EXISTENCE OF SOLUTIONS FOR EQUATIONS INVOLVING ITERATED FUNCTIONAL SERIES

V. MURUGAN AND P. V. SUBRAHMANYAM

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Theorems on the existence and uniqueness of differentiable solutions for a class of iterated functional series equations are obtained. These extend earlier results due to Zhang.

1. Introduction

The study of iterated functional equations dates back to the classical works of Abel, Babbage, and others. This paper offers new theorems on the existence and uniqueness of solutions to the iterated functional series equation

$$\sum_{i=1}^{\infty} \lambda_i H_i(f^i(x)) = F(x), \tag{1.1}$$

where λ_i's are nonnegative numbers and $f^0(x) = x$, $f^k(x) = f(f^{k-1}(x))$, $k \in \mathbb{N}$. In (1.1) the functions F, H_i and constants λ_i ($i \in \mathbb{N}$) are given and the unknown function f is to be found. The above equation is more general than those considered by Dhombres [2], Mukherjea and Ratti [3], Nabeya [4], and Zhang [5].

2. Preliminaries

This section collects the standard terminology and results used in the sequel (see [5]).

Let $I = [a,b]$ be an interval of real numbers. $C^1(I,I)$, the set of all continuously differentiable functions from I into I, is a closed subset of the Banach Space $C^1(I,\mathbb{R})$ of all continuously differentiable functions from I into $\mathbb{R}$ with the norm $\|\cdot\|_{c^1}$ defined by $\|\phi\|_{c^1} = \|\phi\|_{c^0} + \|\phi'\|_{c^0}$, $\phi \in C^1(I,\mathbb{R})$ where $\|\phi\|_{c^0} = \max_{x\in I} |\phi(x)|$ and ϕ' is the derivative of ϕ. Following Zhang [5], for given constants $M \geq 0$, $M^* \geq 0$, and $\delta > 0$, we define the families of functions

$$\begin{aligned} \mathcal{R}^1(I,M,M^*) = \{\phi \in C^1(I,I) : \phi(a) = a,\ \phi(b) = b,\ 0 \leq \phi'(x) \leq M\ \forall x \in I, \\ |\phi'(x_1) - \phi'(x_2)| \leq M^* |x_1 - x_2|\ \forall x_1, x_2 \in I\} \end{aligned} \tag{2.1}$$

and $\mathcal{F}^1_\delta(I,M,M^*) = \{\phi \in \mathcal{R}^1(I,M,M^*) : \delta \leq \phi'(x) \leq M \text{ for all } x \in I\}$.

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In this context it is useful to note the following proposition.

Proposition 2.1. *Let $\delta > 0$, $M \geq 0$, and $M^* \geq 0$. Then*

(i) *for $M < 1$, $\mathcal{R}^1(I,M,M^*)$ is empty and for $M = 1$, $\mathcal{R}^1(I,M,M^*)$ contains only the identity function;*

(ii) *for $\delta > 1$, $\mathcal{F}^1_\delta(I,M,M^*)$ is empty and for $\delta = 1$, $\mathcal{F}^1_\delta(I,M,M^*)$ contains only the identity function.*

Proof. (i) Let $\phi \in \mathcal{R}^1(I,M,M^*)$, where $0 \leq M < 1$. Clearly ϕ is a strict contraction with Lipschitz constant M on I. So ϕ has a unique fixed point contrary to the assumption that ϕ has at least two fixed points a and b.

If $\phi \in \mathcal{R}^1(I,1,M^*)$, then by the mean-value theorem and the hypothesis that $\phi'(x) \leq 1$ for all $x \in I$, $\phi(b) - \phi(x) \leq b - x$ and $\phi(x) - \phi(a) \leq x - a$ for all $x \in I$. Since $\phi(a) = a$ and $\phi(b) = b$, ϕ must necessarily be the identity function.

(ii) Let $\phi \in \mathcal{F}^1_\delta(I,M,M^*)$, where $\delta > 1$. Then by the mean-value theorem, $\phi(b) - \phi(a) > b - a$. This contradicts that a and b are fixed points of ϕ. The argument for the case when $\delta = 1$ is similar to the case when $M = 1$. □

In view of the above proposition, one cannot seek solutions of equations such as (1.1) in $\mathcal{R}^1(I,M,M^*)$ without imposing conditions on M. The following lemmata of Zhang [5] will be used in the sequel.

Lemma 2.2 (Zhang [5]). *Let ϕ, $\psi \in \mathcal{R}^1(I,M,M^*)$. Then, for $i = 1,2,\ldots$,*

(1) $|(\phi^i)'(x)| \leq M^i$ *for all* $x \in I$,

(2) $|(\phi^i)'(x_1) - (\phi^i)'(x_2)| \leq M^*(\sum_{j=i-1}^{2i-2} M^j)|x_1 - x_2|$ *for all* $x_1,x_2 \in I$,

(3) $\|\phi^i - \psi^i\|_{c^0} \leq (\sum_{j=1}^{i} M^{j-1})\|\phi - \psi\|_{c^0}$,

(4) $\|(\phi^i)' - (\psi^i)'\|_{c^0} \leq iM^{i-1}\|\phi' - \psi'\|_{c^0} + Q(i)M^*(\sum_{j=1}^{i-1}(i-j)M^{i+j-2})\|\phi - \psi\|_{c^0}$, *where* $Q(1) = 0$, $Q(s) = 1$ *if* $s = 2,3,\ldots$.

Lemma 2.3 (Zhang [5]). *Let $\phi \in \mathcal{F}^1_\delta(I,M,M^*)$. Then*

$$\left|\left(\phi^{-1}\right)'(x_1) - \left(\phi^{-1}\right)'(x_2)\right| \leq \frac{M^*}{\delta^3}\left|x_1 - x_2\right| \quad \forall x_1,x_2 \in I. \tag{2.2}$$

Lemma 2.4 (Zhang [5]). *Let ϕ_1, ϕ_2 be two homeomorphisms from I onto itself and $|\phi_i(x_1) - \phi_i(x_2)| \leq M^*|x_1 - x_2|$ for all $x_1,x_2 \in I$, $i = 1,2$. Then*

$$||\phi_1 - \phi_2||_{c^0} \leq M^*||\phi_1^{-1} - \phi_2^{-1}||_{c^0}. \tag{2.3}$$

The following results are well known.

Lemma 2.5 (see [1]). *For each $n \in \mathbb{N}$, let f_n be a real-valued function on $I = [a,b]$ which has derivative f_n' on I. Suppose that the infinite series $\sum_{n=1}^{\infty} f_n$ converges for at least one point of I and that the series of derivatives $\sum_{n=1}^{\infty} f_n'$ converges uniformly on I. Then there exists a real-valued function f on I such that $\sum_{n=1}^{\infty} f_n$ converges uniformly on I to f. In addition, f has a derivative on I and $f' = \sum_{n=1}^{\infty} f_n'$.*

Lemma 2.6. *Let $f : J \to J$ be a differentiable function on an interval J in $\mathbb{R}$ satisfying the inequality $0 < a \le f'(x) \le b$, $x \in J$, for some a, b in $\mathbb{R}$. Then the inverse function f^{-1} exists and is differentiable on J. Further, for all $x \in J$,*

$$b^{-1} \le \left(f^{-1}\right)'(x) \le a^{-1}. \tag{2.4}$$

Lemma 2.7. *For $n \in \mathbb{N}$, $x \in \mathbb{R}$,*

$$\sum_{i=1}^{n} ix^{i-1} = \begin{cases} \dfrac{(n+1)x^n}{(x-1)} - \dfrac{x^{n+1}-1}{(x-1)^2}, & x \neq 1, \\ \dfrac{n(n+1)}{2}, & x = 1, \end{cases} \tag{2.5}$$

and further

$$\sum_{i=1}^{n-1} (n-i)x^{n+i-2} = \begin{cases} x^{n-1}\left(\dfrac{x^n - 1}{(x-1)^2} - \dfrac{n}{x-1}\right), & x \neq 1, \\ \dfrac{n(n-1)}{2}, & x = 1. \end{cases} \tag{2.6}$$

3. Existence

In this section, we prove in detail a theorem on the existence of solutions for the functional series equation (1.1).

Theorem 3.1. *Suppose (λ_n) is a sequence of nonnegative numbers with $\lambda_1 > 0$ and $\sum_{i=1}^{\infty} \lambda_i = 1$. Let $F \in \mathcal{F}_\delta^1(I, \lambda_1 \eta M, M^*)$, $H_1 \in \mathcal{F}_\eta^1(I, L_1, L_1')$, and $H_i \in \mathcal{R}^1(I, L_i, L_i')$ for $i = 2, 3, \dots$, where δ, $\eta > 0$ and M, M^*, L_i, $L_i' \ge 0$ for all $i \in \mathbb{N}$.*

Assume further that

(i) $M > 1$,

(ii) $K_0 = (1/(M-1)) \sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} M^{i-1}(M^i - 1)$ *and* $\gamma = \lambda_1 \eta - K_0 M^2 > 0$,

(iii) $\sum_{i=1}^{\infty} \lambda_i L_i' M^{i-1}(M^i - 1) < \infty$.

Then the functional series equation $\sum_{i=1}^{\infty} \lambda_i H_i(f^i(x)) = F(x)$ has a solution f in $\mathcal{R}^1(I, M, M')$ where $M' = (M^* + K_1' M^2)/\gamma$ and $K_1' = \sum_{i=1}^{\infty} \lambda_i L_i' M^{2(i-1)}$.

Proof. For each $\phi \in \mathcal{R}^1(I, M, M')$, define the function

$$(L\phi)(x) = \sum_{i=1}^{\infty} \lambda_i H_i\left(\phi^{i-1}(x)\right) \quad \text{for } x \in I. \tag{3.1}$$

Since $\lambda_i \ge 0$, $\sum_{i=1}^{\infty} \lambda_i = 1$, and $|H_i(x)| \le \max\{|b|, |a|\}$, $(L\phi)(x)$ is well defined for all $x \in I$. Further $(L\phi)(a) = a$ and $(L\phi)(b) = b$. Since ϕ and H_i are differentiable with $o \le H_i'(x) \le L_i$ and $0 \le (\phi^{i-1}(x))' \le M^{i-1}$,

$$0 \le \lambda_i H_i'\left(\phi^{i-1}(x)\right)\left(\phi^{i-1}(x)\right)' \le \lambda_i L_i M^{i-1} \quad \forall x \in I,\ i \in \mathbb{N}. \tag{3.2}$$

and $\sum_{i=1}^{\infty}\lambda_i L_i M^{i-1}$ converges in view of (ii). By Weierstrass M-test,

$$\sum_{i=1}^{\infty}\lambda_i H_i'\big(\phi^{i-1}(x)\big)\big(\phi^{i-1}(x)\big)' \tag{3.3}$$

converges uniformly on I. From Lemma 2.5, $L\phi$ is differentiable on I and

$$(L\phi)'(x)=\sum_{i=1}^{\infty}\lambda_i H_i'\big(\phi^{i-1}(x)\big)\big(\phi^{i-1}(x)\big)' \quad \forall x\in I,\ \phi\in\mathcal{R}^1(I,M,M'). \tag{3.4}$$

Since $0<\eta\le H_1'(x)\le L_1$, it is clear that $0<\lambda_1\eta\le(L\phi)'(x)\le\sum_{i=1}^{\infty}\lambda_i L_i M^{i-1}$. Writing $K_1=\sum_{i=1}^{\infty}\lambda_i L_i M^{i-1}$, we note that

$$0<\lambda_1\eta\le(L\phi)'(x)\le K_1. \tag{3.5}$$

From Lemma 2.6, for x in I,

$$0<K_1^{-1}\le\big((L\phi)^{-1}\big)'(x)\le(\lambda_1\eta)^{-1}. \tag{3.6}$$

In short, $L\phi: I\to I$ is a nondecreasing self-diffeomorphism. For $x_1,x_2\in I$,

$$\begin{aligned}
&\big|(L\phi)'(x_1)-(L\phi)'(x_2)\big|\\
&\quad=\sum_{i=1}^{\infty}\lambda_i\big|H_i'\big(\phi^{i-1}(x_1)\big)\big(\phi^{i-1}\big)'(x_1)-H_i'\big(\phi^{i-1}(x_2)\big)\big(\phi^{i-1}\big)'(x_2)\big|\\
&\quad\le\sum_{i=1}^{\infty}\lambda_i\Big\{\big|H_i'\big(\phi^{i-1}(x_1)\big)\big|\,\big|\big(\phi^{i-1}\big)'(x_1)-\big(\phi^{i-1}\big)'(x_2)\big|\\
&\qquad\qquad+\big|H_i'\big(\phi^{i-1}(x_1)\big)-H_i'\big(\phi^{i-1}(x_2)\big)\big|\,\big|\big(\phi^{i-1}\big)'(x_2)\big|\Big\}\\
&\quad\le\left\{\sum_{i=2}^{\infty}\lambda_i L_i M'\left(\sum_{j=i-2}^{2(i-2)}M^i\right)+\sum_{i=1}^{\infty}\lambda_i L_i' M^{2(i-1)}\right\}|x_1-x_2|\\
&\qquad\qquad\text{(by the definition of } H_i\text{'s and using Lemma 2.2)}\\
&\quad=\left\{\frac{M'}{M-1}\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}M^{i-1}\big(M^i-1\big)+\sum_{i=1}^{\infty}\lambda_i L_i' M^{2(i-1)}\right\}|x_1-x_2|\\
&\quad=\big(K_0M'+K_1'\big)|x_1-x_2|.
\end{aligned} \tag{3.7}$$

Thus,

$$\big|(L\phi)'(x_1)-(L\phi)'(x_2)\big|\le K_2|x_1-x_2| \quad \forall x_1,x_2\in I, \tag{3.8}$$

where $K_2 = K_0 M' + K_1'$, $K_0 = 1/(M-1)\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}M^{i-1}(M^i-1)$, and $K_1' = \sum_{i=1}^{\infty}\lambda_i L_i' M^{2(i-1)}$. From Lemma 2.3, it follows that

$$\left|\left((L\phi)^{-1}\right)'(x_1) - \left((L\phi)^{-1}\right)'(x_2)\right| \le \frac{K_2}{\lambda_1^3\eta^3}\left|x_1 - x_2\right| \quad \forall x_1, x_2 \in I. \tag{3.9}$$

We define $T : \mathcal{R}^1(I, M, M') \to C^1(I, I)$ by

$$(T\phi)(x) = \left((L\phi)^{-1}\right)(F(x)) \quad \forall \phi \in \mathcal{R}^1(I, M, M'),\ x \in I. \tag{3.10}$$

Clearly $(T\phi)(a) = a$, $(T\phi)(b) = b$, and by (3.6) we have

$$\delta K_1^{-1} \le (T\phi)'(x) = \left((L\phi)^{-1}\right)'(F(x))F'(x) \le M \quad \forall x \in I. \tag{3.11}$$

So T is a sense-preserving diffeomorphism of I onto I. For $x_1, x_2 \in I$,

$$\begin{aligned}
&\left|(T\phi)'(x_1) - (T\phi)'(x_2)\right| \\
&\quad = \left|\left((L\phi)^{-1}\right)'(F(x_1)) - \left((L\phi)^{-1}\right)'(F(x_2))\right| \\
&\quad \le \left|\left((L\phi)^{-1}\right)'(F(x_1))\right|\left|F'(x_1) - F'(x_2)\right| \\
&\qquad + \left|\left((L\phi)^{-1}\right)'(F(x_1)) - \left((L\phi)^{-1}\right)'(F(x_2))\right|\left|F'(x_2)\right| \\
&\quad \le \frac{M^*}{\lambda_1\eta}\left|x_1 - x_2\right| + \frac{K_2}{\lambda_1^3\eta^3}\left|F(x_1) - F(x_2)\right|\lambda_1\eta M \\
&\quad \le \frac{M^*}{\lambda_1\eta}\left|x_1 - x_2\right| + \frac{K_2\lambda_1^2\eta^2M^2}{\lambda_1^3\eta^3}\left|x_1 - x_2\right| \quad \left(\text{as } \left|F'(x)\right| \le \lambda_1\eta M\right) \\
&\quad = \left(\frac{M^* + K_2M^2}{\lambda_1\eta}\right)\left|x_1 - x_2\right| \\
&\quad = \left(\frac{M^* + K_0M'M^2 + K_1'M^2}{\lambda_1\eta}\right)\left|x_1 - x_2\right|.
\end{aligned} \tag{3.12}$$

Since $M'(\lambda_1\eta - K_0M^2) = M^* + K_1'M^2$,

$$\left|(T\phi)'(x_1) - (T\phi)'(x_2)\right| \le M'\left|x_1 - x_2\right| \quad \forall x_1, x_2 \in I. \tag{3.13}$$

It implies that $T\phi \in \mathcal{R}^1(I, M, M')$.

Next we show that T is continuous. For arbitrary functions $\phi_i \in \mathcal{R}^1(I, M, M')$, we denote $f_i = T\phi_i$, $i = 1, 2$. Then $|f_i'(x)| \le M$, $|f_i'(x_1) - f_i'(x_2)| \le M'|x_1 - x_2|$, and

$|(f_i^{-1})'(x)| \le K_1/\delta$ for $x, x_1, x_2 \in I$ and $i = 1, 2$. Hence,

$$\begin{aligned}
&\|f_1' - f_2'\|_{c^0} \\
&\quad = \|f_1' - (f_1(f_1^{-1}(f_2)))'\|_{c^0} \\
&\quad = \max_{x\in I}\{|f_1'(x) - f_1'(f_2(x))(f_1^{-1})'(f_2(x))f_2'(x)|\} \\
&\quad = \max_{x\in I}\{|f_1'(x)(f_2^{-1}(f_2(x)))' - f_1'(f_2(x))(f_1^{-1})'(f_2(x))f_2'(x)|\} \\
&\quad \le M\max_{x\in I}\{|f_1'(x)(f_2^{-1})'(f_2(x)) - f_1'(f_2(x))(f_1^{-1})'(f_2(x))|\} \\
&\quad \le M\max_{x\in I}\{|f_1'(x)||(f_2^{-1})'(f_2(x)) - (f_1^{-1})'(f_2(x))| \\
&\qquad\qquad + |f_1'(x) - f_1'(f_1^{-1}(f_2(x)))||(f_1^{-1})'(f_2(x))|\} \\
&\quad \le M^2\max_{x\in I}\Big\{|(f_2^{-1})'(f_2(x)) - (f_1^{-1})'(f_2(x))| \\
&\qquad\qquad + \frac{MK_1M'}{\delta}\max_{x\in I}|x - (f_1^{-1})'(f_2(x))|\Big\}.
\end{aligned} \tag{3.14}$$

Thus,

$$\|f_1' - f_2'\|_{c^0} \le M^2\|(f_1^{-1})' - (f_2^{-1})'\|_{c^0} + \frac{MK_1M'}{\delta}\|f_1^{-1} - f_2^{-1}\|_{c^0}. \tag{3.15}$$

Besides, by Lemma 2.4, we have

$$\|f_1 - f_2\|_{c^0} \le M\|f_1^{-1} - f_2^{-1}\|_{c^0}. \tag{3.16}$$

From (3.15) and (3.16), it follows that

$$\begin{aligned}
&\|T\phi_1 - T\phi_2\|_{c^1} \\
&\quad = \|f_1 - f_2\|_{c^1} = \|f_1 - f_2\|_{c^0} + \|f_1' - f_2'\|_{c^0} \\
&\quad \le M\|f_1^{-1} - f_2^{-1}\|_{c^0} + M^2\|(f_1^{-1})' - (f_2^{-1})'\|_{c^0} + \frac{MK_1M'}{\delta}\|f_1^{-1} - f_2^{-1}\|_{c^0}.
\end{aligned} \tag{3.17}$$

Thus,

$$\|T\phi_1 - T\phi_2\|_{c^1} \le E_1\|f_1^{-1} - f_2^{-1}\|_{c^1}, \tag{3.18}$$

where $E_1 = \max\{M + K_1MM'/\delta, M^2\}$. Furthermore, since $F \in \mathcal{F}_\delta^1(I, \lambda_1\eta M, M^*)$, an application of Lemma 2.3 gives

$$|(F^{-1})'(x_1) - (F^{-1})'(x_2)| \le \frac{M^*}{\delta^3}|x_1 - x_2| \quad \forall x_1, x_2 \in I. \tag{3.19}$$

Now

$$\begin{aligned}||f_1^{-1}-f_2^{-1}||_{c^1} &= ||F^{-1}\circ(L\phi_1)-F^{-1}\circ(L\phi_2)||_{c^1}\\ &= ||F^{-1}\circ(L\phi_1)-F^{-1}\circ(L\phi_2)||_{c^0}\\ &\quad+||((F^{-1})'(L\phi_1))(L\phi_1)'-((F^{-1})'(L\phi_2))(L\phi_2)'||_{c^0}.\end{aligned} \tag{3.20}$$

Using Lemma 2.6 and the fact that $F\in\mathcal{F}_\delta^1(I,\lambda_1\eta M,M^*)$,

$$\begin{aligned}&||f_1^{-1}-f_2^{-1}||_{c^1}\\ &\quad=\frac{1}{\delta}||L\phi_1-L\phi_2||_{c^0}+||((F^{-1})'(L\phi_1))(L\phi_1)'-((F^{-1})'(L\phi_2))(L\phi_1)'||_{c^0}\\ &\qquad+||((F^{-1})'(L\phi_2))((L\phi_1)'-(L\phi_2)')||_{c^0}\\ &\quad\le\frac{1}{\delta}||L\phi_1-L\phi_2||_{c^0}+\frac{K_1M^*}{\delta^3}||L\phi_1-L\phi_2||_{c^0}\\ &\qquad+\frac{1}{\delta}||(L\phi_1)'-(L\phi_2)'||_{c^0}\quad(\text{by}(3.5)\text{ and }(3.19))\\ &\quad\le\left(\frac{1}{\delta}+\frac{K_1M^*}{\delta^3}\right)||(L\phi_1)'-(L\phi_2)'||_{c^1}.\end{aligned} \tag{3.21}$$

Thus,

$$||f_1^{-1}-f_2^{-1}||_{c^1}\le E_2||(L\phi_1)'-(L\phi_2)'||_{c^1}, \tag{3.22}$$

where $E_2=1/\delta+K_1M^*/\delta^3$. By the definition of $L\phi$, we have

$$\begin{aligned}&||L\phi_1-L\phi_2||_{c^0}\\ &\qquad\le\sum_{i=1}^{\infty}\lambda_i||H_i(\phi_1^{i-1})-H_i(\phi_2^{i-1})||_{c^0}\\ &\qquad\le\sum_{i=2}^{\infty}\lambda_iL_i||\phi_1^{i-1}-\phi_2^{i-1}||_{c^0}\quad(0\le H_i'(x)\le L_i,\ x\in I,\ i=1,2,\dots)\\ &\qquad\le\sum_{i=2}^{\infty}\lambda_iL_i\left(\sum_{j=1}^{i-1}M^{j-1}\right)||\phi_1-\phi_2||_{c^0}\quad(\text{by Lemma 2.2})\\ &\qquad\le\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}\left(\sum_{j=1}^{i}M^{j-1}\right)||\phi_1-\phi_2||_{c^0}.\end{aligned} \tag{3.23}$$

Thus,

$$||L\phi_1-L\phi_2||_{c^0}\le\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}\left(\frac{M^i-1}{M-1}\right)||\phi_1-\phi_2||_{c^0}. \tag{3.24}$$

Further,

$$
\begin{aligned}
&\|(L\phi_1)' - (L\phi_2)'\|_{c^0}\\
&\quad \le \sum_{i=2}^{\infty}\lambda_i\|(H_i'(\phi_1^{i-1}))(\phi_1^{i-1})' - (H_i'(\phi_2^{i-1}))(\phi_2^{i-1})'\|_{c^0}\\
&\quad \le \sum_{i=2}^{\infty}\lambda_i\Big\{\|[H_i'(\phi_1^{i-1}) - H_i'(\phi_2^{i-1})](\phi_1^{i-1})'\|_{c^0}\\
&\qquad\qquad + \|H_i'(\phi_2^{i-1})[(\phi_1^{i-1})' - (\phi_2^{i-1})']\|_{c^0}\Big\}\\
&\quad \le \sum_{i=2}^{\infty}\lambda_i\{M^{i-1}L_i'\|\phi_1^{i-1} - \phi_2^{i-1}\|_{c^0} + L_i\|(\phi_1^{i-1})' - (\phi_2^{i-1})'\|_{c^0}\}\\
&\qquad \text{(using the fact that } H_i \in \mathcal{R}^1(I,L_i,L_i'),\ i\in\mathbb{N}, \text{ and by Lemma 2.2)}\\
&\quad \le \sum_{i=2}^{\infty}\lambda_i M^{i-1}L_i'\left(\sum_{j=1}^{i-1}M^{j-1}\right)\|\phi_1 - \phi_2\|_{c^0} + \sum_{i=2}^{\infty}\lambda_i L_i(i-1)M^{i-1}\|\phi_1' - \phi_2'\|_{c^0}\\
&\qquad + \sum_{i=2}^{\infty}\lambda_i L_i Q(i-1)M'\left(\sum_{j=1}^{i-2}(i-j-1)M^{i+j-3}\right)\|\phi_1 - \phi_2\|_{c^0}.
\end{aligned}
\tag{3.25}
$$

Upon relabelling the subscripts in the above, we get

$$
\begin{aligned}
&\|(L\phi_1)' - (L\phi_2)'\|_{c^0}\\
&\quad \le \sum_{i=1}^{\infty}\lambda_{i+1}M^iL_{i+1}'\left(\sum_{j=1}^{i}M^{j-1}\right)\|\phi_1 - \phi_2\|_{c^0} + \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}iM^i\|\phi_1' - \phi_2'\|_{c^0}\\
&\qquad + \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}Q(i)M'\left(\sum_{j=1}^{i-1}(i-j)M^{i+j-2}\right)\|\phi_1 - \phi_2\|_{c^0}.
\end{aligned}
\tag{3.26}
$$

From Lemma 2.7,

$$
\begin{aligned}
&\|(L\phi_1)' - (L\phi_2)'\|_{c^0}\\
&\quad \le \sum_{i=1}^{\infty}\lambda_{i+1}M^iL_{i+1}'\left(\frac{M^i - 1}{M - 1}\right)\|\phi_1 - \phi_2\|_{c^0} + \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}iM^i\|\phi_1' - \phi_2'\|_{c^0}\\
&\qquad + \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}Q(i)M'M^{i-1}\left\{\frac{M^i - 1}{(M-1)^2} - \frac{i}{M-1}\right\}\|\phi_1 - \phi_2\|_{c^0}.
\end{aligned}
\tag{3.27}
$$

From (3.22) and (3.24), it follows that

$$\begin{aligned} &||L\phi_1 - L\phi_2||_{c^1} \\ &\quad = ||L\phi_1 - L\phi_2||_{c^0} + ||(L\phi_1)' - (L\phi_2)'||_{c^0} \\ &\quad \le \sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} \left(\frac{M^i - 1}{M - 1} \right) ||\phi_1 - \phi_2||_{c^0} \\ &\qquad + \sum_{i=1}^{\infty} \lambda_{i+1} M^i L'_{i+1} \left(\frac{M^i - 1}{M - 1} \right) ||\phi_1 - \phi_2||_{c^0} + \sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} i M^i ||\phi'_1 - \phi'_2||_{c^0} \\ &\qquad + \sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} Q(i) M' M^{i-1} \left\{ \frac{M^i - 1}{(M-1)^2} - \frac{i}{M-1} \right\} ||\phi_1 - \phi_2||_{c^0}. \end{aligned} \tag{3.28}$$

We can more conveniently rewrite this as $\|L\phi_1 - L\phi_2\|_{c^1} \le \sum_{i=1}^{\infty} \lambda_{i+1} A_{i+1} \|\phi_1 - \phi_2\|_{c^1}$, where $A_{i+1} = \max\{((M^i - 1)/(M - 1))(L_{i+1} + M^i L'_{i+1}) + L_{i+1} Q(i) M' M^{i-1} [(M^i - 1)/(M - 1)^2 - i/(M - 1)];\ L_{i+1} i M^i\}$. By hypotheses (ii) and (iii) of the theorem and with the fact that $i \le (M^i - 1)/(M - 1)$, it is easy to see that the series $\sum_{i=1}^{\infty} \lambda_{i+1}(L_{i+1} + M^i L'_{i+1})((M^i - 1)/(M - 1))$, $\sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} i M^i$, and $\sum_{i=1}^{\infty} \lambda_{i+1} L_{i+1} Q(i) M' M^{i-1} \{(M^i - 1)/(M - 1)^2 - i/(M - 1)\}$ converge. Since the convergence of $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ for a_n, $b_n \ge 0$ for all $n \in \mathbb{N}$ implies that of $\sum_{n=1}^{\infty} \max\{a_n, b_n\}$, we conclude that $\sum_{i=1}^{\infty} \lambda_{i+1} A_{i+1}$ converges. We denote it by E_3. Thus we have

$$||L\phi_1 - L\phi_2||_{c^1} \le E_3 ||\phi_1 - \phi_2||_{c^1}. \tag{3.29}$$

From (3.18), (3.22), and (3.29), it follows that

$$||T\phi_1 - T\phi_2||_{c^1} \le E_1 E_2 E_3 ||\phi_1 - \phi_2||_{c^1}. \tag{3.30}$$

Consequently, $T : \mathcal{R}^1(I, M, M') \to \mathcal{R}^1(I, M, M')$ is a continuous operator.

Next we show that $\mathcal{R}^1(I, M, M')$ is a convex compact subset of $C^1(I, \mathbb{R})$. The routine proof that $\mathcal{R}^1(I, M, M')$ is a closed convex subset of $C^1(I, \mathbb{R})$ is omitted.

For $\phi \in \mathcal{R}^1(I, M, M')$, $\|\phi\|_{c^1} = \|\phi\|_{c^0} + \|\phi'\|_{c^0} \le \max\{|a|, |b|\} + M$ and for x in I, $0 \le \phi'(x) \le M$. So $\mathcal{R}^1(I, M, M')$ is an equicontinuous family of functions bounded in the norm $\|\cdot\|_{c^1}$. Since $|\phi'(x_1) - \phi'(x_2)| \le M' |x_1 - x_2|$ for all $x_1, x_2 \in I$ and $\phi \in \mathcal{R}^1(I, M, M')$, $\{\phi' : \phi \in \mathcal{R}^1(I, M, M')\}$ is also an equicontinuous family. From Arzela-Ascoli theorem and Lemma 2.5, we conclude that $\mathcal{R}^1(I, M, M')$ is a compact convex subset of $C^1(I, \mathbb{R})$.

T is a continuous map on $\mathcal{R}^1(I, M, M')$ into itself and by Schauder's fixed point theorem T has a fixed point in $\mathcal{R}^1(I, M, M')$. Thus there is a function $\phi \in \mathcal{R}^1(I, M, M')$ such that $(T\phi)(x) = \phi(x)$. So $(L\phi)^{-1}(F(x)) = \phi(x)$ and $\sum_{i=1}^{\infty} \lambda_i H_i(\phi^i(x)) = F(x)$. Thus ϕ is a solution of the functional series equation (1.1) in $\mathcal{R}^1(I, M, M')$. □

Additionally, we note that if $E_1 E_2 E_3 < 1$, then T is a contraction mapping on the closed subset $\mathcal{R}^1(I, M, M')$ of $C^1(I, \mathbb{R})$. So by Banach's contraction principle, T has a unique fixed point, which gives a solution of (1.1). This is restated in the following theorem.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, suppose that the number $E_1E_2E_3$ is less than 1, where*

$$\begin{aligned} E_1 &= \max\left\{M+\frac{K_1MM'}{\delta}, M^2\right\}, \qquad E_2=\frac{1}{\delta}+\frac{K_1M^*}{\delta^3}, \\ E_3 &= \sum_{i=1}^{\infty}\lambda_{i+1}A_{i+1}, \qquad K_1=\sum_{i=1}^{\infty}\lambda_iL_iM^{i-1}, \\ A_{i+1} &= \max\left\{\left(\frac{M^i-1}{M-1}\right)\left(L_{i+1}+M^iL'_{i+1}\right)\right. \\ &\quad \left. + L_{i+1}Q(i)M'M^{i-1}\left[\frac{M^i-1}{(M-1)^2}-\frac{i}{M-1}\right], L_{i+1}iM^i\right\}. \end{aligned} \tag{3.31}$$

Then (1.1) has a unique solution in $\mathcal{R}^1(I,M,M')$.

Remark 3.3. When we are seeking a solution of (1.1) with $\lambda_1>0$ and $\sum_{i=1}^{\infty}\lambda_i=1$ for given functions $F\in\mathcal{F}_\delta^1(I,\lambda_1\eta,M^*)$, $H_1\in\mathcal{F}_\eta^1(I,L_1,L_1')$, by Proposition 2.1, $\lambda_1\eta M=\lambda_1\eta\geq 1$. Since $\lambda_1\leq 1$ and $\eta\leq 1$, $\lambda_1\eta=1$. So $\lambda_1=1=\eta$. Further $\lambda_i=0$ for $i=2,3,\ldots$. Thus F and H_1 are identity functions, and our equation reduces to $f(x)=x$.

Example 3.4. Consider the functional series equation

$$\left(\frac{55}{27}-e^{1/27}\right)f(x)+\sum_{i=2}^{\infty}\frac{1}{i!27^i}\sin^i\left[\frac{\pi}{2}f^i(x)\right]=\frac{1}{2(e^{1/2}-1)}\int_0^x e^{|t-1/2|}dt, \quad x\in[0,1]. \tag{3.32}$$

Here we have

$$\begin{aligned} \lambda_1 &= \frac{55}{27}-e^{1/27}, \qquad H_1(x)=x, \\ \lambda_i &= \frac{1}{i!27^i}, \quad H_i(x)=\sin^i\left(\frac{\pi}{2}x\right), \qquad \text{for } i=2,3,\ldots, \end{aligned} \tag{3.33}$$

and $F(x)=1/2(e^{1/2}-1)\int_0^x e^{|t-1/2|}dt$, $x\in I=[0,1]$. Choose

$$\begin{aligned} &M=3, \qquad M^*=\frac{e^{1/2}}{2(e^{1/2}-1)}, \qquad \delta=\frac{1}{2(e^{1/2}-1)}, \qquad \eta=1, \\ &L_1=1, \qquad L_1'=0, \qquad L_i=\frac{\pi}{2}i, \quad L_i'=\frac{\pi^2}{4}i(i-1), \quad i=2,3,\ldots. \end{aligned} \tag{3.34}$$

Then $F(0)=0$, $F(1)=1$, $\delta=1/2(e^{1/2}-1)\leq F'(x)\leq e^{1/2}/2(e^{1/2}-1)\leq(55/27-e^{1/27})3=\lambda_1\eta M$, and $|F'(x_1)-F'(x_2)|\leq e^{1/2}/2(e^{1/2}-1)|x_1-x_2|$ for $x,x_1,x_2\in I$. So $F\in\mathcal{F}_\delta^1(I,\lambda_1\eta M,M^*)$, $H_1(x)\in\mathcal{F}_1^1(I,L_1,L_1')$, and $H_i(x)\in\mathcal{R}^1(I,L_i,L_i')$ for $i=2,3,\ldots$. We note that F' is not differentiable on $[0,1]$. Now $\sum_{i=2}^{\infty}\lambda_i = \sum_{i=2}^{\infty} 1/i!27^i = e^{1/27} - 28/27$ and so $\sum_{i=1}^{\infty}\lambda_i = 1$. Also

$$\begin{aligned} K_0M^2 &= \frac{1}{M-1}\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}M^{i+1}(M^i-1) \\ &= \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{(i+1)!27^{i+1}}\frac{\pi}{2}(i+1)3^{i+1}(3^i-1) = \frac{\pi}{4}\sum_{i=1}^{\infty}\left(\frac{3^{2i+1}}{i!3^{3(i+1)}} - \frac{3^{i+1}}{i!3^{3(i+1)}}\right) \\ &= \frac{\pi}{36}\left[\sum_{i=1}^{\infty}\frac{1}{3!3^i} - \sum_{i=1}^{\infty}\frac{1}{3!3^{2i}}\right] = \frac{\pi}{36}(e^{1/3} - 1 - e^{1/9} + 1) = \frac{\pi}{36}(e^{1/3} - e^{1/9}). \end{aligned} \tag{3.35}$$

Thus we have $\lambda_1\eta > K_0M^2$. Since $L_1' = 0$,

$$\begin{aligned} \sum_{i=1}^{\infty}\lambda_i L_i' M^{i-1}(M^i-1) &= \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}'M^i(M^{i+1}-1) \\ &= \sum_{i=1}^{\infty}\frac{1}{(i+1)!27^{i+1}}\frac{\pi^2}{4}i(i+1)3^i(3^{i+1}-1) \\ &= \frac{\pi^2}{4}\sum_{i=1}^{\infty}\left(\frac{3^{2i+1}}{3^{3(i+1)}} - \frac{3^i}{3^{3(i+1)}}\right)\frac{1}{(i-1)!} < \frac{\pi^2}{4}\sum_{i=1}^{\infty}\frac{1}{(i-1)!} = \frac{\pi^2}{4}e. \end{aligned} \tag{3.36}$$

As the positive series $\sum_{i=1}^{\infty}\lambda_i L_i' M^{i-1}(M^i-1)$ converges and $M > 1$, $K_1' = \sum_{i=1}^{\infty}\lambda_i L_i' M^{2(i-1)}$ is finite. Since all the hypotheses of Theorem 3.1 are satisfied we conclude that there is a solution for (3.32) in $\mathcal{R}^1(I,M,M')$ for $M' = (M^* + K_1'M^2)/(\lambda_1\eta - K_0M^2)$.

Example 3.5. Consider the functional series equation

$$(8e^4 - e + 2)f(x) + \sum_{i=2}^{\infty}\frac{(f^i(x))^i}{i!} = \frac{8e^4(e^x-1)}{e-1}, \quad x \in I = [0,1]. \tag{3.37}$$

Setting

$$\begin{aligned} \lambda_1 &= \frac{8e^4 - e + 2}{8e^4}, \qquad F(x) = \frac{e^x - 1}{e-1}, \\ \lambda_i &= \frac{1}{i!8e^4}, \quad H_i(x) = x^i, \quad i = 2,3,\ldots,\ x \in I, \end{aligned} \tag{3.38}$$

equation (3.37) can be rewritten as $\sum_{i=1}^{\infty}\lambda_i H_i(f^i(x)) = F(x)$. Clearly $F(0) = 0$, $F(1) = 1$, $1/(e-1) \le F'(x) \le e/(e-1)$, and $|F''(x)| \le e/(e-1)$ for $x \in I$.

Upon choosing

$$\begin{aligned} M &= 2, \qquad M^* = \frac{e}{e-1}, \qquad \delta = \frac{1}{e-1}, \\ \eta &= 1, \qquad L_i = i, \quad L_i' = i(i-1), \quad i \in \mathbb{N}, \end{aligned} \tag{3.39}$$

it is readily seen that $\lambda_1\eta M = ((8e^4 - e + 2)/8e^4)2 > e/(e-1)$. So $F \in \mathcal{F}^1_\delta(I,\lambda_1\eta M, M^*)$, $H_1(x) \in \mathcal{F}^1_\eta(I,L_1,L_1')$, and $H_i(x) \in \mathcal{R}^1(I,L_i,L_i')$ for $i = 2,3,\ldots$. Also,

$$\begin{aligned} K_0M^2 &= \frac{1}{M-1}\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}M^{i+1}(M^i-1) \\ &= \frac{1}{M-1}\sum_{i=1}^{\infty}\frac{1}{(i+1)!8e^4}(i+1)2^{i+1}(2^i-1) \\ &\le \sum_{i=1}^{\infty}\frac{1}{i!4e^4}4^i = \frac{1}{4e^4}(e^4-1) \le \frac{1}{4}. \end{aligned} \tag{3.40}$$

Thus $\lambda_1\eta = \lambda_1 > 1/2 > K_0M^2$. Further,

$$\begin{aligned} &\sum_{i=1}^{\infty}\lambda_i L_i' M^{i-1}(M^i-1) \\ &\quad= \sum_{i=2}^{\infty}\lambda_i L_i' M^{i-1}(M^i-1) = \sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}'M^{i+1}(M^{i+1}-1) \\ &\quad= \sum_{i=1}^{\infty}\frac{1}{(i+1)!8e^4}(i+1)i2^i(2^{i+1}-1) = \frac{1}{8e^4}\sum_{i=1}^{\infty}\frac{1}{(i-1)!}(2^{2i}-2^i) \\ &\quad= \frac{1}{e^4}\sum_{i=1}^{\infty}\frac{4^{i-1}}{(i-1)!} - \frac{1}{4e^4}\sum_{i=1}^{\infty}\frac{2^{i-1}}{(i-1)!} = 1 - \frac{1}{4e^2}. \end{aligned} \tag{3.41}$$

Since all the hypotheses of Theorem 3.1 are satisfied, we conclude that there is a solution for the given equation (3.37) in $\mathcal{R}^1(I,M,M')$, where $M' = (M^* + K_1'M^2)/(\lambda_1\eta - K_0M^2)$ and $K_1' = \sum_{i=1}^{\infty}\lambda_i L_i' M^{2(i-1)}$.

COROLLARY 3.6. *Suppose (λ_n) is a sequence of nonnegative numbers with $\lambda_1 > 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$. Let $F \in \mathcal{F}^1_\delta(I,\lambda_1 M, M^*)$, where $\delta > 0$, M, $M^* \ge 0$.*

Assume further that

(i) $M > 1$,

(ii) $K_0 = (1/(M-1))\sum_{i=1}^{\infty}\lambda_{i+1}M^{i-1}(M^i-1)$ *and* $\gamma = \lambda_1 - K_0M^2 > 0$.

Then the functional series equation $\sum_{i=1}^{\infty}\lambda_i f^i(x) = F(x)$ has a solution f in $\mathcal{R}^1(I,M,M')$, where $M' = M^*/\gamma$.

Proof. The proof follows from Theorem 3.1 upon setting $H_i(x) \equiv x$ for each $i \in \mathbb{N}$. □

Example 3.7. Consider the following functional series equation

$$\sum_{i=1}^{\infty}\frac{26}{27^i}f^i(x) = \frac{2\pi}{3\sqrt{3}}\sin x, \quad x \in I = \left[0, \frac{\pi}{3}\right]. \tag{3.42}$$

Here we have

$$\lambda_i = \frac{26}{27^i}, \quad F(x) = \frac{2\pi}{3\sqrt{3}}, \quad H_i(x) = x \quad \forall x \in I,\ i \in \mathbb{N}. \tag{3.43}$$

Upon choosing

$$M = 3, \qquad M^* = \frac{\pi}{3}, \qquad \eta = 1, \qquad \delta = \frac{\pi}{3\sqrt{3}}, \tag{3.44}$$

it is readily seen that $\delta \le F'(x) \le 2\pi/3\sqrt{3}$ and $|F''(x)| \le \pi/3$ for all $x \in I$. Now $\lambda_1 \eta M = (26/27)(3) > 2\pi/3\sqrt{3}$ and so $F \in \mathcal{F}^1_\delta(I,\lambda_1\eta M, M^*)$. Clearly $\sum_{i=1}^{\infty}\lambda_i = 1$, $K_0M^2 = 13/24 < \lambda_1$, and $M' = M^*/(\lambda_1 - K_0M^2) = 72\pi/91$.

Thus by Corollary 3.6, there is a function f in $\mathcal{R}^1(I,3,72\pi/91)$ satisfying the functional series equation (3.42).

The main theorem of Zhang [5] can be deduced as a corollary to Theorem 3.1.

Corollary 3.8 (Zhang [5]). *Given positive constants δ, M, M^*, $n \in \mathbb{N}$, we suppose that $M > 1$ and $\lambda_1 > K_0M^2$, where $K_0 = 1/(M-1)\sum_{i=1}^{n}\lambda_{i+1}M^{i-1}(M^i - 1)$. Then for each $F \in \mathcal{F}^1_\delta(I,\lambda_1 M, M^*)$, there is a solution for the equation $\sum_{i=1}^{n}\lambda_i f^i(x) = F(x)$, $\lambda_1 > 0$, $\lambda_i \ge 0$, $i = 2,3,\dots,n$, $\sum_{i=1}^{n}\lambda_i = 1$ in $\mathcal{R}^1(I,M,M')$, where $M' = M^*(\lambda_1 - K_0M^2)^{-1}$.*

Proof. Setting $\lambda_i = 0$ for all $i > n$ in Corollary 3.6, the result follows. □

The following theorem proves the existence of a solution for an iterative functional series equation. Given $\delta > 0$, M, $M^* \ge 0$, we define

$$\begin{aligned}\mathcal{G}^1_\delta(I,M,M^*) = \{\phi \in C^1(I,\mathbb{R}) : \delta \le \phi'(x) \le M \ \forall x \in I \text{ and} \\ |\phi'(x_1) - \phi'(x_2)| \le M^*|x_1 - x_2| \ \forall x_1, x_2 \in I\}.\end{aligned} \tag{3.45}$$

Clearly $\mathcal{F}^1_\delta(I,M,M^*) \subseteq \mathcal{G}^1_\delta(I,M,M^*)$.

Theorem 3.9. *Suppose (λ_n) is a sequence of nonnegative numbers with $\lambda_1 > 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$. Let $F \in \mathcal{G}^1_\delta(I,\lambda_1\eta M, M^*)$, $H_1 \in \mathcal{F}^1_\eta(I,L_1,L_1')$, and $H_i \in \mathcal{R}^1(I,L_i,L_i')$ for $i = 2,3,\dots$, where δ, $\eta > 0$ and M, M^*, L_i, $L_i' \ge 0$ for all $i \in \mathbb{N}$.*

Assume further that

(i) $M_1 = ((b-a)/(F(b)-F(a)))M > 1$,

(ii) $K_0 = (1/(M_1 - 1))\sum_{i=1}^{\infty}\lambda_{i+1}L_{i+1}M_1^{i-1}(M_1^i - 1)$ *and* $\gamma = \lambda_1\eta - K_0M_1^2 > 0$,

(iii) $\sum_{i=1}^{\infty}\lambda_i L_i' M_1^{i-1}(M_1^i - 1) < \infty$.

Then the functional series equation

$$\sum_{i=1}^{\infty}\lambda_i H_i(f^i(x)) = \frac{(b-a)F(x) - bF(a) + aF(b)}{F(b) - F(a)}, \quad x \in I, \tag{3.46}$$

has a solution f in $\mathcal{R}^1(I,M_1,M_1')$, where $M_1' = (M_1^* + K_1'M_1^2)/\gamma$, $M_1^* = ((b-a)/(F(b) - F(a)))M_1$, and $K_1' = \sum_{i=1}^{\infty}\lambda_i L_i' M_1^{2(i-1)}$.

Proof. For a function $F \in \mathcal{G}^1_\delta(I,\lambda_1\eta M, M^*)$, the mapping $\tilde{F}$ defined by

$$\tilde{F}(x) = \frac{(b-a)F(x) - bF(a) + aF(b)}{F(b) - F(a)} \quad \forall x \in I \tag{3.47}$$

is readily seen to belong to $\mathcal{F}^1_{\delta_1}(I,\lambda_1\eta M_1,M_1^*)$, where $\delta_1 = ((b-a)/(F(b)-F(a)))\delta$. From Theorem 3.1, it now follows that $\sum_{i=1}^{\infty}\lambda_i H_i(\phi^i(x)) = \tilde{F}(x)$ for some $\phi \in \mathcal{R}^1(I,M_1,M_1')$. □

Corollary 3.10. *Let δ, $\eta > 0$, $M > 1$, L, $\lambda_i \geq 0$, $i \in \mathbb{N}$ with $\lambda_1 > 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$. Suppose that $\gamma = \lambda_1\eta - K_0M^2 > 0$, where $K_0 = (L/(M-1))\sum_{i=1}^{\infty}\lambda_{i+1}M^{i-1}(M^i-1)$. If $F \in \mathcal{F}^1_\delta(I,\lambda_1\eta M,M)$, $H_1 \in \mathcal{F}^1_\eta(I,L,L)$, and $H_i \in \mathcal{R}^1(I,L,L)$ for $i = 2,3,\ldots$, then there is a solution function ϕ for the equation $\sum_{i=1}^{\infty}\lambda_i H_i(\phi^i(x)) = F(x)$ in $\mathcal{R}^1(I,M,M')$, where $M' = M(1+K_1'M)/\gamma$ and $K_1' = L\sum_{i=1}^{\infty}\lambda_i M^{2(i-1)}$.*

Corollary 3.11. *Let δ, $\eta > 0$, $M > 1$, $\lambda_i \geq 0$, $i \in \mathbb{N}$ with $\lambda_1 > 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$. Suppose that $\gamma = \lambda_1\eta - K_0M^2 > 0$, where $K_0 = (1/(M-1))\sum_{i=1}^{\infty}\lambda_{i+1}M^i(M^i-1)$. If $F \in \mathcal{F}^1_\delta(I,\lambda_1\eta M, M)$, $H_1 \in \mathcal{F}^1_\eta(I,M,M)$, and $H_i \in \mathcal{R}^1(I,M,M)$ for $i = 2,3,\ldots$, then there is a solution function ϕ for the equation $\sum_{i=1}^{\infty}\lambda_i H_i(\phi^i(x)) = F(x)$ in $\mathcal{R}^1(I,M,M')$, where $M' = M(1+K_1'M)/\gamma$ and $K_1' = \sum_{i=1}^{\infty}\lambda_i M^{2i-1}$.*

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References

[1] R. G. Bartle, *The Elements of Real Analysis*, John Wiley & Sons, New York, 1964.
[2] J. G. Dhombres, *Itération linéaire d'ordre deux*, Publ. Math. Debrecen **24** (1977), no. 3-4, 277–287 (French).
[3] A. Mukherjea and J. S. Ratti, *On a functional equation involving iterates of a bijection on the unit interval*, Nonlinear Anal. **7** (1983), no. 8, 899–908.
[4] S. Nabeya, *On the functional equation $f(p+qx+rf(x)) = a+bx+cf(x)$*, Aequationes Math. **11** (1974), 199–211.
[5] W. Zhang, *Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n}\lambda_i f^i(x) = F(x)$*, Nonlinear Anal. **15** (1990), no. 4, 387–398.

V. Murugan: Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
E-mail address: murugan@iitm.ac.in

P.V. Subrahmanyam: Department of Mathematics, Indian Institute of Technology Madras, Chennai 600 036, India
E-mail address: pvs@iitm.ac.in