n - x_0\|. \tag{3.2}$$

It is clear that every nonexpansive map is an SL map. Several examples of functions can be constructed which are SL maps but not nonexpansive. We include here the following simple example. We further remark that Theorem 1.1 does not apply to the function defined below.

Example 3.2. Let $C = [0,3/5]$ with the usual norm and consider the map $T(x) = x^2$. It is easy to see that T is an SL map but not nonexpansive. Moreover, T is 1-χ-contractive and has a fixed point.

We now prove a result which contains Theorem 1.1, as a special case, and is applicable to the above example.

THEOREM 3.3. *Let C be a nonempty closed bounded convex subset of a Banach X such that $\epsilon_\alpha(X) < 1$ and $T : C \to KC(X)$ a continuous,* SL, *1-χ-contractive map. If T satisfies*

$$T(x) \subset I_C(x) \quad \forall x \in C, \tag{3.3}$$

then T has a fixed point.

Proof. We follow the arguments given in [3]. Let $x_0 \in C$ be fixed. Define, for each $n \geq 1$, a mapping $T_n : C \to KC(X)$ by

$$T_n(x) := \frac{1}{n}x_0 + \left(1-\frac{1}{n}\right)T(x), \tag{3.4}$$

where $x \in C$. Then T_n is $(1 - 1/n)$-χ-contractive. Also $T_n(x) \subset I_C(x)$ for all $x \in C$. Now Lemma 2.1 guarantees that each T_n has a fixed point $x_n \in C$. As a result, we have $\lim_{n\to\infty} d(x_n, T(x_n)) = 0$. Let $\{n_\alpha\}$ be an ultranet of the positive integers $\{n\}$. Set $A = A(C, \{x_{n_\alpha}\})$. We claim that

$$T(x) \cap I_A(x) \neq \varnothing \tag{3.5}$$

for all $x \in A$. To prove our claim, let $x \in A$. Since $T(x_{n_\alpha})$ is compact, we can find $y_{n_\alpha} \in T(x_{n_\alpha})$ such that

$$||x_{n_\alpha} - y_{n_\alpha}|| = d(x_{n_\alpha}, T(x_{n_\alpha})). \tag{3.6}$$

We also have $z_{n_\alpha} \in T(x)$ such that

$$||y_{n_\alpha} - z_{n_\alpha}|| = d(y_{n_\alpha}, T(x)). \tag{3.7}$$

We can assume that $z = \lim_\alpha z_{n_\alpha}$. Clearly, $z \in T(x)$. We show that $z \in I_A(x) = \{x + \lambda(y - x) : \lambda \geq 0,\ y \in A\}$. Since T is an SL map and $\{x_{n_\alpha}\}$ is asymptotically T-regular, it follows that

$$\limsup_\alpha H(T(x_{n_\alpha}), T(x)) \leq \limsup_\alpha ||x_{n_\alpha} - x|| \tag{3.8}$$

for all $x \in A$. Now

$$\begin{aligned} ||y_{n_\alpha} - z_{n_\alpha}|| &= d(y_{n_\alpha}, T(x)) \\ &\leq H(T(x_{n_\alpha}), T(x)) \end{aligned} \tag{3.9}$$

and so

$$\begin{aligned} \lim_\alpha ||x_{n_\alpha} - z|| &= \lim_\alpha ||y_{n_\alpha} - z_{n_\alpha}|| \\ &\leq \limsup_\alpha ||x_{n_\alpha} - x|| = r, \end{aligned} \tag{3.10}$$

where $r = r(C, \{x_{n_\alpha}\})$. Notice also that $z \in T(x) \subset I_C(x)$ and so $z = x + \lambda(y - x)$ for some $\lambda \geq 0$ and $y \in C$. Without loss of generality, we may assume that $\lambda > 1$. Now

$$y = \frac{1}{\lambda} z + \left(1 - \frac{1}{\lambda}\right) x \tag{3.11}$$

and so

$$\lim_\alpha ||x_{n_\alpha} - y|| \leq \frac{1}{\lambda} \lim_\alpha ||x_{n_\alpha} - z|| + \left(1 - \frac{1}{\lambda}\right) \lim_\alpha ||x_{n_\alpha} - x|| \leq r. \tag{3.12}$$

This implies that $y \in A$ and so $z \in I_A(x)$. This proves our claim. By Lemma 2.3, we have

$$r_C(A) \leq \lambda r, \tag{3.13}$$

where $\lambda := 1 - \Delta_{X,\alpha}(1^-) < 1$. Now choose $x_1 \in A$ and for each $\mu \in (0,1)$, define a mapping $T_\mu : A \to KC(X)$ by

$$T_\mu(x) = \mu x_1 + (1-\mu)T(x). \tag{3.14}$$

Then each T_μ is a χ-condensing and satisfies

$$T_\mu(x) \cap I_A(x) \neq \varnothing \tag{3.15}$$

for all $x \in A$. Now Lemma 2.2 guarantees that T_μ has a fixed point. As a result, we can get an asymptotically T-regular sequence $\{x_n^1\}$ in A. Proceeding as above, we obtain

$$T(x) \cap I_{A^1}(x) \neq \varnothing \tag{3.16}$$

for all $x \in A^1 := A(C, \{x_{n_\alpha}^1\})$ and $r_C(A^1) \le \lambda r_C(A)$. By induction, for each $m \ge 1$, we can find an asymptotically T-regular sequence $\{x_n^m\}_n \subseteq A^{m-1}$. Using the ultranet $\{x_{n_\alpha}^m\}_\alpha$, we construct $A^m := A(C, \{x_{n_\alpha}^m\})$ with $r_C(A^m) \le \lambda^m r_C(A)$. Choose $x_m \in A^m$. Then $\{x_m\}_m$ is a Cauchy sequence. Indeed, for each $m \ge 1$, we have

$$\begin{aligned} \|x_{m-1} - x_m\| &\le \|x_{m-1} - x_n^m\| + \|x_n^m - x_m\| \\ &\le \operatorname{diam}(A^{m-1}) + \|x_n^m - x_m\|, \end{aligned} \tag{3.17}$$

for all $n \ge 1$. Now taking lim sup, we see that

$$\begin{aligned} \|x_{m-1} - x_m\| &\le \operatorname{diam} A^{m-1} + \limsup_n \|x_n^m - x_m\| \\ &\le 3r_C(A^{m-1}) \\ &\le 3\lambda^{m-1} r_C(A). \end{aligned} \tag{3.18}$$

Taking the limit as $m \to \infty$, we get $\lim_{m\to\infty} \|x_{m-1} - x_m\| = 0$. This implies that $\{x_m\}$ is a Cauchy sequence and so is convergent. Let $x = \lim_{m\to\infty} x_m$. Finally, we show that x is a fixed point of T. Since T is an SL map, for $m \ge 1$, we have

$$\limsup_n H(T(x_n^m), T(x_m)) \le \limsup_n \|x_n^m - x_m\|. \tag{3.19}$$

Now, we have for $m \ge 1$,

$$\begin{aligned} d(x_m, T(x_m)) \le \|x_m - x_n^m\| &+ d(x_n^m, T(x_n^m)) \\ &+ H(T(x_n^m), T(x_m)). \end{aligned} \tag{3.20}$$

This implies that

$$\begin{aligned} d(x_m, T(x_m)) &\le 2\limsup_n \|x_m - x_n^m\| \\ &\le 2\lambda^{m-1} r_C(A). \end{aligned} \tag{3.21}$$

Taking the limit as $m \to \infty$, we have $\lim_{m\to\infty} d(x_m, T(x_m)) = 0$ and so $x \in T(x)$. This completes the proof. □

Theorem 3.3 fails if the assumption that T is an SL map is dropped.

Example 3.4. Let B be the closed unit ball of l_2. Define $T : B \to B$ by

$$T(x) = T(x_1, x_2, \dots) = \left(\sqrt{1 - \|x\|^2}, x_1, x_2, \dots\right). \tag{3.22}$$

Then T is 1-χ-contractive without a fixed point (see [1, 2]). We can show that this map is not SL if we consider the following sequence $\{x^{(n)}\}$ in B:

$$\begin{aligned} x^{(1)} &= \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{16}}, \dots\right), \\ x^{(2)} &= \left(0, \frac{1}{\sqrt{2}\sqrt{2}}, \frac{1}{\sqrt{2}\sqrt{2}}, \frac{1}{\sqrt{2}\sqrt{4}}, \frac{1}{\sqrt{2}\sqrt{4}}, \frac{1}{\sqrt{2}\sqrt{8}}, \frac{1}{\sqrt{2}\sqrt{8}}, \dots\right), \\ x^{(3)} &= \left(0, \frac{1}{\sqrt{3}\sqrt{2}}, \frac{1}{\sqrt{3}\sqrt{2}}, \frac{1}{\sqrt{3}\sqrt{2}}, \frac{1}{\sqrt{3}\sqrt{4}}, \frac{1}{\sqrt{3}\sqrt{4}}, \frac{1}{\sqrt{3}\sqrt{4}}, \frac{1}{\sqrt{3}\sqrt{8}}, \frac{1}{\sqrt{3}\sqrt{8}}, \frac{1}{\sqrt{3}\sqrt{8}}, \dots\right), \end{aligned} \tag{3.23}$$

and so on.

Corollary 3.5. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ satisfying the nonstrict Opial condition and $T : C \to KC(X)$ a nonexpansive map. If T satisfies*

$$T(x) \subset I_C(x) \quad \forall x \in C, \tag{3.24}$$

then T has a fixed point.

Proof. This follows immediately from [2, Theorem 4.5] and Theorem 3.3. □

Next we present some fixed point results for $*$-nonexpansive maps.

Theorem 3.6. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ and $T : C \to KC(X)$ a $*$-nonexpansive, 1-χ-contractive map. If T satisfies*

$$T(x) \subset I_C(x) \quad \forall x \in C, \tag{3.25}$$

then T has a fixed point.

Proof. Define

$$P_T(x) = \{u_x \in T(x) : d(x, u_x) = d(x, T(x))\} \tag{3.26}$$

for $x \in C$. Since $T(x)$ is compact, $P_T(x)$ is nonempty for each x. Furthermore, P_T is convex and compact valued since T is so. Also, P_T is nonexpansive because T is $*$-nonexpansive. Let B be a bounded subset of C. Then it is easy to see that $P_T(C)$ is a bounded set and $\chi(P_T(B)) \le \chi(B)$. Thus P_T is 1-χ-contractive. P_T also satisfies

$$P_T(x) \subset I_C(x) \quad \forall x \in C. \tag{3.27}$$

Now Theorem 3.3 guarantees that P_T has a fixed point. Hence T has a fixed point. □

Similarly, we get the following corollary, which extends [11, Theorem 2] to non-self-multimaps and to spaces satisfying the nonstrict Opial condition.

Corollary 3.7. *Let C be a nonempty closed bounded convex subset of a Banach space X such that $\epsilon_\alpha(X) < 1$ satisfying the nonstrict Opial condition and $T : C \to KC(X)$ a $*$-nonexpansive map. If T satisfies*

$$T(x) \subset I_C(x) \quad \forall x \in C, \tag{3.28}$$

then T has a fixed point.

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Naseer Shahzad: Department of Mathematics, Faculty of Sciences, King Abdul Aziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail address: nshahzad@kau.edu.sa

Amjad Lone: Department of Mathematics, College of Sciences, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia
E-mail address: amlone@kku.edu.sa